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SLOPE ESTIMATOR FOR THE LINEAR ERROR-IN-VARIABLES MODEL

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ABSTRACT

It is well known that in the presence of errors-in-variable the ordinary least squares (OLS) estimator of the parameters of the regression model is inappropriate. This is true even if the ratio of error variances (λ) is known. Wald's grouping method could deal with such problem but it lacks efficiency and is subject to identifiability problem. The main aim of the paper is to introduce a reflection based grouping method to improve the efficiency of the Wald's estimator under flexible assumption on λ. We compare the relative performance of the proposed reflection grouping (RG) estimator with the OLS, ML, Wald's and Geary's estimators by simulation studies under various conditions. The simulation results show that the RG estimator is more consistent and efficient than the other estimators.

Key Words: Linear regression models, Measurement error, Reflection of points; Ratio of error variances; Method of cumulants; Instrumental variable, and method of moments.
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1. Introduction

The error-in-variables problem in the simple linear regression model is also known as the measurement error (ME) problem. The ME poses a serious problem, as it directly impacts on estimators and their standard error (see Fuller, 2006, p. 3). In practice, it is rare to measure the variables precisely, for example, medical variables such as blood pressure and blood chemistries, agricultural variables such as soil nitrogen and rainfall etc.

In the conventional notation, let x be the true explanatory variable which is called the *latent* variable. Let m be the observable, or *manifest* explanatory variable. Similarly let η be the true value of the response and y be the associated manifest variable.

If the *latent* variables x_j and η_j are measured without error then their linear relationship is expressed as

$$\eta_j = \beta_0 + \beta_1 x_j, \quad j = 1, 2, \ldots, n. \tag{1.1}$$

If there is ME in both response and explanatory variables, the actual observed values, m and y are not the true values, and we define

$$m_j = x_j + u_j, \quad y_j = \eta_j + e_j \quad j = 1, 2, \ldots, n, \tag{1.2}$$

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where η_j is the jth realisation of the *latent* response variable, x_j is the jth value of the *latent* explanatory variable, e_j is the ME in the response variable and u_j is the ME in the explanatory variable. It is assumed that, $e_j \sim N(0, \sigma_{ee})$, and $u_j \sim N(0, \sigma_{uu})$. The model with the fixed x is called the *functional model*, whereas, the model with independent identically distributed random variable x is called *structural model*. The simple regression model with ME in both variables and without equation error is expressed as

$$y_j = \beta_0 + \beta_1 m_j + v_j, \quad j = 1, 2, \ldots, n, \tag{1.3}$$

where $v_j = e_j - \beta_1 u_j$. Note in equation (1.3) m_j and v_j are not independent, and hence least squares method is not valid for the above model.

Wald (1940) considered the problem of error in both explanatory and response variables. He proposed an estimation method based on dividing the observations of both the response and explanatory variables into two groups, G1 and G2. The G1 contains the first half of the ordered observations and G2 contains the second half. He showed that the slope of the line joining the group means provided consistent estimator for the slope parameter of the simple linear regression model. Properties of this estimator can be found in Gupta and Amanullah (1970). Similarly, Bartlett (1949) developed another grouping method where the available observations were divided into three, instead of two, groups. Gibson and Jowett (1957) investigated optimum ways of grouping the observations, and offered advice on how to place the data into these three groups to obtain the most efficient estimate of the slope. Another approach to deal with ME the instrumental variable (IV). Fundamentally, the IV method involves finding a variable z_j that is correlated with the manifest explanatory variable m_j, but is uncorrelated with the random error component, u_j. It is difficult to obtain a good IV which meets the above criteria. The disadvantage of Wald's method is the loss of efficiency (cf. Theil and Yzeren, 1956). The purpose of the present paper is to increase the efficiency of Wald's method using a different measure to find the instrumental variable.

2. Existing estimation methods

In this section we introduce estimators based on the principle of grouping of observations, maximum likelihood, and cumulants.

2.1. Grouping method

In 1940 Wald pointed out that a consistent estimator of β_1 may be calculated if the following assumptions are met:

1. The random variables $e_1, \ldots, e_n$ have the same distribution and they are uncorrelated, that is, $E(e_i e_j) = 0$ for $i \neq j$. The variance of e_j is finite.
2. The random variables $u_1, \ldots, u_n$ have the same distribution and they are uncorrelated, that is, $E(u_i u_j) = 0$ for $i \neq j$. The variance of u_j is finite.
3. The random variables e_j and u_j are uncorrelated, that is, $E(e_j u_j) = 0$.
4. The limit inferior of $\left|\{\sum_{j=1}^{k} x_j - \sum_{j=k+1}^{n} x_j\}/n\right| > 0$, where n is even ($n = 2, 4, 6, \ldots, \infty$), and $k = \frac{n}{2}$.

