# OBSERVED DATA BASED ESTIMATOR AND BOTH OBSERVED DATA AND PRIOR INFORMATION BASED ESTIMATOR UNDER ASYMMETRIC LOSSES

Zahirul Hoque, Department of Statistics, College of Business and Economics, United Arab Emirates University, Al Ain, UAE
Jacek Wesolowski, Warsaw University of Technology, Poland
Shahjahan Khan, University of Southern Queensland, Australia

## SYNOPTIC ABSTRACT

In Bayesian approach of statistical analyses we incorporate the prior information about the parameter of the model with the observed data. This prior information is in the form of a prior distribution of the parameter. If the prior information is available as a constant value of the parameter rather than its' prior distribution, the Bayesian approach cannot be pursued. However, there are estimation methods that incorporate such prior information with the observed data. The expectation is that the incorporation of such additional information in the estimation process would result in a better estimator than that based on the observed data alone. In some cases this may be true, but in many other cases the risk of worse consequences cannot be ruled out. This paper studies the performance of the observed data based unrestricted estimator (UE), and both observed data and prior information based preliminary test estimator (PTE) of the univariate normal mean under the linex loss function. The risk functions of both UE and PTE are derived. The moment generating function (MGF) of PTE is derived which turns out to be a component of the risk function. From the MGF the first two moments of PTE are obtained and found to be identical to that obtained by using different approaches in Khan and Saleh (2001) and Zellner (1986). Under the linex loss criterion the performance of the PTE is compared with that of UE. It is revealed that if the uncertain non-sample prior information about the value of the mean is not too far from its true value, PTE outperforms UE.

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Key Words and Phrases: asymmetric loss; non-sample prior information; maximum likelihood and preliminary test estimators; risk function.

## 1. INTRODUCTION

The popularity of the squared error loss function is due to its mathematical and interpretational convenience. To compare the performance of different estimators of unknown parameters this loss function is used in many studies. For a recent account on the topic readers may see Khan and Saleh (2001) and the references therein. In spite of the wide popularity of this symmetric loss, many authors have recognized its inappropriateness in various problems (cf. Varian, 1975). As pointed out by Zellner (1986), the admissibility of an estimator may depend quite sensitively on features of the loss function, such as symmetry, is not generally appreciated. Due to the symmetric nature of the squared error loss it cannot differentiate overestimation from underestimation, and hence attaches equal weights to both.

In real life situations there are numerous cases where underestimation of a parameter leads to more (or less) severe consequences than overestimation. In dam construction, for example, underestimation of the peak water level is more serious than overestimation. On the other hand, for a manufacturing company, overestimation of the mean life of the product for the purposes of customers warranty is more serious than underestimation. As the squared error loss function is unable to assign appropriate unequal weights for underestimation and overestimation of any parameter, the use of this loss function is inappropriate in such cases and hence not useful.

In an applied study of real estate assessment, Varian (1975) introduced a very useful non-symmetric loss function called the linex loss function that has both linear and exponential components. The linex loss function assigns unequal weights to the underestimation and overestimation by introducing a shape parameter. For small values of the shape parameter the linex loss function is approximately symmetric and not much different from the squared error loss function. The linex loss function is more general than the squared error loss function as the latter is a special case of the former.

Zellner (1986) studied the properties of estimation and prediction procedures under the linex loss function. He showed that some usual estimators that are admissible under the squared error loss function are inadmissible under the linex loss function. For example, he proved that the UE, $\overline{X}$, of the univariate normal mean is inadmissible under the linex loss function, as the risk of the estimator $\overline{X} - a\sigma^2/2n$ is less than that of the UE, where $a$ is the shape parameter of the linex loss function, $\sigma^2$ is the population variance and $n$ is the size of the sample. In the case of unknown $\sigma^2$, he suggested using it's UE. Later, Rojo (1987) generalized Zellner's result and showed that under the linex loss function any estimator of $\theta$, of the form $c\overline{x} + d$, is admissible if either $0 \le c \le 1$, or $c = 1$ and $d = a\sigma^2/2n$. Otherwise, $c\overline{x} + d$ is inadmissible. Further contributions to this area include Parsian, Farispour, and Nematullahi (1993), Pandey and Rai (1992) and Bhattacharaya, Samaniego, and Vestrup (2002), to mention a few.

The exclusive sample information based MLE, popularly known as UE, of the population mean is uniformly minimum variance unbiased and minimax (with respect to the squared error loss criterion) estimator. The natural expectation is that the use of additional information such as non-sample prior information along with the sample information would result in a better estimator than the exclusive sample information based estimator. Based on both sample and non-sample prior information, Bancroft (1944) defined the PTE and showed that with respect to the squared error loss function it outperforms the UE under certain conditions. Khan and Saleh (2001) introduced a coefficient of distrust $d$ $(0 \leq d \leq 1)$, a measure of degree of lack of trust on the non-sample prior information, to the definition of the PTE. They called their new estimator the shrinkage PTE (SPTE) and showed that with respect to squared error loss function the SPTE outperforms the UE in certain subspace of the parameter space. For $d=0$ the SPTE becomes the PTE.

