On the Size Corrected Tests in Improved Estimation

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Abstract

In this paper we propose shrinkage preliminary test estimator (SPTE) of the coefficient vector in the multiple linear regression model based on the size corrected Wald (W), likelihood ratio (LR) and Lagrangian multiplier (LM) tests. The correction factors used are those obtained from degrees of freedom corrections to the estimate of the error variance and those obtained from the second order Edgeworth approximations to the exact distributions of the test statistics. The bias and weighted mean squared error (WMSE) functions of the estimators are derived. With respect to WMSE, the relative efficiencies of the SPTEs relative to the maximum likelihood estimator are calculated. This study shows that the amount of conflict can be substantial when the three tests are based on the same asymptotic chi-square critical value. The conflict among the SPTEs is due to the asymptotic tests not having the correct significance level. The Edgeworth size corrected W, LR and LM tests reduce the conflict remarkably.

1 Introduction

The multiple linear regression model is the most widely used statistical tool for the practitioners in many disciplines, and hence the estimation of its parameters is very important. A common and popular estimator of the p-dimensional regression coefficient

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vector $\boldsymbol{\beta}$ is the maximum likelihood estimator (MLE). Customarily, the MLE is based exclusively on the sample responses, and is known as the unrestricted estimator (UE). This estimator obviously disregards any other kind of non-sample prior information about $\boldsymbol{\beta}$.

Under the general linear hypothesis, $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$, where H is a $q \times p$ matrix of full rank and $\boldsymbol{h}$ is a known q-vector, the estimator of $\boldsymbol{\beta}$ is known as the restricted estimator (RE). With respect to the squared error loss function the RE performs better than the UE when the null hypothesis holds. Otherwise, the UE is better than the RE. Therefore, it is desirable to develop an improved estimator which is a compromise between the UE and RE. This can be done by using the shrinkage preliminary test estimator (SPTE) which is a function of the UE, RE and an appropriate test for testing the hypothesis H_0 (see Saleh and Han, 1990; Khan and Saleh, 2001). The SPTE performs reasonably well in the neighborhood of the null hypothesis as compared to the UE and RE. It is well known that the shrinkage estimator (SE) is a competitor of the SPTE when the parameter vector moves away from the subspace of the restriction (see Saleh and Han, 1990). However, by definition, the application of the SE is restricted by the constrained $q \geq 3$. Therefore, the SPTE is preferable to the SE particularly, when this condition does not meet in practice.

The widely used tests for testing $H_0 : H\boldsymbol{\beta} = \boldsymbol{h}$ are: Wald (W; Wald, 1943), likelihood ratio (LR; Aitchison and Silvey, 1958; Silvey, 1959) and Lagrangian multiplier (LM; Rao, 1947). Savin (1976) and Berndt and Savin (1977) show that the inequality relationship $W \geq LR \geq LM$ exists among the values of the test statistics for testing linear restrictions on the coefficients of certain linear models. The exact sampling distributions of these test statistics can be complicated, so in practice the critical regions of the tests are based on asymptotic approximations, based on the the critical value for an α level test of H_0, denoted as $\chi^2_q(\alpha)$. Mukherjhe (2002) shows that optimality properties hold for the LR test in terms of second-order local maximinity and for LM test in terms of third-order 'average' local power. Using a numerical study he also proves that the results may be valid even for moderately large samples.

The above three large sample tests do not have the correct significance level. Evans and Savin (1982) have shown that the tests differ with respect to size and power, and there may also have conflict between their conclusions. However, when the chi-square critical values are adjusted by computable correction factors the conflict between the W, LR and LM tests reduces substantially (see Evans and Savin, 1982). The correction factors used are those derived by Gallant (1975) from degrees of freedom corrections to the estimate of the error variance and those derived by Rothenberg (1977) from second order Edgeworth approximations to the exact distributions of the test statistics. For comparison among the three tests, particularly with respect to the power property, readers may also see Chandra and Mukherjhe (1984, 1985) and the references therein.

In various contexts, Billah (1997); Billah and Saleh (1998, 2000a,b); and more recently Kibria (2002) have compared the performances of the shrinkage preliminary test estimators under the large sample tests. The results of these studies show that the SPTEs under these tests give different WMSE. The difference may be because the large sample tests used in the formation of the SPTE do not have the correct significance level. It should be worth mentioning that the SPTE based on the test with incorrect significance level may give misleading WMSE. Hence, much care should be given before using a test in the formation of SPTE.

In this paper we propose the SPTE based on the corrected W, LR and LM tests in the context of the univariate multiple linear regression model. We investigate whether the corrections to the tests, as stated by Evans and Savin (1982), reduce the conflict among the SPTEs of the regression coefficient vector $\boldsymbol{\beta}$. Conflict is defined as the difference between the largest and the smallest relative efficiencies (with respect to WMSE) of the SPTEs relative to the UE. We also calculate the conflict among the estimators based on the Edgeworth size corrected tests.

The organization of the paper is as follows. The model and some preliminaries are outlined in Section 2. In Section 3, the bias and WMSE functions of the SPTE under the large sample W, LR and LM tests are stated. The bias and WMSE functions of the SPTE under the size corrected W, LR and LM tests are derived in Sections 4 and 5. Some results have been discussed in Section 6 and the final section includes the conclusion.