Then divide the observations into two groups based on the ranks of the manifest explanatory variable m_j, those above the median of m_j into one group G_1 and those below the median into another group G_2. Wald considers the expression

$$a_1 = \frac{(m_1 + \ldots + m_k) - (m_{k+1} + \ldots + m_n)}{n}, \; b_1 = \frac{(y_1 + \ldots + y_k) - (y_{k+1} + \ldots + y_n)}{n}.$$

Then Wald's estimators of β_1 and β_0 are given by

$$\begin{aligned} \hat{\beta}_1 &= \frac{a_1}{b_1} = \frac{(y_1 + \ldots + y_k) - (y_{k+1} + \ldots + y_n)}{(m_1 + \ldots + m_k) - (m_{k+1} + \ldots + m_n)} = \frac{\bar{y}_2 - \bar{y}_1}{\bar{m}_2 - \bar{m}_1} \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{m}, \end{aligned}$$

where $\bar{y} = \frac{\sum_{j=1}^{n} y_j}{n}$, and $\bar{m} = \frac{\sum_{j=1}^{n} m_j}{n}$.

Johnston (1972, p. 284) showed how Wald's grouping method is based an instrumental variable. If the number of sample observations is even then define a z matrix as

$$z' = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & -1 & 1 & \ldots & -1 \end{bmatrix},$$

where the second row included minus or plus one according to the value of the manifest explanatory variable m_j is below or above the median of m_j. If we rewrite the model (2.1) in matrix for m as

$$\eta = x\beta,$$

where $x' = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ x_1 & x_2 & x_3 & \ldots & x_n \end{bmatrix}$, and $\beta = (\beta_0, \beta_1)'$, then the instrumental variable estimator of β is defined by

$$\hat{\beta} = (z'x)^{-1} z'\eta = \begin{bmatrix} n & 0 \\ 0 & \frac{n}{2}(\bar{x}_2 - \bar{x}_1) \end{bmatrix}^{-1} \begin{bmatrix} n\bar{\eta} \\ \frac{n}{2}(\bar{\eta}_2 - \bar{\eta}_1) \end{bmatrix} = \begin{bmatrix} \bar{\eta} \\ \frac{\bar{\eta}_2 - \bar{\eta}_1}{\bar{x}_2 - \bar{x}_1} \end{bmatrix}.$$

The estimator of the slope is

$$\hat{\beta}_1 = \frac{\bar{\eta}_2 - \bar{\eta}_1}{\bar{x}_2 - \bar{x}_1}.$$

According to Johnston (1972, p. 284) $\bar{\eta}$ is the estimator of $\beta_0 + \beta_1 E(x)$, and so

$$\hat{\beta}_0 = \bar{\eta} - \hat{\beta}_1 \bar{x}.$$

It is suggested that one should omit the central observation before computations if n is odd.

Wald countered the problem of consistency, the groups are not independent of the error terms if they did not base on the order of the true values. He proved that the grouping by the observed values is the same as grouping with respect to the true values. However, there are some criticisms in literature about Wald estimator, but these criticisms lacked consensus. Neyman and Scott (1951) pointed out that the Wald estimator is consistent for β_1 in the structural relation situation if and only if

$$Pr[m_{p_1} - e < x \leq m_{p_1} - \mu] = Pr[m_{1-p_2} - e < x < m_{p_1} - \mu] = 0,$$

where m_{p_1} and m_{1-p_2} are the p_1 and $(1-p_2)$ percentile points of $F(m)$, the distribution of m, and $e-\mu$ is the range of u.

This condition means that we must know the range of the error in m, and in order to satisfy the condition the range should be finite, otherwise the condition becomes $Pr[-\infty < x < \infty] = 0$ which is never satisfied. Often relies on the central limit theorem and assumes that u is normally distributed, where it has an infinite range, then the above condition be unsatisfied when the errors u_j are normally distributed (see Madansky, 1959). Wald estimator is consistent under very specific conditions except that the errors are not normally distributed (cf. Gupta and Amanullah, 1970). While Pakes (1982) claimed that the work of Gupta and Amanullah (1970) is needless, where the Wald's estimator is inconsistent. However, according to Theil and Yzeren (1956) the Wald's method is valuable, though there is loss of efficiency. Johnston (1972, p. 284) stated 'Under fairly general conditions the Wald estimator is consistent but likely to have a large sampling variance'. Moreover, Fuller (2006, p. 74) mentioned that the Wald's method was often interpreted improperly. In fact, there are many discussions on improving the efficiency of the grouping method by dividing the observations to more than two groups and groups of unequal size (see Nair and Banerjee, 1942, Bartlett, 1949, Dorff and Gurland, 1961, and Ware, 1972). Under the normality assumption the grouping estimator is the maximum likelihood estimator (see Chang and Huang, 1997). In practice, the grouping method is still important, and the grouping estimator is the maximum likelihood estimator under the normality assumption (Chang and Huang 1997, Cheng and Van Ness, 1999, p. 130).