In this paper the MGF of the PTE is obtained. This MGF is instrumental to the derivation of the sampling distribution of the PTE. From the MGF the first moment of the PTE is obtained. This moment is used to derive the risk function of the PTE under the linex loss. The performance of the PTE relative to the UE is investigated. A table of maximum and minimum guaranteed efficiencies of the PTE relative to the UE is provided for selected sample sizes and size of the preliminary test. It is observed that if the non-sample prior information about the value of the population mean is not too far from its true value the PTE outperforms the UE. Similar to the form of the linex loss function, the form of the risk function of the PTE is also asymmetric. However, for very small value of the shape parameter of the linex loss function the form of the risk function of the PTE is almost symmetric.

The layout of this paper is as follows. The linex loss function is briefly described in Section 2. Some useful lemmas are proved in Section 3. The estimators of the univariate normal mean and their risk functions under the linex loss function are derived in Section 4. The analysis of the risk functions and a table of maximum and minimum guaranteed efficiencies of the PTE relative to the UE are presented in Section 5. Some concluding remarks are presented in Section 6.

## 2. LINEX LOSS FUNCTION

The linex loss function, proposed by Varian (1975), of $\theta^*$ for estimating any parameter $\theta$ is given by

$$L(\delta) = b[\exp(a\delta) - a\delta - 1] \qquad \forall\ a \neq 0, b > 0 \tag{1}$$

where $\delta = (\theta^* - \theta)$ is the estimation error. The two parameters $a$ and $b$ in $L(\delta)$ serve to determine the shape and scale, respectively, of $L(\delta)$. A positive value of $a$ implies that overestimation is more serious than underestimation and a negative value of $a$ represents the reverse situation. The magnitude of $a$

reflects the degree of asymmetry about $\delta = 0$. This asymmetric loss function grows approximately linearly on one side of $\delta = 0$ and grows approximately on

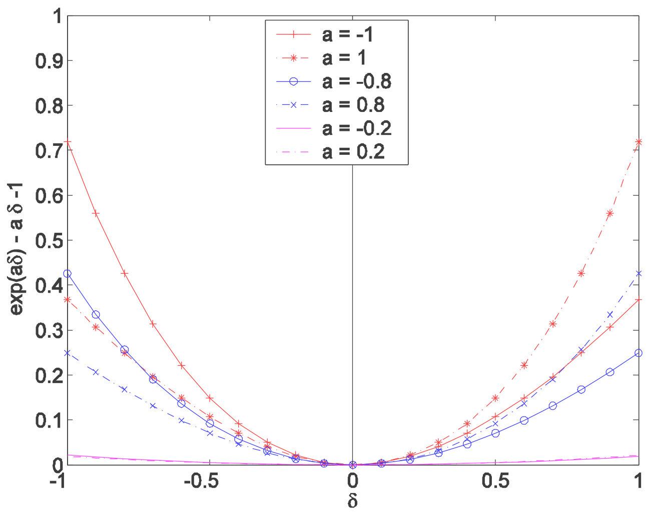


FIGURE 1. The plot of the linex loss functions for selected values of $a$.

exponentially on the other side. If $a \to 0$, then the linex loss function reduces to the squared error loss function.

Figure 1 displays the form of linex loss function for selected values of $a$ against a range of values of $\delta$. It is clear that if $a = 1$ the growth of the loss is approximately linear for negative values of $\delta$, while for positive values of $\delta$ it is approximately exponential. For $a = -1$, the situation is reversed. As $a$ approaches *0*, the growth pattern of linex loss becomes similar for both positive and negative errors of estimation and approaches the quadratic loss. Hence, the linex loss function is more general than the quadratic loss function. Further details about this loss function can be found in Zellner (1986).

## 3. SOME PRELIMINARIES

Let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size $n$ from a univariate normal distribution with unknown mean $\mu$ and variance $\sigma^2$. In this section we derive three important results that are essential to derive the risk functions of the UE and PTE of $\mu$ under the linex loss function. The usual UE of $\mu$ is $\overline{X}$ and an unbiased estimator of $\sigma^2$ is $S^2 = \sum_{i=1}^{n} (X_i - \overline{X})^2 / (n-1)$.

**Lemma 2:** *If Z ~ N(0,1), and Z and S are independent then for any Borel measurable function* $\phi : \Re \times (0, \infty) \to \Re$ *and for any* $c \in \Re$,

$$\mathrm{E}[\exp(cZ)\phi(Z,S) = \exp(c^2/2)\,\mathrm{E}[\phi(Z+c,S)]$$

*provided* $\exp(cZ)\phi(Z,S)$ *is integrable.*

**Proof:** By definition

$$\mathrm{E}\left[\exp(cZ)\phi(Z,S)\right] = \mathrm{E}\left[\mathrm{E}(\exp(cZ)\phi(Z,S) \mid S)\right]$$

$$\mathrm{E}\left[\exp(cZ)\phi(Z,S)\right] = \mathrm{E}\left(\frac{1}{\sqrt{2\pi}}\int_{\Re}\phi(z,S)\,\exp(cz - z^2/2)\,dz\right)$$

$$= \exp(c^2/2)\mathrm{E}\left(\frac{1}{\sqrt{2\pi}}\int_{\Re}\phi(z,S)\,\exp\left(-\frac{1}{2}(z-c)^2\right)dz\right) \qquad (3)$$