2 The Model and Some Preliminaries

Consider the linear multiple regression model

$$\boldsymbol{Y} = X\boldsymbol{\beta} + \boldsymbol{e}, \tag{2.1}$$

where $\boldsymbol{Y}$ is an n-vector of the response variable, X is an $n \times p$ matrix of non-stochastic independent variables, $\boldsymbol{\beta}$ is a p-vector of regression coefficients and $\boldsymbol{e}$ is the error vector having the same dimension of $\boldsymbol{Y}$. It is assumed that X is of full rank, and $n \geq p$. Also assume that the errors follow normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\sigma^2 I$, where I is the identity matrix of order n.

Let us express the non-sample prior information about $\boldsymbol{\beta}$ in the following form of the null hypothesis:

$$H_0 : H\boldsymbol{\beta} = \boldsymbol{h}. \tag{2.2}$$

The maximum likelihood estimator (MLE) for $\boldsymbol{\beta}$ is the ordinary least squares estimator given by $\widetilde{\boldsymbol{\beta}} = (X'X)^{-1}X'\boldsymbol{Y} = C^{-1}X'\boldsymbol{Y}$, where $C = X'X$. Furthermore, the MLE of σ^2 is $\widetilde{\sigma}^2 = \frac{1}{n}(\boldsymbol{Y} - X\widetilde{\boldsymbol{\beta}})'(\boldsymbol{Y} - X\widetilde{\boldsymbol{\beta}})$. The MLE is also known as the unrestricted estimator (UE). The bias and mean squared error matrix (MSEM) of the UE of $\boldsymbol{\beta}$ are 0 and $\sigma^2 C^{-1}$, respectively.

The estimators of $\boldsymbol{\beta}$ and σ^2 under the null hypothesis (2.2) are called the restricted estimators (RE), and are given by $\widehat{\boldsymbol{\beta}} = \widetilde{\boldsymbol{\beta}} - C^{-1}H'(HC^{-1}H')^{-1}(H\widetilde{\boldsymbol{\beta}} - \boldsymbol{h})$ and $\widehat{\sigma}^2 = \frac{1}{n}(\boldsymbol{Y} - X\widehat{\boldsymbol{\beta}})'(\boldsymbol{Y} - X\widehat{\boldsymbol{\beta}})$, respectively. The bias and MSEM of the RE of $\boldsymbol{\beta}$ are $\mathrm{B}(\widehat{\boldsymbol{\beta}}) = \boldsymbol{\eta} = -C^{-1}H'(HC^{-1}H')^{-1}(H\boldsymbol{\beta} - \boldsymbol{h})$ and $\mathrm{MSEM}(\widehat{\boldsymbol{\beta}}) = \sigma^2 C^{-1} - \sigma^2 \Lambda + \boldsymbol{\eta}\boldsymbol{\eta}'$, respectively, where $\Lambda = C^{-1}H'(HC^{-1}H')^{-1}HC^{-1}$.

Under the quadratic loss function, the RE of $\boldsymbol{\beta}$ performs better than the UE when the null hypothesis in (2.2) holds. However, as $H\boldsymbol{\beta}$ deviates further from $\boldsymbol{h}$, the RE may be considerably biased, inefficient and inconsistent while the performance of the UE remains steady over such departures. Therefore, it is desirable to use the shrinkage preliminary test estimator (SPTE) which provides a smooth transition between the UE and RE under uncertain prior information $H\boldsymbol{\beta} = \boldsymbol{h}$. Let us assume that ζ is the test statistic for testing the hypothesis in (2.2). Then, a simple form of the SPTE of $\boldsymbol{\beta}$ is as follows:

$$\widehat{\boldsymbol{\beta}}_{\zeta}^{SPTE} = \widetilde{\boldsymbol{\beta}} - (1-d)I(\zeta \le \zeta_\alpha)(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}), \tag{2.3}$$

where ζ_α is the critical value of the test ζ at α significance level, $I(\cdot)$ is an indicator function which assumes value unity when the inequality in the argument holds, and 0 otherwise. Here d is the degree of distrust on the null hypothesis that ranges from 0 to 1, and needs to be determined by the experimenter. Ideally, the value of d should not be too far from 0 (cf. Khan and Saleh, 2001). If the experimenter has complete trust on the data and believes that the parameter belongs to the restricted subspace when H_0 is accepted, he/she should use $d = 0$. For an optimum choice of the value of d readers are referred to Saleh and Han (1990). When $I(\cdot) = 1$ and $d = 0$, the SPTE becomes the RE. If either $I(\cdot) = 0$ or $d = 1$, the SPTE is the UE.

The SPTE falls in the area of inference with uncertain prior information and has been studied by Judge and Bock (1978), Berger (1980), Copas (1983), Anderson (1984), Saleh and Sen (1986, 1978, 1984), Ahmed and Saleh (1989), Gupta and Saleh (1996), and Khan *et al.* (2002), among others.