2.2 Maximum likelihood method

The likelihood method can have one or more solutions, and might be a saddle point, a local maximum, or a local minimum of the likelihood function. Lindley (1947) mentioned that the likelihood equations are consistent if the ratio of error variances λ is known. The ML estimator of β_1 is given by

$$\hat{\beta}_{ML} = \left[(S_y^2 - \lambda S_m^2) + \sqrt{(S_y^2 - \lambda S_m^2)^2 + 4\lambda S_{ym}^2} \right] / 2S_{ym}.$$

The most common criticisms of this method is the misspecification of λ. This method deals only with models that do not include an equation error which means that all data should fall exactly on a straight line, which is rare to happen in practice.

2.3 Cumulants method

This method is closely related to the method of higher-order moments, and both methods lead to similar estimators. Geary (1949) wrote a series of papers on the method of moments. He introduced a treatment for the ME model under assumptions that the latent variable x is not normally distributed and all moments exist. Geary's estimators (G_a, G_b) or cumulant method estimators do not work if (x_j, η_j) are jointly normally distributed, because all cumulants of order ≥ 3 are zero in normal systems. The Geary's estimators are given by

$$\hat{\beta}_{1G_a} = \frac{k(1,3)}{k(2,2)}, \quad \text{and} \quad \hat{\beta}_{1G_b} = \frac{k(2,2)}{k(3,1)},$$

where $k(\cdot,\cdot)$ represents an appropriate cumulant (see Fuller 2006, p. 72, for details). The cumulant estimates deals only with the non-normal structural model (cf Cheng and Van Ness, 1999, p. 127).

3. Proposed reflection method

The reflection grouping (RG) method is constructed based on the ranks of a new variable d_j which is located at the middle of the manifest explanatory variable m_j and its *reflection* m_j^*. The difference from Wald's original method is to use the ranks of the transformed variable d_j to dividing the observations into two groups instead of using the ranks of the manifest explanatory variable. Moreover, assume the additional assumption that the ratio of error variances $\lambda = \sigma_{ee}\sigma_{uu}^{-1}$ is known as $\lambda < 1$, $\lambda = 1$, or $\lambda > 1$. To avoid the unwanted and troublesome influence of the ME in the explanatory variable, the idea of *reflection* of the manifest variable is used for all the values of explanatory variable. The *reflection* of the points is taken about the fitted regression line of the manifest variables. This is essentially done by a transformation of the observed values of the explanatory variable to their *reflection* on the Euclidean plane. In the conventional notation, the *reflection* of the explanatory variable $m_j = x_j + u_j$ (with ME u_j) for $j = 1, 2, \ldots, n$, can be defined as

$$m_j^* = m_j \cos 2\psi + (y_j - \hat{\beta}_{0m}) \sin 2\psi, \tag{3.1}$$

where $\hat{\beta}_{0m}$ is the least square estimate of the intercept parameter, ψ is the angle measure defined as $\psi = \arctan \hat{\beta}_{1m}$ in which $\hat{\beta}_{1m}$ is the least square estimate of the slope parameter in the manifest model, and cos, and sin are the usual trigonometric cosine and sine functions respectively. For the definition of *reflection* of points on the Euclidean plane (see Vaisman, 1997, p. 164-169).

The general idea of using the reflection is that the true value of the latent explanatory variable x is located at the middle of the manifest variable m and its reflection m^*, if the ratio of error variances $\lambda = 1$. The reflection group estimator takes different form depending on the value of λ. We consider the following three cases for $\lambda = 1$, $\lambda < 1$, and $\lambda > 1$. Therefore we suggest a new variable $d_j = (m_j + m_j^*)/2$ when $\lambda = 1$, but if $\lambda < 1$ we use another variable $d_{1j} = (d_j + m_j^*)/2$, and $d_{2j} = (d_j + m_j)/2$ if $\lambda > 1$. The main advantage of using the proposed variable d_j for grouping is to reduce the sum squares of error. We propose modifications to the Wald's method to introduce three different forms of the reflection grouping estimator for the slope β_1 for varying values of λ.