Consider $U = Z - c$ . The Jacobian of the transformation is $|J|=1$. Therefore,

$$\mathrm{E}\left[\exp(cZ)\phi(Z,S)\right] = \exp(c^2/2)\mathrm{E}\left(\frac{1}{\sqrt{2\pi}}\int_{\Re}\phi(u+c,S)\,\exp(-u^2/2)\,du\right)$$

$$= \exp(c^2/2)\;\mathrm{E}\left[\phi(\mathrm{Z}+\mathrm{c},\mathrm{S})\right]. \qquad (4)$$

This completes the proof of Lemma 5.

**Lemma 5:** *If $f_{t,k,\delta}(\cdot)$ is the density function of non-central Student's t distribution with k d.f. and non-centrality parameter δ, and $f_{F(1,k,\delta^2)}(\cdot)$ is the density function of non-central F distribution with (1, k) d.f. and non-centrality parameter $\delta^2$ then for any $x > 0$*

$$f_{t(k,\delta)}(x) + f_{t(k,\delta)}(-x) = 2x\,f_{F(1,k,\delta^2)}(x^2).$$

**Proof:** The density function of the non-central Student's $t$ variable with $k$ d.f. and non-centrality parameter $\delta$ is given by (cf. Evans, Hasting, and Peacock 2000, p. 184)

$$f_{t(k,\delta)}(x) = \frac{k^{k/2}\exp(-\delta^2/2)}{\Gamma(k/2)\sqrt{\pi}\;(k+x^2)^{(k+1)/2}}\sum_{i=0}^{\infty}\Gamma\left(\frac{k+1+i}{2}\right)\frac{(x\delta)^i}{i!}\left(\frac{2}{k+x^2}\right)^{i/2}. \qquad (6)$$

Consider now $f_{t(k,\delta)}(x) + f_{t(k,\delta)}(-x)$ for an arbitrary $x \ge 0$. Then (6) implies that the terms of the series with odd powers of $x$ cancel and the terms with even powers of $x$ will double. Thus,

$$f_{t(k,\delta)}(x) + f_{t(k,\delta)}(-x) = \frac{2k^{k/2}\exp\left(-\frac{\delta^2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{\pi}\;(k+x^2)^{\frac{k+1}{2}}}\sum_{i=0}^{\infty}\Gamma\left(\frac{k+1}{2}+i\right)\frac{x^{2i}\delta^{2i}}{(2i)!}\left(\frac{2}{k+x^2}\right)^{i/2}$$

$$= \frac{2k^{k/2}\exp\left(-\frac{\delta^2}{2}\right)}{\Gamma\left(\frac{k}{2}\right)\sqrt{\pi}\;(k+x^2)^{\frac{k+1}{2}}}\sum_{i=0}^{\infty}\Gamma\left(\frac{k+1+i}{2}\right)\frac{(x^2)^i(\delta^2)^i 2^i}{2^i\, i!\;(2\mathrm{i}-1)!!\,(k+x^2)^i}$$

$$f_{t(k,\delta)}(x)+f_{t(k,\delta)}(-x)=2x\frac{k^{k/2}\exp\left(-\frac{\delta^2}{2}\right)(x^2)^{-1/2}}{\Gamma\left(\frac{k}{2}\right)(k+x^2)^{\frac{k+1}{2}}}\sum_{i=0}^{\infty}\frac{\Gamma\left(\frac{k+1}{2}+i\right)}{i!\;\Gamma\left(\frac{1}{2}+\mathrm{i}\right)}\left\{\frac{x^2\delta^2}{2(k+x^2)}\right\}^i. \tag{7}$$

Here $(2i-1)!!=(2i-1)\times(2i-3)\times\cdots\times5\times3\times1$. Using the expression of non-central $F$ density from EHP(2000, p. 95) we can write

$$f_{t,k,\delta}(x)+f_{t(k,\delta)}(-x)=2x\,f_{F(1,k,\delta^2)}(x^2). \tag{8}$$

This completes the proof of Lemma 5.