3 Estimators Under the Large Sample W, LR and LM Tests

To test the null hypothesis in (2.2) the usual F statistic is

$$F = \frac{(RRSS - URSS)}{URSS}\frac{m}{q}, \tag{3.1}$$

where $m = n - p$, $URSS = (\boldsymbol{Y} - X\widetilde{\boldsymbol{\beta}})'(\boldsymbol{Y} - X\widetilde{\boldsymbol{\beta}})$ is the unrestricted residual sum of squares and $RRSS = (\boldsymbol{Y} - X\widehat{\boldsymbol{\beta}})'(\boldsymbol{Y} - X\widehat{\boldsymbol{\beta}})$ is the restricted residual sum of squares. Under the alternative hypothesis the distribution of (3.1) is a non-central F with (q, m) degrees of freedom $(d.f.)$ and with the non-centrality parameter given by

$$\Delta = \frac{1}{2\sigma^2}(H\boldsymbol{\beta} - \boldsymbol{h})'(HC^{-1}H')^{-1}(H\boldsymbol{\beta} - \boldsymbol{h}). \tag{3.2}$$

Alternately, the W, LR and LM tests are extensively used in practice. These test statistics are:

$$W = (H\widetilde{\boldsymbol{\beta}} - \boldsymbol{h})'(\widetilde{\sigma}^2 HC^{-1}H')^{-1}(H\widetilde{\boldsymbol{\beta}} - \boldsymbol{h}) \tag{3.3}$$

$$LR = n[\ln\widehat{\sigma}^2 - \ln\widetilde{\sigma}^2] \tag{3.4}$$

$$LM = (H\widetilde{\boldsymbol{\beta}} - \boldsymbol{h})'(\widehat{\sigma}^2 HC^{-1}H')^{-1}(H\widetilde{\boldsymbol{\beta}} - \boldsymbol{h}). \tag{3.5}$$

The above test statistics can also be written as follows (see Ullah and Zinde-Walsh, 1984)

$$W = \frac{nq}{m}F \tag{3.6}$$

$$LR = n\ln\left(1 + \frac{qF}{m}\right) \tag{3.7}$$

$$LM = \frac{nqF}{m + qF}. \tag{3.8}$$

The SPTE of $\boldsymbol{\beta}$ under the above tests can be defined as follows:

$$\widehat{\boldsymbol{\beta}}_W^{\text{SPTE}} = \widetilde{\boldsymbol{\beta}} - \delta I(W \leq \chi^2_\alpha(q))(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}) \tag{3.9}$$

$$\widehat{\boldsymbol{\beta}}_{LR}^{\text{SPTE}} = \widetilde{\boldsymbol{\beta}} - \delta I(LR \leq \chi^2_\alpha(q))(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}) \tag{3.10}$$

$$\widehat{\boldsymbol{\beta}}_{LM}^{\text{SPTE}} = \widetilde{\boldsymbol{\beta}} - \delta I(LM \leq \chi^2_\alpha(q))(\widetilde{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}) \tag{3.11}$$

where $\delta = (1-d)$ and $\chi^2_\alpha(q)$ is the chi-square critical value of the tests at α significance level with q d.f.

Let us define the loss function as $(\widehat{\boldsymbol{\beta}}_\zeta^{\text{SPTE}} - \boldsymbol{\beta})'\mathcal{W}(\widehat{\boldsymbol{\beta}}_\zeta^{\text{SPTE}} - \boldsymbol{\beta})$ for a given non-singular matrix $\mathcal{W}$. Then the WMSE is given by

$$WMSE = E[(\widehat{\boldsymbol{\beta}}_\zeta^{\text{SPTE}} - \boldsymbol{\beta})'\mathcal{W}(\widehat{\boldsymbol{\beta}}_\zeta^{\text{SPTE}} - \boldsymbol{\beta})].$$

Assuming $\mathcal{W} = \sigma^{-2}C$, the direct calculation from Judge and Bock (1978) leads to the following theorems.

Theorem 3.1: *The bias functions of the SPTE of $\boldsymbol{\beta}$ under the large sample W, LR and LM tests are respectively:*

$$\text{B}(\widehat{\boldsymbol{\beta}}_W^{\text{SPTE}}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_1^W; \Delta\right) \tag{3.12}$$

$$\text{B}(\widehat{\boldsymbol{\beta}}_{LR}^{\text{SPTE}}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_1^{LR}; \Delta\right)) \tag{3.13}$$

$$\text{B}(\widehat{\boldsymbol{\beta}}_{LM}^{\text{SPTE}}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_1^{LM}; \Delta\right) \tag{3.14}$$

