Problem Solving Activities in a Constructivist Framework: Exploring how Students Approach Difficult Problems

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The paper describes results of a teaching experiment with five high school (Year 10 and 11) students. Four qualitative characteristics were established: the first step of solution, main information extracted from the problem, generalisation from a problem and completion of solution. From these characteristics the corresponding quantitative indices were introduced and analysed. The structure of two of them, specific S_{FS} and common S_{HP}, are given in detail. Investigation of quantitative indices and their qualitative characteristics gives an opportunity to find out more about interrelations between different stages of the problem-solving process.

Introduction

Problem solving, in various forms and contexts, takes significant time in classroom activities throughout primary and secondary school. This area attracts the interest of both mathematics educators, seeking powerful activities in teaching and learning, and researchers, considering problem solving as a research domain. As a consequence many fundamental research papers on different aspects of problem solving appeared recently. For some of them the concept of a problem was the main focus, in others, students' performance in problem solving and analysis of their thinking strategies were considered. In this paper we attempted to analyse the mathematical behaviour of gifted high school students while they were involved in problem solving activities of non-trivial problems. Indeed, most of students are able to solve standard problems and routine exercises, but with harder ones, the situation looks completely different. It is obvious that even the ablest students often experience difficulties in this kind of activity. However, why does this happen? Which factors were the crucial ones in students' failure to solve a hard problem? How could teachers encourage students in the most effective way, if they fail to address a problem adequately? The answers to these questions and similar ones are extremely important for developing both further theoretical frameworka for the topic and practical implications in work with gifted students. Most high-profile students regularly participate in numerous mathematical competitions. To achieve the best results their training should be grounded on a sound theoretical base. But we have to recognise the fact that the stages of solving hard problems are hidden from researchers in many circumstances and the most talented students find their solutions so natural that explanations are not required. So, how can students' abilities and skills to move toward a solution be investigated, evaluated, and further developed? What is a student's perception of a hard problem? What are the obstacles to finding solutions for such a problem? Our aim was to find out more about the nature of this process. These questions have formed the basis of research the authors focused on for a long period of time (Passmore, 2007; Yevdokimov, 2005a, 2005b, 2006). This paper highlights the linkage between mathematical problems and student cognition. How do they depend and influence each other? Which factors have the most essential influence on students' mathematical thinking and reasoning? In particular, how much helpful information can the statement of a problem provide for students? What part of the problem solving process is the most difficult for them and how we can evaluate students' work on different stages of such activities? The paper attempts to answer these questions.

Theoretical Framework

Mathematical tasks that are different from routine exercises are usually called non-trivial problems. They can be classified in many ways, for example, by level of difficulty. There are other ways to define non-trivial problems in the mathematics education literature. Morton (1927) defined a problem as any mathematical question where the person attempting an answer must select the operations. Krutetskii (1976) used the terminology "task complexity" as equivalent to intellectual complexity of a problem. According to Charles and Lester (1982), problems were classified as standard, non-standard, real-world problems, and puzzles. Hembree (1992) pointed out that "distinctions among definitions of a problem relate to the effort that solvers must make toward solution" (p. 244). Williams and Clarke (1997) identified six dimensions of task complexity – linguistic, contextual, operational, conceptual, intellectual, and representational complexity. We define *a* *hard problem* as one that encourages the use of flexible methods, stimulated guessing, and use of unusual strategies towards a solution. Our conceptual framework is based on the construct of a mathematical problem and its solution implemented by students. It was influenced by the papers of Stein, Grover, and Henningsen (1996) and Henningsen and Stein (1997). The framework, shown in Figure 1, defines a hard mathematical task as a learning activity, the purpose of which is "to focus students' attention on a particular mathematical concept, idea, or skill" (Henningsen & Stein, 1997, p. 528).