**Lemma 9**: *For any positive integers m and n*

$$\frac{\partial f_{F(m,n,D)}(x)}{\partial D}=-\frac{1}{2}f_{F(m,n,D)}(x)+\frac{m}{2(m+2)}f_{F(m+2,n,D)}\left(\frac{mx}{m+2}\right),\quad x,D\in\Re$$

*where* $f_{F(k,l,D)}$ *denotes the density function of the non-central F with (k,l) d.f. and non-centrality parameter D.*

**Proof:** The density function of the non-central $F$ variable with $(m,n)$ d.f. and non-centrality parameter $D$ is given by (cf. EHP 2000, p.95)

$$f_{F(m,n,D)}(x)=\frac{\exp\left(-\frac{D}{2}\right)m^{\frac{m}{2}}n^{\frac{n}{2}}x^{\frac{m}{2}-1}}{\Gamma\left(\frac{n}{2}\right)(n+mx)^{\frac{m+n}{2}}}\sum_{j=0}^{\infty}\left(\frac{mxD}{2(n+mx)}\right)^j\frac{\Gamma\left(\frac{m+n}{2}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)j!}. \tag{10}$$

Differentiating both sides of (10) with respect to $D$, we get

$$\frac{\partial f_{F(m,n,D)}(x)}{\partial D}=\frac{1}{2}f_{F(m,n,D)}(x)+\frac{\exp(-D/2)m^{m/2}n^{n/2}}{\Gamma(n/2)}\frac{x^{m/2-1}}{(n+mx)^{(m+n)/2}}$$

$$\times\sum_{j=1}^{\infty}\left(\frac{mx}{2(n+mx)}\right)^j\frac{D^{j-1}}{(j-1)!}\frac{\Gamma\left(\frac{m+n}{2}+j\right)}{\Gamma\left(\frac{m}{2}+j\right)}$$

$$=\frac{1}{2}f_{F(m,n,D)}(x)+\frac{\exp(-D/2)m^{m/2}n^{n/2}}{\Gamma(n/2)}\frac{x^{m/2-1}}{(n+mx)^{(m+n)/2}}$$

$$\times\sum_{i=0}^{\infty}\left(\frac{mx}{2(n+mx)}\right)^{i+1}\frac{D^i}{i!}\frac{\Gamma\left(\frac{m+n}{2}+i\right)}{\Gamma\left(\frac{m+2}{2}+i\right)}$$

$$= \frac{1}{2} f_{F(m,n,D)}(x) + \frac{\exp(-D/2) m^{m/2+1} n^{n/2}}{2\Gamma(n/2)} \frac{x^{(m+2)/2-1}}{(n+mx)^{(m+2+n)/2}}$$

$$\times \sum_{i=0}^{\infty} \left( \frac{mxD}{2(n+mx)} \right)^{i} \frac{\Gamma\left( \frac{m+2+n}{2} + i \right)}{i!\ \Gamma\left( \frac{m+2}{2} + i \right)}$$

$$= -\frac{1}{2} f_{F(m,n,D)}(x) + \frac{\exp(-D/2) m^{\frac{m+2}{2}} n^{n/2} \left( \frac{mx}{m+2} \right)^{\frac{m+2}{2}-1} \left( \frac{m+2}{m} \right)^{m/2}}{2\ \Gamma(n/2) \left( n + (m+2) \frac{mx}{m+2} \right)^{(m+2+n)/2}}$$

$$\times \sum_{i=0}^{\infty} \left( \frac{(m+2) \frac{mxD}{m+2}}{2\left\{ n + (m+2) \frac{mx}{m+2} \right\}} \right)^{i} \frac{\Gamma\left( \frac{m+2+n}{2} + i \right)}{i!\ \Gamma\left( \frac{m+2}{2} + i \right)}$$

$$= -\frac{1}{2} f_{F(m,n,D)}(x) + \frac{m \exp(-D/2)(m+2)^{\frac{m+2}{2}} n^{n/2} \left( \frac{mx}{m+2} \right)^{\frac{m+2}{2}-1}}{2(m+2)\Gamma(n/2) \left( n + (m+2) \frac{mx}{m+2} \right)^{(m+2+n)/2}}$$

$$\times \sum_{i=0}^{\infty} \left( \frac{(m+2) \frac{mx}{m+2} D}{2\left\{ n + (m+2) \frac{mx}{m+2} \right\}} \right)^{i} \frac{\Gamma\left( \frac{m+2+n}{2} + i \right)}{i!\ \Gamma\left( \frac{m+2}{2} + i \right)}. \tag{11}$$

Therefore,

$$\frac{\partial f_{F(m,n,D)}(x)}{\partial D} = -\frac{1}{2} f_{F(m,n,D)}(x) + \frac{m}{2(m+2)} f_{F(m+2,n,D)} \left( \frac{mx}{(m+2)} \right). \tag{12}$$

This completes the proof of Lemma 9.

## 4. THE ESTIMATORS AND RISKS

The exclusive sample information based UE of $\mu$ is $\tilde{\mu} = \overline{X}$. The risk function of the exclusive sample information based UE $\tilde{\mu}$ of $\mu$ under the linex loss function is stated in the following theorem.

**Theorem 13:** *The risk function of the UE of* $\mu$ *under the linex loss function is given by*

$$R[\tilde{\mu};\mu] = \exp(a_1^{\,2}/2) - 1$$

*where* $a_1 = a\sigma/\sqrt{n}$.

For proof of the theorem readers may see Zellner (1986).