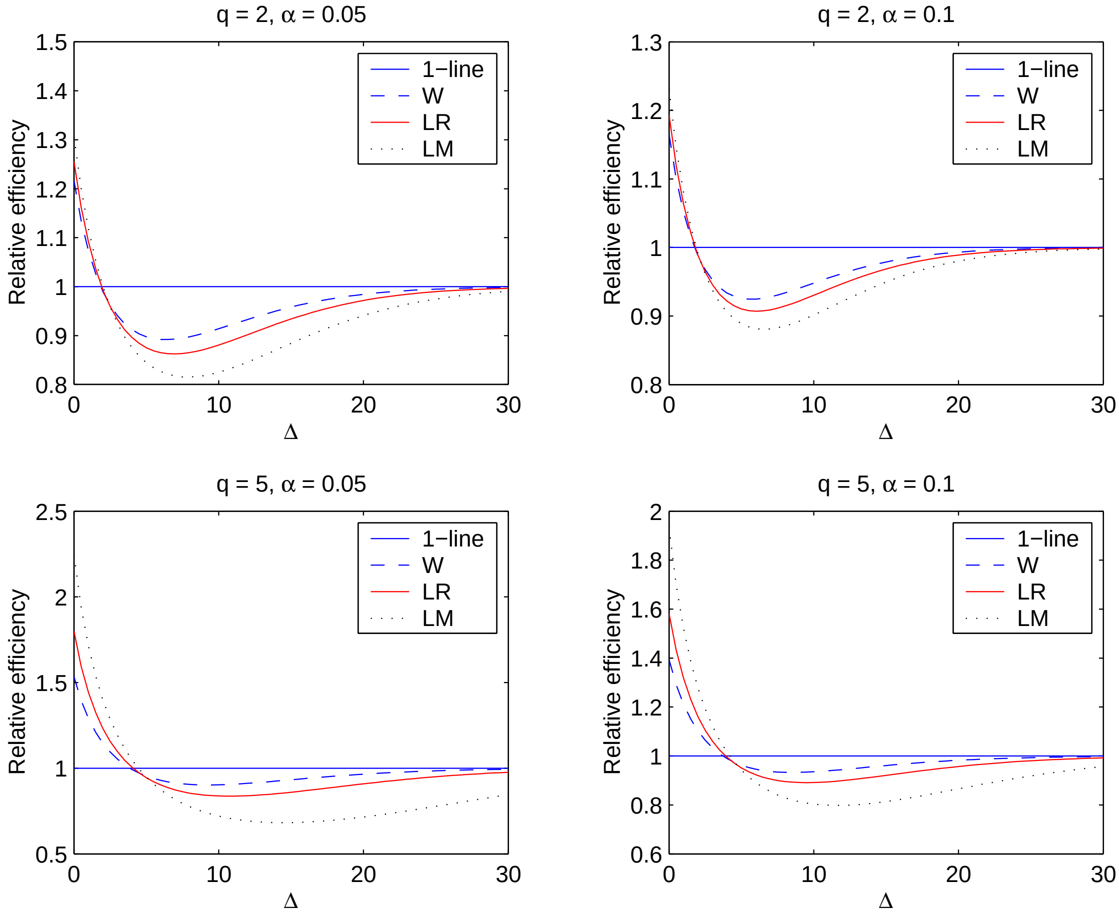

Figure 1: The relative efficiency of the SPTE based on the large sample W, LR and LM tests for $n = 25$, $d = 0.1$, $p = 8$ and selected values of q and α.

where $h_i^W = \frac{m\chi_\alpha^2}{n(q+2i)}$, $h_i^{LR} = \frac{m}{(q+2i)}\left(e^{\frac{\chi_\alpha^2}{n}} - 1\right)$, $h_i^{LM} = \frac{m\chi_\alpha^2}{(q+2i)\left(n-\chi_\alpha^2\right)}$, $i = 1$; *and* $G_{a,b}(h;\Delta)$ *is the cumulative distribution function of the non-central F-distribution with* (a,b) *d.f., with non-centrality parameter* Δ *and is evaluated at* h.

Theorem 3.2: *The WMSE functions of the SPTE of* $\boldsymbol{\beta}$ *under the large sample* W, LR *and* LM *tests are respectively as follows:*

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_W^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_1^W;\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_1^W;\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_2^W;\Delta\right) \quad (3.15)$$

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_{LR}^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_1^{LR};\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_1^{LR};\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_2^{LR};\Delta\right) \quad (3.16)$$

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_{LM}^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_1^{LM};\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_1^{LM};\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_2^{LM};\Delta\right) \tag{3.17}$$

where $\delta^* = (1-d^2)$; h_i^W, h_i^{LR} *and* h_i^{LM}, $i = 1, 2$, *are defined in Theorem 3.1.*

An identical analytical comparison between these WMSE functions can be found in Billah (1997), and Billah and Saleh (1998)

4 Estimators Under the Modified W, LR and LM Tests

The modified tests are obtained from degrees of freedom corrections to the estimate of the error variance and those derived from second order Edgeworth approximations to the exact distributions. The degrees of freedom corrections have been suggested by Gallant (1975) for the non-linear models and Rothenberg (1977) has suggested the Edgeworth corrections for the multivariate linear regression model. Evans and Savin (1982) has investigated the performances of such corrections in the context of size and power properties of the W, LR and LM tests in the univariate multiple linear regression model. The results of Evans and Savin (1982) show that the modifications reduce the conflict among the tests with respect to size and power.

For testing the null hypothesis in (2.2) the modified test statistics are:

$$W* = qF \tag{4.1}$$

$$LR* = (m + \frac{q}{2} - 1)\ln\left(1 + \frac{qF}{m}\right) \tag{4.2}$$

$$LM* = \frac{(m+q)qF}{m+qF}. \tag{4.3}$$

The modified statistic for W is obtained by replacing n by m and the modified LM statistic by replacing n by $m+q$. These degrees of freedom corrections correct the bias in the respective estimators of the error variance σ^2. The modified statistic for LR is obtained by replacing n by $m + q/2 - 1$. This correction to the LR statistic is the Edgeworth size correction which ensures that the LR test has the correct significance level to order $1/m$ (c.f. Evans and Savin (9)). The inequality relation $W* \geq LR* \geq LM*$ does not satisfy for the modified tests for all m and q. The bias and WMSE functions of the SPTE under the modified tests are derived in Theorems 4.1 and 4.2 respectively.