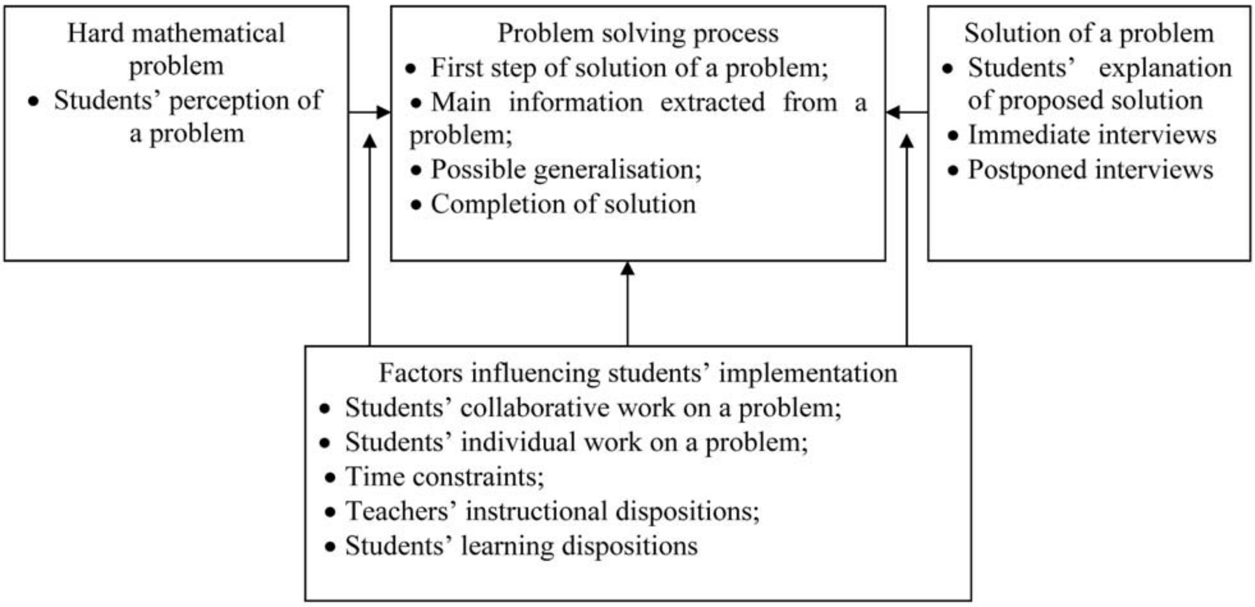

Figure 1. Mathematical task as implemented by students.

More exactly, we consider the modified third phase of the Henningsen and Stein (1997) conceptual framework with respect to hard problems in a constructivist framework. We followed von Glasersfeld's (1995) idea that "learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors" (p. 120).

In this framework mathematical tasks pass through three stages: as perceived by students in the beginning of their work, as solved by students during their work, and as explained by students on the basis of their work. The first stage sounds similar to the first Polya step (understanding the problem). However, taking into account students' understanding the problem, we paid much attention to their perception of the problem from a psychological point of view. Our hypothesis was that it could have a significant impact on students' performance in problem solving, and, therefore, should be taken into consideration. The second stage consists of four dimensions. Each of them represents a different qualitative characteristic of solving hard problems. All characteristics refer to the thinking processes in which students engage. The aim of the second stage was to analyse different dimensions and establish their role and impact on thinking processes in problem solving, as shown in Figure 2.

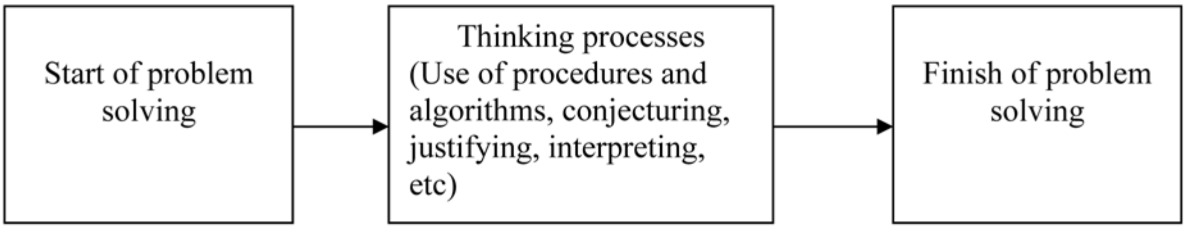

Figure 2. Simplified scheme of problem solving.