Suppose that the non-sample prior information about the value of $\mu$ is available from experts knowledge or previous experience of the researchers. According to Fisher this non-sample prior information can be expressed in the form of the null hypothesis $H_0 : \mu = \mu_0$ (cf. Khan and Saleh, 2001). This hypothesized value of $\mu$ is known as the restricted estimator (RE). Under the null hypothesis, the RE outperforms the UE. Otherwise, the UE outperforms the RE. Therefore, it is a natural expectation that an estimator that uses both sample and non-sample prior information about the value of $\mu$, will outperform both the UE and RE. As we are not sure that the non-sample prior information is quite true, Fisher suggested to remove the uncertainty by performing an appropriate statistical test on the null hypothesis. To test the null hypothesis against the alternative hypothesis $H_1 : \mu \neq \mu_0$ an appropriate test is the likelihood ratio test, and the test statistic is given by $t = \sqrt{n}(\tilde{\mu} - \mu_0)/S$. Under the null hypotheses the distribution of *t* is central Student's *t* distribution with $\nu = n-1$ degrees of freedom (d.f.), and under $H_1$ it follows a non-central Student's *t* distribution with the same d.f. and non-centrality parameter $\Delta = \sqrt{n}(\mu - \mu_0)/\sigma$. Following Bancroft (1944), the PTE of $\mu$ is defined as

$$\hat{\mu}^{\text{PTE}} = \tilde{\mu} - (\tilde{\mu} - \mu_0)\, I(|\, t\,| < t_{\alpha/2}) \tag{14}$$

where $t_{\alpha/2}$ is the $\alpha$-level critical value of *t*-statistic and $I(A)$ is the indicator function of the set *A* which takes the value 1 when the argument holds and 0, otherwise. The risk function of the PTE of $\mu$ is stated in the following theorem.

**Theorem 15:** *The risk function of the PTE of* $\mu$ *under the linex loss function is given by*

$$\begin{aligned}\mathrm{R}(\hat{\mu}^{\text{PTE}};\mu] = {} & \exp(-a_1\Delta)\, G_{1,\nu}(c^2;\Delta^2) + \exp(a_1^{\,2}/2)\left[1 - G_{1,\nu}(c^2;(\Delta + a_1)^2)\right] \\ & + a_1\Delta G_{3,\nu}(c^2/3;\Delta^2) - 1\end{aligned}$$

*where* $c^2$ *is the* $\alpha$*-level critical value of the F distribution with* $(1,\nu)$ *d.f. and* $G_{a,b}(c^2;\theta)$ *is the cumulative distribution function of the non-central F variable with* $(a,b)$ *d.f. and non-centrality parameter* $\theta$*, evaluated at c.*

**Proof:** The risk function of the PTE, $\hat{\mu}^{\text{PTE}}$, of $\mu$ under the linex loss function is

$$\mathrm{R}[\hat{\mu}^{\text{PTE}};\mu] = \mathrm{E}[\exp(a\Phi)] - a\,\mathrm{E}[\Phi] - 1 \tag{16}$$

where $\Phi = \hat{\mu}^{PTE} - \mu$ .

The first term of the right hand side (R.H.S.) of (16) can be expressed as

$$\mathrm{E}[\exp(a\Phi)] = \exp(a(\mu_0 - \mu)\, P(|\,t\,| < c) + \mathrm{E}[\exp(a(\tilde{\mu} - \mu)\, P((|\,t\,| \geq c)] \qquad (17)$$

where $c = t_{\alpha/2}$ is the $\alpha$ -level critical value of the Student's *t* statistic for two-sided test.

The first term of the R.H.S. of (17) is

$$\begin{aligned}\exp(a(\mu_0 - \mu)\, P(|\,t\,| < c) &= \exp(a(\mu_0 - \mu))\, P(t^2 < c^2) \\ &= \exp(-a_1\Delta)\, G_{1,\nu}(c^2; \Delta^2) \qquad (18)\end{aligned}$$

where $G_{1,\nu}(c^2;\Delta^2)$ is the cumulative distribution function of a non-central *F* distribution with $(1,\nu)$ d.f. and non-centrality parameter $\Delta^2$. The second term of the R.H.S. of (17) can be written as

$$\mathrm{E}[\exp(a(\tilde{\mu} - \mu)\; I(|t| \geq c)] = \mathrm{E}\big[\exp(a_1 Z)\;\; I(|\sigma(Z + \Delta)S^{-1}| \geq c)\big] \qquad (19)$$

Applying Lemma 1 with $\phi(X,Y) = I(|\,\sigma(X+\Delta)Y^{-1}| \geq c)$, in (19) we get

$$\begin{aligned}\mathrm{E}[\exp(a(\tilde{\mu} - \mu)\; I(|t| \geq c)] &= \exp(a_1{}^2/2)\, \mathrm{E}[I(|\sigma(Z + a_1 + \Delta)S^{-1}| \geq c)] \\ &= \exp(a_1{}^2/2)\big[1 - F_{t(\nu,\Delta+a1)}(c) - F_{t(\nu,\Delta+a1)}(-c)\big] \\ &= \exp(a_1{}^2/2)\big[1 - G_{1,\nu}(c^2;(\Delta + a_1)^2)\big] \qquad (20)\end{aligned}$$

where $F_{t(\nu,\Delta+a_1)}$ is the cumulative distribution function of non-central *t* variable with $\nu$ d.f. and non-centrality parameter $\Delta + a_1$ evaluated at *c*. Combining (18) and (20) the first term of the right hand side (16) is obtained as

$$\mathrm{E}[\exp(a\Phi)] = \exp(-a_1\Delta)G_{1,\nu}(c^2;\Delta^2) + \exp(a_1{}^2/2)[1 - G_{1,\nu}(c^2;(\Delta + a_1)^2)]. \qquad (21)$$