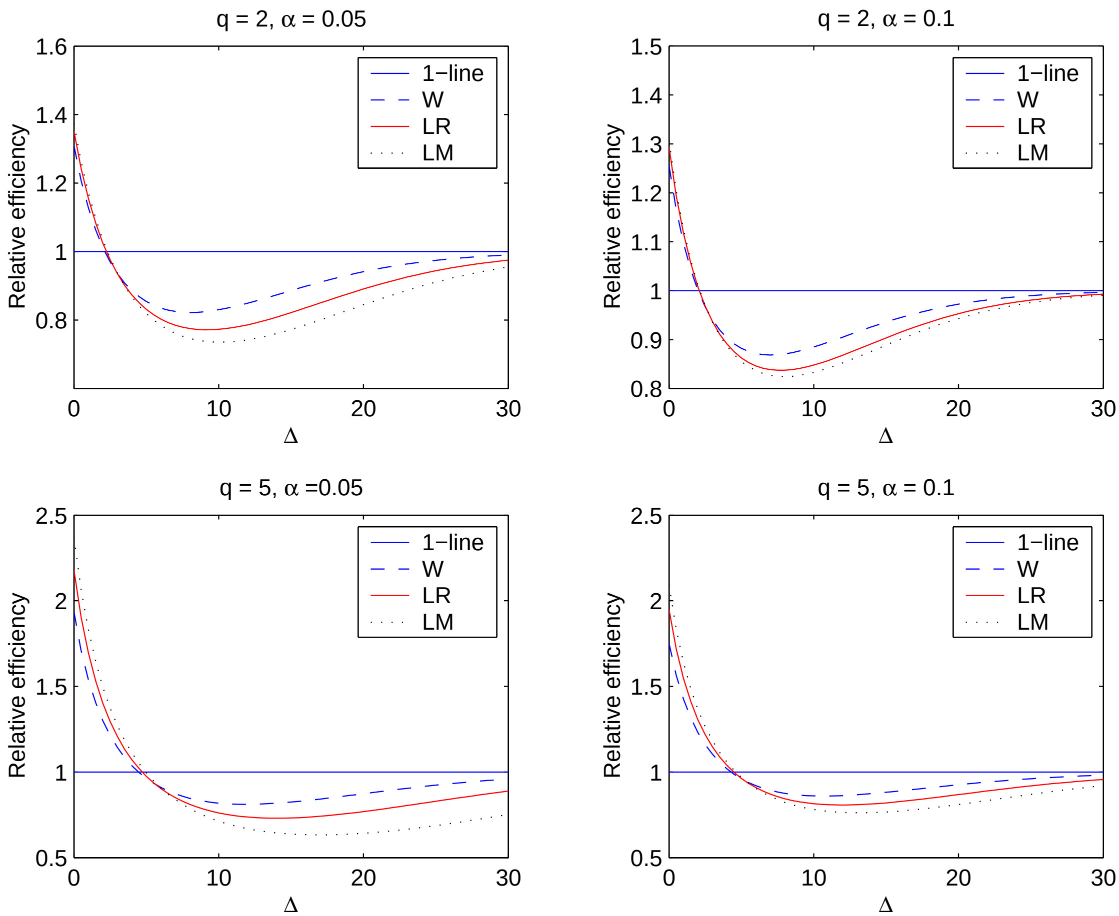

Figure 2: The relative efficiency of the SPTE based on the modified W, LR and LM tests for $n = 25$, $d = 0.1$, $p = 8$ and selected values of q and α.

Theorem 4.1: *The bias functions of the SPTE of $\boldsymbol{\beta}$ under the modified W, LR and LM tests are respectively:*

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{W*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_1^{W*};\Delta\right) \tag{4.4}$$

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LR*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_{E1}^{LR*};\Delta\right) \tag{4.5}$$

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LM*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h_1^{LM*};\Delta\right) \tag{4.6}$$

where $h_i^{W*} = \frac{\chi_\alpha^2}{(q+2i)}$, $h_{Ei}^{LR*} = \frac{m}{(q+2i)}\left(e^{\chi_\alpha^2/(m+\frac{q}{2}-1)} - 1\right)$, $h_i^{LM*} = \frac{m\chi_\alpha^2}{(q+2i)\left(m+q-\chi_\alpha^2\right)}$, $i = 1$; *and* $G_{a,b}(h;\Delta)$ *is the cumulative distribution function of the non-central F-distribution with* (a,b) *d.f., with non-centrality parameter* Δ *and is evaluated at* h.

Theorem 4.2: *The WMSE functions of the SPTE of* $\boldsymbol{\beta}$ *under the modified* W, LR *and* LM *tests are respectively:*

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_{W*}^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_1^{W*};\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_1^{W*};\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_2^{W*};\Delta\right) \quad (4.7)$$

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_{LR*}^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_{E1}^{LR*};\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_{E1}^{LR*};\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_{E2}^{LR*};\Delta\right) \quad (4.8)$$

$$\text{WMSE}\left(\widehat{\boldsymbol{\beta}}_{LM*}^{\text{SPTE}}\right) = p - q\delta^* G_{q+2,m}\left(h_1^{LM*};\Delta\right) + 2\delta\Delta G_{q+2,m}\left(h_1^{LM*};\Delta\right) - \delta^*\Delta G_{q+4,m}\left(h_2^{LM*};\Delta\right) \quad (4.9)$$

where h_i^{W*}, h_{Ei}^{LR*} *and* h_i^{LM*}, $i = 1,2$, *are defined in Theorem 4.1.*

As we will see, the modified tests reduce the conflict among the relative efficiencies of the SPTEs as compared to the large sample tests. However, the conflict may still remain substantial. In the next section we propose the SPTE under the Edgeworth size corrected tests.