(a) **When $\lambda = 1$**

Let the instrumental variable T_1 be based on the ranks of the variable $d_j = (m_j + m_j^*)/2$. The entries in the second row of T_1' is -1 if the value of d_j is less then the median of $d_j's$, and $+1$ otherwise. A typical representation of T_1' is

$$T_1' = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & -1 & 1 & \ldots & -1 \end{bmatrix}.$$

The RG estimator of β_1 and β_0 is given by

$$\hat{\beta}_{RG1} = (T_1'm)^{-1}T_1'y = \begin{bmatrix} n & 0 \\ 0 & \frac{n}{2}(\bar{m}_{12} - \bar{m}_{11}) \end{bmatrix}^{-1} \begin{bmatrix} n\bar{y} \\ \frac{n}{2}(\bar{y}_{12} - \bar{y}_{11}) \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \frac{\bar{y}_{12} - \bar{y}_{11}}{\bar{m}_{12} - \bar{m}_{11}} \end{bmatrix}, \text{and}$$

$$\hat{\beta}_{1RG1} = \frac{\bar{y}_{12} - \bar{y}_{11}}{\bar{m}_{12} - \bar{m}_{11}}, \quad \hat{\beta}_{0RG1} = \bar{y} - \hat{\beta}_{1RG1}\bar{m}$$

(b) **When $\lambda < 1$**

Similarly, let the instrumental variable T_2 be based on the ranks of the variable $d_{j2} = (d_j + m_j)/2$. The entries in the second row of T_2' is -1 if the value of d_{j2} is less then the median of d_{j2}, and $+1$ otherwise. The RG estimator of β_1 and β_0 becomes

$$\hat{\beta}_{RG2} = (T_2'm)^{-1}T_2'y = \begin{bmatrix} n & 0 \\ 0 & \frac{n}{2}(\bar{m}_{22} - \bar{m}_{21}) \end{bmatrix}^{-1} \begin{bmatrix} n\bar{y} \\ \frac{n}{2}(\bar{y}_{22} - \bar{y}_{21}) \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \frac{\bar{y}_{22} - \bar{y}_{21}}{\bar{m}_{22} - \bar{m}_{21}} \end{bmatrix}, \text{and}$$

$$\hat{\beta}_{1RG2} = \frac{\bar{y}_{22} - \bar{y}_{21}}{\bar{m}_{22} - \bar{m}_{21}}, \quad \hat{\beta}_{0RG2} = \bar{y} - \hat{\beta}_{1RG2}\bar{m}$$

(c) **When $\lambda > 1$**

Finally, let the instrumental variable T_3 be based on the ranks of the variable $d_{j1} = (z_j + m_j^*)/2$. The entries in the second row of T_3' is -1 if the value of d_{j1} is less then the median of d_{j1}, and $+1$ otherwise. The RG estimator of β_0 and β_1 becomes

$$\hat{\beta} = (T_3'm)^{-1}T_3'y = \begin{bmatrix} n & 0 \\ 0 & \frac{n}{2}(\bar{m}_{32} - \bar{m}_{31}) \end{bmatrix}^{-1} \begin{bmatrix} n\bar{y} \\ \frac{n}{2}(\bar{y}_{32} - \bar{y}_{31}) \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \frac{\bar{y}_{32} - \bar{y}_{31}}{\bar{m}_{32} - \bar{m}_{31}} \end{bmatrix}, \text{and}$$

$$\hat{\beta}_{1RG3} = \frac{\bar{y}_{32} - \bar{y}_{31}}{\bar{m}_{32} - \bar{m}_{31}}, \quad \hat{\beta}_{0RG3} = \bar{y} - \hat{\beta}_{1RG3}\bar{m}$$

We should omit the central observation before computations if n is odd. Although the second row of T_1', T_2', T_3' (that is the sequence of -1 and $+1$) appears to be similar, they will be different when the method is applied to any real data set.

The aims of the proposed RG method are to deal with the situation of misspecification λ which makes the maximum likelihood estimator biased, and increasing the efficiency of the Wald's method.

4. Simulation studies

We perform large scale simulation study to compare between the proposed (RG) estimator and other existing estimators for both normal model when $0.1 < \lambda < 9$, and non-normal model when $\lambda = 1, < 1$, and > 1.

4.1 Normal structural model

This study compares between the proposed RG estimator, ML, and OLS estimators of the slope parameter β_1 of the normal structural model for sample size $n = 40$, and when λ is correct and misspecified. The data is based on $10,000$ replications, $x \sim N(0, 49)$, and the parameters settings are $\beta_1 = 1$, $\beta_0 = 0$, when $0.1 < \lambda < 9$.