Now we compute the second term of the right hand side of (16). From (21) the MGF of the PTE of $\mu$ is

$$\begin{aligned}&\mathrm{E}[\exp(a\{(\tilde{\mu} - \mu) - (\tilde{\mu} - \mu_0)I(|\,t\,| < c)\})] \\ &= \exp(-a_1\Delta)\left[\int_{-c}^{0} f_{t(\nu,\Delta)}(x)dx + \int_{0}^{c} f_{t(\nu,\Delta)}(x)dx\right] \\ &\quad + \exp(a_1{}^2/2)\left[1 - \int_{-c}^{0} f_{t(\nu,\Delta-a_1)}(x) - \int_{0}^{c} f_{t(\nu,\Delta-a_1)}(x)dx\right] \\ &= m(a), \text{ (say)} \qquad (22)\end{aligned}$$

Writing $g_{t(\nu,\Delta)}(x) = f_{t(\nu,\Delta)}(x) + f_{t(\nu,\Delta)}(-x)$ for any *x>0* in (22) we get

$$m(a) = \exp(-a_1\Delta)\int_0^c g_{t(\nu,\Delta)}(x)dx + \exp(a_1{}^2/2)\int_c^\infty g_{t(\nu,\Delta-a_1})(x)dx. \qquad (23)$$

Applying Lemma 2 in (23) we get

$$m(a) = \exp(-a_1\Delta)\int_0^c 2x\, f_{F(1,\nu,\Delta^2)}(x^2)dx + \exp(a_1^2/2)\int_c^\infty 2x\, f_{F\left(1,\nu,(-\Delta-a_1)^2\right)}(x^2)dx$$

$$= \exp(-a_1\Delta)\int_0^{c^2} f_{F(1,\nu,\Delta^2)}(y)dy + \exp(a_1^2/2)\int_{c^2}^\infty f_{F\left(1,\nu,(-\Delta-a_1)^2\right)}(y)dy. \qquad (24)$$

Differentiating both sides of (24) with respect to $a$, then using Lemma 3 and finally changing the variable $y/3 = t$ in the left integral we get

$$\frac{\sigma}{\sqrt{n}}\left[-\Delta\exp(-a_1\Delta)\int_0^{c^2} f_{F(1,\nu,\Delta^2)}(y)dy + a_1\exp(a_1^2/2)\int_{c^2}^\infty f_{F\left(1,\nu,(-\Delta-a_1)^2\right)}(y)dy\right.$$

$$\left. + 2(\Delta+a_1)\exp\left(-\frac{a_1^2}{2}\right)\int_{c^2}^\infty -\left(\frac{1}{2}f_{F(1,\nu,(\Delta+a_1)^2)}(y) + \frac{1}{6}f_{F(3,\nu,(\Delta+a_1)^2)}\left(\frac{y}{3}\right)\right)dy\right]$$

$$= \frac{\sigma}{\sqrt{n}}\left[-\Delta\exp(-a_1\Delta)\int_0^{c^2} f_{F(1,\nu,\Delta^2)}(y)dy + a_1\exp(a_1^2/2)\int_{c^2}^\infty f_{F\left(1,\nu,(\Delta+a_1)^2\right)}(y)dy\right.$$

$$- (\Delta+a_1)\exp(-a_1^2/2)\int_{c^2}^\infty f_{F\left(1,\nu,(\Delta+a_1)^2\right)}(y)dy + (\Delta+a_1)$$

$$\left.\times\exp(-a_1^2/2)\int_{c^2/3}^\infty f_{F\left(3,\nu,(\Delta+a_1)^2\right)}(y)dy\right]$$

$$= m'(a) \qquad (25)$$

Putting $a = 0$ in (25),

$$\mathrm{E}(\Phi) = -(\mu-\mu_0)\,G_{3,\nu}(c^2/3;\Delta^2) \qquad (26)$$

which is the bias function of the PTE of $\mu$.

Collecting (21) and (26), and substituting in (16) the risk function of the PTE of $\mu$ under the linex loss function is obtained as

$$\mathrm{R}[\hat{\mu}^{\mathrm{PTE}};\mu] = \exp(-a_1\Delta)G_{1,\nu}(c^2;\Delta^2) + \exp(a_1^2/2)[1-G_{1,\nu}(c^2;(\Delta+a_1)^2]$$

$$+ a_1\Delta\, G_{3,\nu}(c^2/3;\Delta^2) - 1. \qquad (27)$$

This completes the proof of the theorem.