The interrelations between these dimensions are complex and need further investigation, which was another aim of the study. They can overlap each other, or even contain one another, influence each other, and, together with factors influencing students' implementation, can change their interrelations with each other. The four dimensions formed a dynamic structure within the framework, as shown in Figure 3.

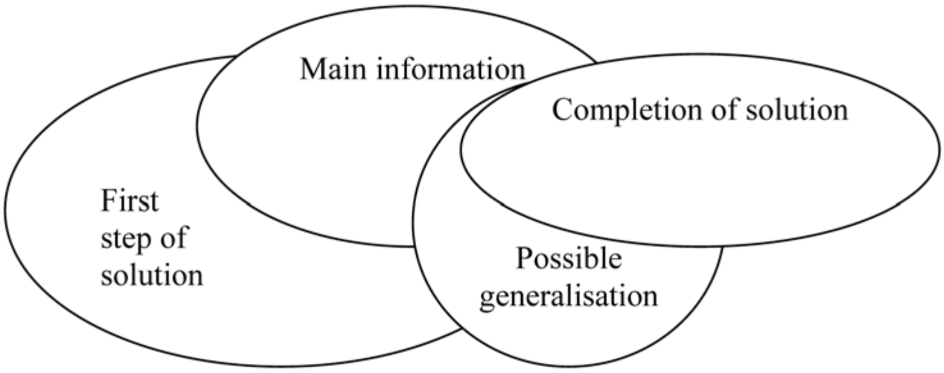

Figure 3. Dynamic structure of interrelations between four dimensions.

Williams (2000) described and categorised students' abilities to solve unfamiliar challenging problems in collaborative work. We focussed mostly on individual student performances, though collaborative work was taken into account. We used two forms of problem solving activities: firstly open problems, and secondly mathematical situations. For open problems proposed for students we followed the Arsac, Germain, and Mante (1988) characterisation:

> The statement of the problem is short, so that it can be easily understood, it fosters discovery and all students are able to start the solution process. The statement of the problem does not suggest the method of solution, or the solution itself, but it creates a situation stimulating the production of conjectures. The problem is set in a conceptual domain, which students are familiar with. Thus, students are able to master the situation rather quickly and to get involved in attempts of conjecturing, planning solution paths and finding counter-examples in a reasonable time.

While solving a certain problem, each student was asked to investigate its "mathematical situation", with his/her own priorities for further inquiry in that problem. Like Brown and Walter (1990), we considered "situation", an issue, which was a localised area of inquiry with features that could be taken as given or challenged and modified. Also, we took into account that current learning perspectives for problem solving activities in a constructivist framework incorporate three important assumptions (Anthony, 1996):

- learning is a process of knowledge construction, not of knowledge recording or absorption;
- learning is knowledge-dependent; people use current knowledge to construct new knowledge;
- the learner is aware of the processes of cognition and can control and regulate them.

Methodology

In order to identify key points of students' performance in problem solving and learn more about their strategies and relation to knowledge construction we distinguished four qualitative problem solving characteristics:

- **FS** – the first step of solution of a problem;
- **MI** – main information extracted from a problem;
- **G** – generalisation possibly required for solution of a problem;
- **C** – completion of the solution of a problem.