5 Estimators Under the Size Corrected W, LR and LM Tests

The conflict among the relative efficiencies of the large sample tests with or without modification is due to the fact that they have the same chi-square critical values despite the fact that the values of the test statistics are not the same in general. Following Evans and Savin (1982), we now consider the correction factors for the chi-square critical values. These correction factors are obtained from the Edgeworth expansions of the exact distributions of the test statistics under the null hypothesis.

The Edgeworth size corrected critical values (to order $1/m$) of the $W*$ and $LM*$

test statistics are respectively (see Evans and Savin (9)):

$$\zeta_{W*} = \chi^2_\alpha(q)\left\{1 + \frac{\chi^2_\alpha(q) - q + 2}{2m}\right\} \tag{5.1}$$

$$\zeta_{LM*} = \chi^2_\alpha(q)\left\{1 + \frac{\chi^2_\alpha(q) - q - 2}{2m}\right\}. \tag{5.2}$$

The tests with these adjusted critical values are known as the Edgeworth size corrected tests. The study of Evans and Savin (9) shows that the size corrected tests give correct significance level and the probability of conflict with respect to size and power is insignificant. The bias and WMSE functions of the SPTE under the Edgeworth size corrected W, LR and LM tests are derived in Theorems 5.1 and 5.2 respectively.

Theorem 5.1: *The bias functions of the SPTE of $\boldsymbol{\beta}$ under the Edgeworth size corrected W, LR and LM tests are respectively:*

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{W*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h^{W*}_{E1}; \Delta\right) \tag{5.3}$$

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LR*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h^{LR*}_{E1}; \Delta\right) \tag{5.4}$$

$$\mathrm{B}(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LM*}) = -\boldsymbol{\eta}\delta G_{q+2,m}\left(h^{LM*}_{E1}; \Delta\right) \tag{5.5}$$

where $h^{W*}_{Ei} = \frac{\chi^2_\alpha}{(q+2i)}\left(1 + \frac{\chi^2_\alpha - q + 2}{2m}\right)$, $h^{LM*}_{Ei} = \frac{m\chi^2_\alpha\left(2m - \chi^2_\alpha + q + 2\right)}{(q+2i)\left(2m^2 + 2mq - \chi^2_\alpha(2m - \chi^2_\alpha + q + 2)\right)}$, h^{LR*}_{Ei} *is defined in Theorem 4.1, $i = 1$; $G_{a,b}(h;\Delta)$ is the cumulative distribution function of the non-central F-distribution with (a, b) d.f., non-centrality parameter Δ and is evaluated at h.*

Theorem 5.2: *The WMSE functions of the SPTE of $\boldsymbol{\beta}$ under the Edgeworth size corrected W, LR and LM tests are respectively:*

$$\begin{aligned}\mathrm{WMSE}\left(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{W*}\right) &= p - q\delta^* G_{q+2,m}\left(h^{W*}_{E1}; \Delta\right) + 2\delta\Delta G_{q+2,m}\left(h^{W*}_{E1}; \Delta\right) \\ &\quad - \delta^*\Delta G_{q+4,m}\left(h^{W*}_{E2}; \Delta\right) \end{aligned}\tag{5.6}$$

$$\begin{aligned}\mathrm{WMSE}\left(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LR*}\right) &= p - q\delta^* G_{q+2,m}\left(h^{LR*}_{E1}; \Delta\right) + 2\delta\Delta G_{q+2,m}\left(h^{LR*}_{E1}; \Delta\right) \\ &\quad - \delta^*\Delta G_{q+4,m}\left(h^{LR*}_{E2}; \Delta\right) \end{aligned}\tag{5.7}$$

$$\begin{aligned}\mathrm{WMSE}\left(\widehat{\boldsymbol{\beta}}^{\mathrm{SPTE}}_{LM*}\right) &= p - q\delta^* G_{q+2,m}\left(h^{LM*}_{E1}; \Delta\right) + 2\delta\Delta G_{q+2,m}\left(h^{LM*}_{E1}; \Delta\right) \\ &\quad - \delta^*\Delta G_{q+4,m}\left(h^{LM*}_{E2}; \Delta\right) \end{aligned}\tag{5.8}$$

where h^{W*}_{Ei} and h^{LM*}_{Ei}, $i = 1, 2$, are defined in Theorem 5.1.