## 5. ANALYSIS OF THE RISK FUNCTIONS

In this section, we discuss some salient features of the risk functions of the UE and PTE of the mean $\mu$ relative to the change of $\Delta$ and $a$.

**The Risk of the UE**

Clearly, the risk function of the UE of $\mu$ is independent of $\delta$, and hence of $\Delta$. However, it depends on the magnitude of $|a|$, but not on its sign. From the functional form of the risk of the UE it is evident that as $\Delta$ grows larger, the risk of the UE also grows larger.

**The Risk of the PTE**

For any non-zero value of $\Delta$, the risk function of the PTE of $\mu$ can be written as

$$R[\hat{\mu}^{PTE};\mu] = R[\tilde{\mu}] + g(\Delta) \tag{28}$$

where

$$g(\Delta) = \exp(-a_1\Delta)G_{1,\nu}(c^2;\Delta^2) + a_1\Delta G_{3,\nu}\left(\frac{c^2}{3};\Delta^2\right) - \exp\left(\frac{a_1^{\ 2}}{2}\right)G_{1,\nu}(c^2;(\Delta+a_1)^2).$$

Under the null hypothesis, $\Delta = 0$ and hence the risk of the PTE of $\mu$ is

$$R[\hat{\mu}^{PTE};\mu] = R[\tilde{\mu}] + G_{1,\nu}(c^2;0) - \exp(a_1^{\ 2}/2)G_{1,\nu}(c^2;a_1^{\ 2}). \tag{29}$$

For any $a \neq 0$, $G_{1,\nu}(c^2:0) - \exp(a_1^{\ 2})G_{1,\nu}(c^2;a_1^{\ 2}) < 0$. Therefore, at $\Delta = 0$, the risk of the PTE is less than that of the UE. This result is known for the PTE of $\mu$ under the squared error loss (cf. Khan and Saleh 2001).

For any positive value of $a$, if the value of $\Delta$ is positive the value of $a_1\Delta G_{3,\nu}(c^2/3,\Delta^2)$ is also positive. Therefore, for positive values of $a$, as $\Delta$ grows larger from zero, the risk of the PTE grows larger and crosses the risk of the UE at

$$\Delta = \frac{\exp(a_1^{\ 2}/2)G_{1,\nu}(c^2;(\Delta+a_1)^2) - \exp(-a_1\Delta)G_{1,\nu}(c^2;\Delta^2)}{a_1 G_{3,\nu}(c^2/3;\Delta^2)} \tag{30}$$

regardless of the value of $a$.

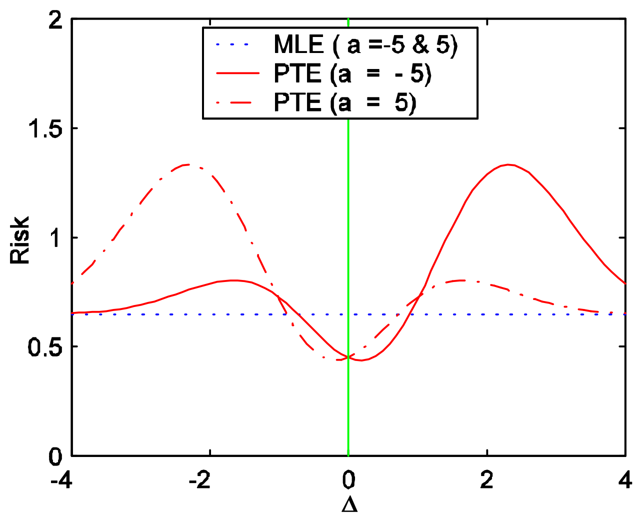


Figure 2: The risk curves of the UE and PTE for $\alpha = 0$, $n$=25, $\sigma = 1$ and selected values of $a$.

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and minimum guaranteed efficiency ($Eff_o$) of the PTE of $\mu$ relative to the UE, and the value of $\Delta$ ($\Delta_o$) at which the minimum guaranteed efficiency attains,

TABLE 1. Maximum and minimum efficiencies of the PTE of $\mu$ relative to the UE for $a = 3$.