We established quantitative indices of students' skills for each of the corresponding qualitative characteristics:

- $\mathbf{S_{FS}}$ – student's skills to find the first step of solution of a problem;
- $\mathbf{S_{MI}}$ – student's skills to find out the main information from a problem;

- **S_G** – student's skills to make generalisation which possibly could be required for solution of a problem;
- **S_C** – student's skills to complete solution of a problem and make conclusion.

Finally, we introduced a common index **S_{HP}** – the level of student's abilities to solve hard problems. We define **S_{HP}** as a variation of AFKS (Yevdokimov, 2006), being considered in the context of a specific learning environment where hard problems are to be solved by students.

The teaching experiment methodology consisted of long-term interactions between teacher/researcher and individual students. These interactions included interviews and teaching episodes. This methodology concentrates on students' conceptual constructions and their cognitive demands. The main goal was to analyse students' constructions in the problem solving process. Interactions between the teacher/researcher and a student were intended to stimulate the student's mental activity. Interviews and teaching episodes provide for intensive interaction between student and teacher, where a teacher assists the student's developmental constructions.

This teaching experiment was conducted in three parts: an interview part, teaching part, and analysis part. It is important to note that development of students' abilities to solve hard problems is directly connected to the teacher's competence to conduct inquiry activities in a classroom. The teacher has to regulate directions of students' inquiry work into the problem solving process and adapt it to the classroom needs. At the same time, "open problems promote the devolution of responsibility from the teacher to students" (Furinghetti & Paola, 2003, p. 399). The teacher's role in this situation, we feel, should follow Mercer's idea (1995) of "the sensitive, supportive intervention of a teacher in the progress of a learner, who is actively involved in some specific task, but who is not quite able to manage the task alone" (p. 48).

These three parts formed a phase of the research, a full cycle taking two months. This was repeated four times per school academic year to verify students' conceptual constructions and trace the dynamics of the changing qualitative characteristics and their quantitative indices for each student. We calculated these indices at the start of the first phase (**$S_{FS,0}$**, **$S_{MI,0}$**, **$S_{G,0}$**, **$S_{C,0}$**) and at the end of all phases of the research (**$S_{FS,1}$**, …, **$S_{C,4}$** respectively), as shown in Figure 4.

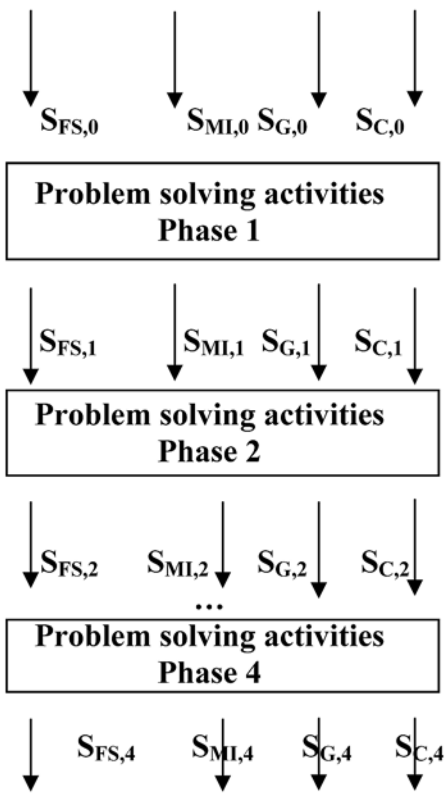

Figure 4. Structure of the teaching experiment.

We compared the common index $\mathbf{S_{HP}}$ with corresponding indices $\mathbf{S_{FS}}$, $\mathbf{S_{MI}}$, $\mathbf{S_G}$, and $\mathbf{S_C}$ to identify their similarities, differences, and mutual influence on each other. During all phases students had been asked to work with testing sheets to solve five problems in the form of tasks. They had to carry out certain problem-solving activities and provide appropriate argumentation for each task (analyse **FS**, **MI, G,** or **C** for a given task, but not all characteristics together for each task). At the same time we had the answers for **FS**, **MI, G,** and **C** in advance for each task on the testing sheets for teachers' use only. Average values of the quantitative indices of students' skills for each qualitative characteristic were calculated on the start and finish of each phase of the experiment. Below are both detailed descriptions of $\mathbf{S_{FS}}$ – the index of the student's skills to find the first step of solution of a problem and $\mathbf{S_{HP}}$ – the common index of the student's abilities to solve hard problems.