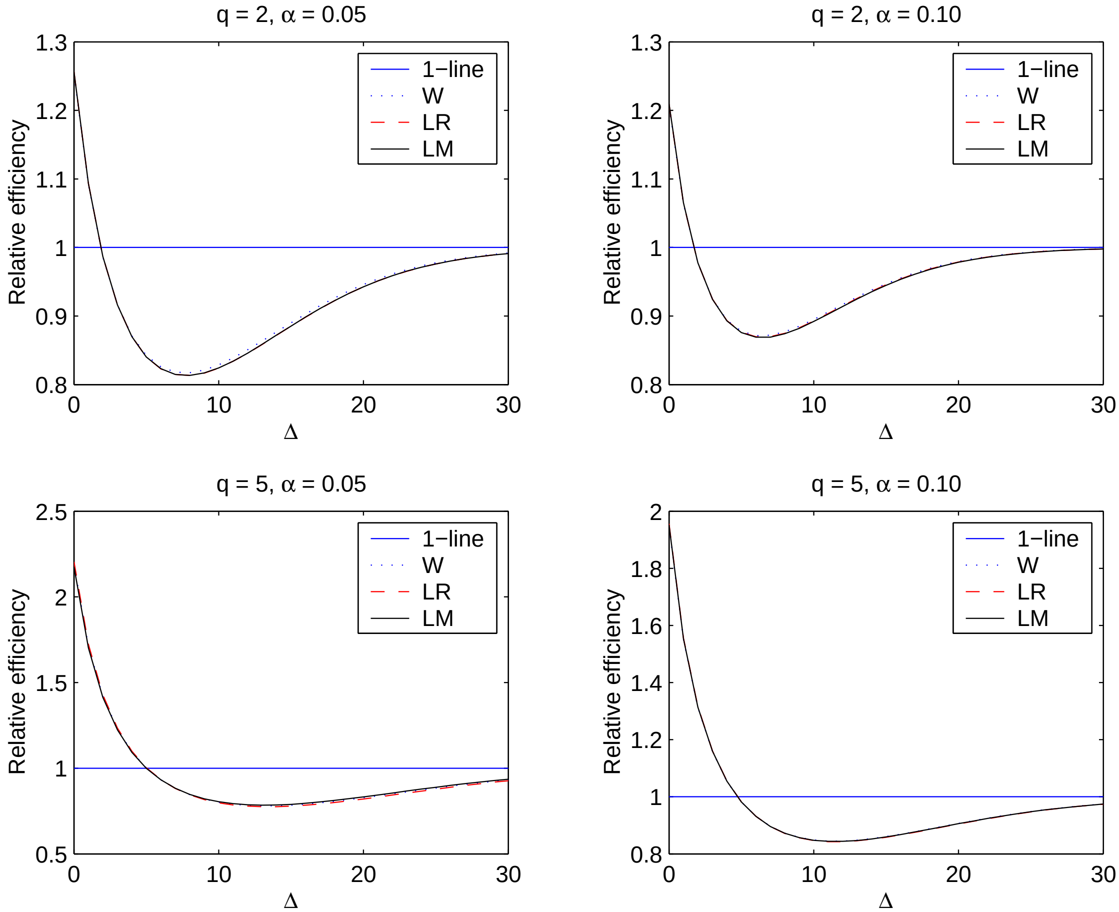

Figure 3: The relative efficiency of the SPTE based on the Edgeworth size corrected W, LR and LM tests for $n = 25$, $d = 0.1$, $p = 8$ and selected values of q and α.

6 Relative Efficiency and Bias Analysis

We calculated the bias and WMSE of the SPTE based on the large sample tests as well as the size corrected tests. The calculations were carried out for $n = 25$, 40 and 100; $p = 8$; $q = 2$, 5 and 8; $\alpha = 0.05$, 0.1, 0.15 and 0.2; $d = 0.1$, 0.25 and 0.5 and for Δ between 0 and 30. If the experimenter wishes to rely on the data completely and trusts that the parameter belongs to the restriction subspace when H_0 is accepted, he/she should use $d = 0$. In practice, the value of d should not be too far from zero. We calculated the relative efficiency of the SPTE relative to the UE with respect to WMSE. For comparison, the conflict among the relative efficiencies is calculated.

Selected results are presented in Table 1 and in Figures 1 to 3.

Table 1 and Figure 1 present the relative efficiency of the estimators and the conflict among them under the large sample tests. The results show that the LM test based SPTE performs the best followed by that based on the LR test, and the W test based SPTE is worse for Δ between 0 to some moderate value (say Δ_0). For $\Delta > \Delta_0$ they perform in reverse order. The conflict among the estimators is considerably large and it increases as q increases. The conflict is as large as 2.02 for $n = 25$, $q = 8$, $\Delta = 0$ and $\alpha = 0.1$. This is due to the fact that the large sample tests do not have the correct significance level.

Selected results for the modified tests based estimators are presented in Table 1 and in Figure 2. From these results we see that the performance patterns of the SPTE based on the modified tests are very similar to those based on the large sample tests. The modified tests reduce the conflict among the estimators as compared to the large sample tests. However, the conflict is still substantial. This is not unexpected because the corrections to the W and LM tests are based on only the bias correction in the estimates of error variance σ^2.