| $\alpha$ | | Sample size, $n$ | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
| 0.05 | $Eff^*$ | 4.2674 | 3.9892 | 3.8713 | 3.8064 | 3.7653 | 3.7369 | 3.7162 |
| | $Eff_o$ | 0.2661 | 0.3076 | 0.3279 | 0.3279 | 0.3401 | 0.3401 | 0.3545 |
| | $\Delta_o$ | -2.6300 | -2.4700 | -2.3850 | -2.3950 | -2.3550 | -2.3250 | -2.3004 |
| 0.10 | $Eff^*$ | 2.5613 | 2.4468 | 2.3978 | 2.3706 | 2.3534 | 2.3415 | 2.3328 |
| | $Eff_o$ | 0.3807 | 0.4202 | 0.4393 | 0.4507 | 0.4585 | 0.4641 | 0.4684 |
| | $\Delta_o$ | -2.3550 | -2.2255 | -2.1515 | -2.1448 | -2.1050 | -2.1105 | -2.0850 |
| 0.15 | $Eff^*$ | 1.9556 | 1.8907 | 1.8629 | 1.8474 | 1.8376 | 1.8308 | 1.8258 |
| | $Eff_o$ | 0.4748 | 0.5112 | 0.5285 | 0.5390 | 0.5462 | 0.5511 | 0.5550 |
| | $\Delta_o$ | -2.1750 | -2.0850 | -2.0390 | -2.0100 | -2.0055 | -2.0025 | -1.9750 |
| 0.20 | $Eff^*$ | 1.6429 | 1.6014 | 1.5835 | 1.5736 | 1.5673 | 1.5629 | 1.5597 |
| | $Eff_o$ | 0.5573 | 0.5900 | 0.6055 | 0.6148 | 0.6211 | 0.6256 | 0.6291 |
| | $\Delta_o$ | -2.0610 | -1.9800 | -1.9500 | -1.9302 | -1.9100 | -1.9000 | -1.8950 |
| 0.25 | $Eff^*$ | 1.4530 | 1.4247 | 1.4125 | 1.4057 | 1.4014 | 1.3984 | 1.3962 |
| | $Eff_o$ | 0.6310 | 0.6597 | 0.6733 | 0.6814 | 0.6868 | 0.6908 | 0.6938 |
| | $\Delta_o$ | -1.9850 | -1.9250 | -1.8950 | -1.8755 | -1.8609 | -1.8550 | -1.8500 |
| 0.30 | $Eff^*$ | 1.3268 | 1.3068 | 1.2982 | 1.2934 | 1.2904 | 1.2883 | 1.2867 |
| | $Eff_o$ | 0.6969 | 0.7215 | 0.7331 | 0.7400 | 0.7446 | 0.7480 | 0.7506 |
| | $\Delta_o$ | -1.9250 | -1.8650 | -1.8459 | -1.8255 | -1.8100 | -1.8080 | -1.7968 |
| 0.35 | $Eff^*$ | 1.2382 | 1.2239 | 1.2177 | 1.2143 | 1.2121 | 1.2105 | 1.2094 |
| | $Eff_o$ | 0.7553 | 0.7759 | 0.7856 | 0.7913 | 0.7952 | 0.7980 | 0.8001 |
| | $\Delta_o$ | -1.8759 | -1.8260 | -1.7992 | -1.7900 | -1.7890 | -1.7645 | -1.7700 |
| 0.40 | $Eff^*$ | 1.1739 | 1.1635 | 1.1590 | 1.1565 | 1.1549 | 1.1538 | 1.1530 |
| | $Eff_o$ | 0.8064 | 0.8232 | 0.8311 | 0.8357 | 0.8389 | 0.8412 | 0.8429 |
| | $\Delta_o$ | -1.8352 | -1.7989 | -1.7777 | -1.7657 | -1.7559 | -1.7325 | -1.7300 |
| 0.45 | $Eff^*$ | 1.1261 | 1.1186 | 1.1154 | 1.1136 | 1.1124 | 1.1116 | 1.1111 |
| | $Eff_o$ | 0.8429 | 0.8637 | 0.8629 | 0.8716 | 0.8700 | 0.8619 | 0.8600 |
| | $\Delta_o$ | -1.8020 | -1.7659 | -1.7592 | -1.7434 | -1.7413 | -1.7400 | -1.7375 |
| 0.50 | $Eff^*$ | 1.0902 | 1.0849 | 1.0826 | 1.0813 | 1.0805 | 1.0799 | 1.0795 |
| | $Eff_o$ | 0.8876 | 0.8978 | 0.9025 | 0.9053 | 0.9072 | 0.9086 | 0.9096 |
| | $\Delta_o$ | -1.7750 | -1.7450 | -1.7236 | -1.7100 | -1.7000 | -1.6959 | -1.7100 |

for selected values of $a, n$ and $\alpha$. For example, if $a = 1$ and $n = 20$, and the experimenter wishes to achieve the minimum guaranteed efficiency 0.6055 of the PTE of $\mu$, the recommended value of $\alpha$ is 0.20.

## 6. CONCLUDING REMARKS

In this paper the MGF of the PTE is obtained which is instrumental to the derivation of the sampling distribution and risk function of the PTE. Moreover, the moments of any order can be obtained from this MGF. The analyses reveal that if the non-sample prior information about the value of the parameter is not too far from its true value, the PTE outperforms the UE. As the sources of non-sample prior information is usually previous studies and experts knowledge it is a natural expectation that such information about the value of the parameter will normally be very close to the true value. In such cases, the PTE is locally admissible over the UE. Similar to the shape of the linex loss function the shape of the risk function of the PTE is also asymmetric. As the shape parameter of the loss function approaches a very small value the shape of the risk function of the PTE approaches symmetry. Therefore, the results in this paper extends the existing known results for the risks under squared error loss function to a wider class of risks under the linex loss function which includes the squared error loss function as a special case. Thus the local admissibility of the PTE of the normal mean is established for a class of asymmetric losses.

## ACKNOWLEDGEMENT

The authors thankfully acknowledge the valuable comments and suggestions of the anonymous referee.

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