Structure of S_{FS}

We follow quantitative methods (Yevdokimov, 2006) on the basis of a formula of elementary probability for the finite number of events (in our terminology – for the finite number of essential levels of student's performance in problem solving):

$$S_{FS} = \frac{\sum_{k=1}^{n} p_k}{n},$$

where either $p_k=1$, if a student's performance was satisfactory, or $p_k=0$, if a student's performance was unsatisfactory, n is a number of levels mentioned above. In the scope of theoretical framework we distinguished six such essential levels, that is, $n=6$ here:

p_1 – a student was able to propose or at least make suggestion about the first step;

p_2 – a student could explain what he or she had done there;

p_3 – this first step leads to the solution of a problem;

p_4 – a student could explain which step of solution should be the next one, in other words, what should be the next step of solution on the basis of the first step proposed by a student;

p_5 – a student could explain (provide) full solution of a problem on the basis of his or her first step;

p_6 – a student provided full solution of a problem, and this solution is the shortest, and can be characterised as one of the best for a problem.

Index $\mathbf{S_{FS}}$ is not a probability value in the proper way, though it has probabilistic sense. We measured changes in $\mathbf{S_{FS}}$ in the range between *0* and *1*, taking into account such factors as studentss experience, constructivism, creativity, and mathematical competence.

Analogous ideas and approaches we used to estimate $\mathbf{S_{MI}}$, $\mathbf{S_G}$, and $\mathbf{S_C}$.

Structure of S_{HP}

To calculate the common index $\mathbf{S_{HP}}$, for each student for a certain task we used the same formula for elementary probability for a finite number of events. However, levels in this case were different from $\mathbf{S_{FS}}$. The solution of any problem was divided into consecutive steps and the step-by-step schemes provided for teacher's use only.

We evaluated student's actual suggestion for each step of the solution with *1*, if he/she could provide clear explanations why he/she did so. Otherwise, a mark for such a step was *0.* The formula was the following

$$S_{HP} = \frac{\sum 1}{N},$$

where N was a number of consecutive steps for a certain task.

Note that N takes different values for each problem and it is also possible for a student to give a different correct solution to a problem from the solution that the teacher has for the $\mathbf{S_{HP}}$ calculation. In such a case, the student's solution is divided into similar consecutive steps and the same scheme is applied to compute $\mathbf{S_{HP}}$.

Findings

Average index evaluations for $\mathbf{S_{FS}}$, $\mathbf{S_{MI}}$, $\mathbf{S_{G}}$, $\mathbf{S_{C}}$, and $\mathbf{S_{HP}}$ are given in the Tables 1 and 2 respectively. At first we calculated average values for each student at each phase. $\Delta S_{FS,k}$ means the difference between two consecutive evaluations $S_{FS,k}$ and $S_{FS,k-1}$, the same notation with other indices. Thus, we could trace dynamics of changes, and which characteristics at a certain phase played more significant role than others. We present average evaluations for five students only to demonstrate the general tendency – how different problem solving characteristics are related each other. The average evaluation for $\mathbf{S_{HP}}$ is in Table 2 due to the different nature of the index.