As mentioned earlier that the Edgeworth size corrected tests have the correct significance level to order $1/m$. Table 1 and Figure 3 present the selected results for the estimators under the Edgeworth size corrected tests. Clearly, the size corrected tests reduce the conflict significantly. For example, when $n = 25$, $q = 8$, $\Delta = 1.5$ and $\alpha = 0.1$, the conflict is 0.038 as compared to 0.789 and 0.495 for the large sample and the modified tests, respectively. The conflict among the SPTEs based on the size corrected tests is negligible as compared to that among the SPTEs based on the large sample and modified tests. The conflict reduces as q decreases. For $n = 25$, $q = 5$, $\Delta = 1.5$ and $\alpha = 0.1$, the conflicts are 0.541, 0.342 and 0.009, respectively for the large sample, modified and Edgeworth size corrected tests.

The bias of the SPTE under the original, modified and Edgeworth size corrected tests are zero at $\Delta = 0$. The inequality relation $\mathrm{B}(\widehat{\boldsymbol{\beta}}_W^{\text{SPTE}}) \leq \mathrm{B}(\widehat{\boldsymbol{\beta}}_{LR}^{\text{SPTE}}) \leq \mathrm{B}(\widehat{\boldsymbol{\beta}}_{LM}^{\text{SPTE}})$ exists among the biases of the SPTEs under the large sample as well as the corrected tests. The conflict among the biases substantially reduces for the size corrected tests as compared to the three large sample tests. The results have not been reported in this paper. However, they are available on request from the first author.

If a test does not have the correct significance level, the bias and WMSE of the SPTE under it may be artificial and hence one should not rely on the performance of the estimator based on the test. Assuming that the Edgeworth size corrected tests based estimators give approximately factual bias and WMSE, our findings show that each of the large sample tests and the modified W and LM tests based estimators suffer from both under-estimation and over-estimation problems. Thus our results illustrate the importance of using the tests with correct significance level in the formation of SPTE.

Table 1: Conflict among the SPTEs under the W, LR and LM tests for $n = 25$, $p = 8$, $\alpha = 0.05$ and selected values of q.

q	Δ	Original			Conf	Modified			Conf	Size-corrected			Conf
		W	LR	LM		W	LR	LM		W	LR	LM	
2	0.0	1.216	1.255	1.306	0.090	1.308	1.350	1.373	0.065	1.347	1.350	1.349	0.003
	0.5	1.135	1.162	1.200	0.065	1.207	1.242	1.262	0.055	1.239	1.242	1.241	0.003
	1.0	1.073	1.091	1.118	0.045	1.127	1.155	1.171	0.044	1.030	1.036	1.044	0.014
	1.5	1.026	1.036	1.053	0.027	1.063	1.083	1.096	0.033	1.081	1.083	1.082	0.002
	2.0	0.990	0.993	1.000	0.010	1.011	1.024	1.033	0.022	1.023	1.024	1.024	0.001
	2.5	0.962	0.959	0.958	0.004	0.969	0.975	0.981	0.012	0.975	0.975	0.975	0.000
	3.0	0.940	0.933	0.925	0.015	0.935	0.935	0.937	0.002	0.935	0.935	0.935	0.000
5	0.0	1.533	1.799	2.245	0.712	1.933	2.177	2.370	0.437	2.165	2.177	2.136	0.041
	0.5	1.395	1.595	1.941	0.546	1.708	1.903	2.064	0.356	1.893	1.903	1.869	0.034
	1.0	1.291	1.442	1.713	0.422	1.537	1.695	1.830	0.293	1.686	1.695	1.667	0.028
	1.5	1.209	1.324	1.537	0.328	1.404	1.531	1.645	0.241	1.525	1.531	1.508	0.023
	2.0	1.145	1.231	1.398	0.253	1.299	1.401	1.496	0.197	1.395	1.401	1.382	0.019
	2.5	1.094	1.157	1.285	0.191	1.213	1.294	1.373	0.160	1.290	1.294	1.279	0.015
	3.0	1.053	1.096	1.192	0.139	1.143	1.206	1.271	0.128	1.203	1.206	1.194	0.012
8	0.0	1.820	2.622	4.652	2.832	2.809	3.673	4.652	1.843	3.661	3.673	3.396	0.277
	0.5	1.623	2.198	3.556	1.933	2.349	2.951	3.624	1.270	2.942	2.951	2.759	0.192
	1.0	1.477	1.904	2.879	1.402	2.028	2.472	2.968	0.940	2.465	2.472	2.331	0.141
	1.5	1.365	1.689	2.420	1.055	1.793	2.132	2.514	0.721	2.127	2.132	2.024	0.108
	2.0	1.277	1.526	2.089	0.812	1.615	1.879	2.182	0.567	1.875	1.879	1.795	0.084
	2.5	1.207	1.399	1.840	0.633	1.475	1.684	1.929	0.454	1.681	1.684	1.618	0.066
	3.0	1.151	1.298	1.645	0.494	1.364	1.530	1.730	0.366	1.528	1.530	1.477	0.053

7 Concluding Remarks

We proposed the SPTE under the size corrected W, LR and LM tests. The bias and WMSE functions of the estimators are derived and compared. Our findings show that the conflict among the SPTEs under the large sample test as well as the modified tests is because the tests do not have the correct significance level. For the Edgeworth size corrected tests the conflict is reduced significantly. Our findings also illustrate the dangers involved in using the large sample W, LR and LM tests in the formation of the SPTE. We recommend the SPTE under any of the tests (leaning towards the W test because of its simplicity) with the Edgeworth correction factors for the chi-square critical values which make the tests have more nearly the correct significance level.

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