Table 1

Average Index Evaluations: S_{FS}, S_{MI}, S_{G}, S_{C}

$S_{FS,0}$	0.15	$S_{MI,0}$	0.27	$S_{G,0}$	0.04	$S_{C,0}$	0.38
$S_{FS,1}$	0.21	$S_{MI,1}$	0.22	$S_{G,1}$	0.1	$S_{C,1}$	0.45
$\Delta S_{FS,1}$	0.06	$\Delta S_{MI,1}$	-0.05	$\Delta S_{G,1}$	0.06	$\Delta S_{C,1}$	0.07
$S_{FS,2}$	0.29	$S_{MI,2}$	0.31	$S_{G,2}$	0.18	$S_{C,2}$	0.42
$\Delta S_{FS,2}$	0.08	$\Delta S_{MI,2}$	0.09	$\Delta S_{G,2}$	0.08	$\Delta S_{C,2}$	-0.03
$S_{FS,3}$	0.57	$S_{MI,3}$	0.48	$S_{G,3}$	0.32	$S_{C,3}$	0.58
$\Delta S_{FS,3}$	0.28	$\Delta S_{MI,3}$	0.17	$\Delta S_{G,3}$	0.14	$\Delta S_{C,3}$	0.16
$S_{FS,4}$	0.76	$S_{MI,4}$	0.68	$S_{G,4}$	0.55	$S_{C,4}$	0.84
$\Delta S_{FS,4}$	0.19	$\Delta S_{MI,4}$	0.2	$\Delta S_{G,4}$	0.23	$\Delta S_{C,4}$	0.26

Table 2

Average Index Evaluations: S_{HP}

$S_{HP,0}$	$S_{HP,1}$	$\Delta S_{HP,1}$	$S_{HP,2}$	$\Delta S_{HP,2}$	$S_{HP,3}$	$\Delta S_{HP,3}$	$S_{HP,4}$	$\Delta S_{HP,4}$
0.14	0.18	0.04	0.25	0.07	0.36	0.11	0.52	0.16

It is important to note that the common index of problem solving abilities $\mathbf{S_{HP}}$ and generalising characteristic $\mathbf{S_{G}}$ have similar tendencies and dynamic changes. The easiest from students' point of view was completion of solution, characteristic $\mathbf{S_{C}}$, however, it did not overlap with generalising skills in most tasks and, therefore, this question needs further investigation. We noticed that, after getting some experience, students' performance in $\mathbf{S_{FS}}$ increased significantly but $\mathbf{S_{MI}}$ did not. However, other indices depended strongly on increasing $\mathbf{S_{MI}}$.

We distinguished three basic strategies which were used by students in their attempts to solve different problems. We called them: the "Blind search", the "Going along the fairway" strategy, and the "Conscious search to find main information". With "Blind search", students made stochastic attempts to solve a problem, they were not able to explain their suggestions and preferences. Very often students tried to check all possible situations in a problem. In "Going along the fairway" they tried to apply the last method that they had previously studied. The third strategy took the leading place in students' work during the third and fourth phases. We observed that, in the case of students' successful answers to the question about the **MI** of a problem, all characteristics were correctly specified in most of other problems. Moreover, in some cases, not universally, but quite often, students began their analysis with **MI**, even if the questions were about other characteristics.

Concluding Remarks

Analysis of the qualitative characteristics and their quantitative indices gives an opportunity to develop knowledge of the problem-solving process from a complex-mental-activity point of view. Schoenfeld (1985) has noted a widespread belief that only the brightest students can succeed at problem solving. Hembree (1992.) argued that this belief is not well-founded. Our results support Hembree's conclusion. We noticed that good mathematics students, though not the brightest ones, after gaining more experience in problem-solving, understand that there are few options for the first step in solving any problem. They can distinguish such situations, though not all students are able to explain their understanding clearly. Furthermore, our results show that illumination and insight do not have any significant impact on students' performance despite the fact that students could not always explain why they made a step in a certain direction. We are inclined to think that students and teachers exaggerate the importance of the "Ah ha! moment" in problem-solving activities. This experiment showed that both qualitative characteristics and their quantitative indices may be used as a powerful diagnostic tool in work with gifted and talented students, for the further development their conceptual constructions, and improvement of their problem-solving skills.

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