It is clear from Figure 1 that the ML estimator does not work well when λ is misspecified.

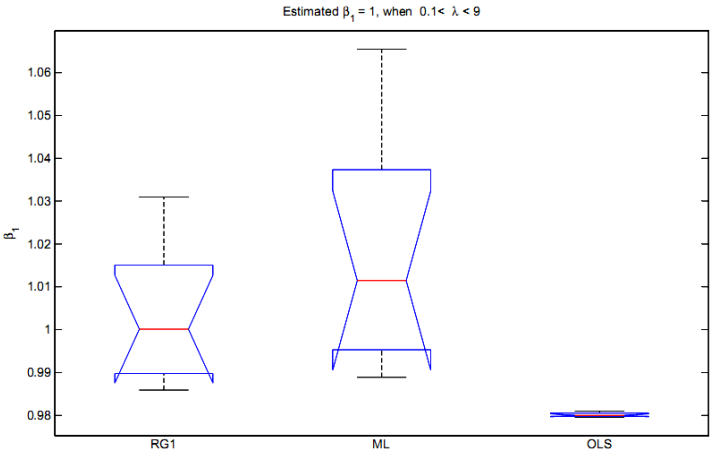

Figure 1: Graph of the estimated slope for three different estimators when $0.1 < \lambda < 9$.

That means without selecting the correct value of λ the ML method overestimates the true slope. The proposed method is better than both the ME and OLS estimators when the ratio of error variances λ is incorrect.

4.2 Non normal structural model

This study provides comparison between the proposed RG estimator and Wald's (W), Geary's (G), and OLS estimators of the slope parameter β_1 of the non-normal structural model for different sample sizes, and when $\lambda = 1$, < 1, and > 1. Also it provides the comparison in terms of the mean absolute error (MAE) of these estimators.

(a) **When** $\lambda = 1$ (d_j is used for $\hat{\beta}_{RG1}$). The data is based on $100,000$ replications, $x \sim$ uniform on $[-5, 5]$, $u \sim N(0, 1)$, $e \sim N(0, 1)$, and the parameters settings are $\beta_1 = 1$, $\beta_0 = 0$ (see Fig. 2).

(b) **When** $\lambda < 1$ (d_{1j} is used for $\hat{\beta}_{RG2}$). The data is based on $100,000$ replications, $x \sim$ uniform on $[-5, 5]$, $u \sim N(0, 1)$, $e \sim N(0, 2.25)$, and the parameters settings are $\beta_1 = 1$, $\beta_0 = 0$ (see Fig. 3).

(c) **When** $\lambda > 1$ (d_{2j} is used for $\hat{\beta}_{RG3}$). The data is based on $100,000$ replications, $x \sim$ uniform on $[-5, 5]$, $u \sim N(0, 2.25)$, $e \sim N(0, 1)$, and the parameters settings are $\beta_1 = 1$, $\beta_0 = 0$ (see Fig. 4).

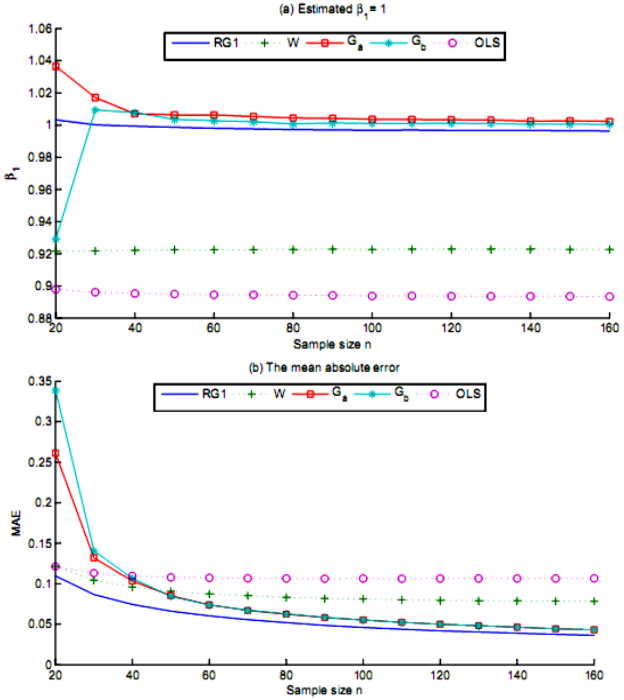

Figure 2: Graph of the estimated slope for five different estimators when $\lambda = 1$.

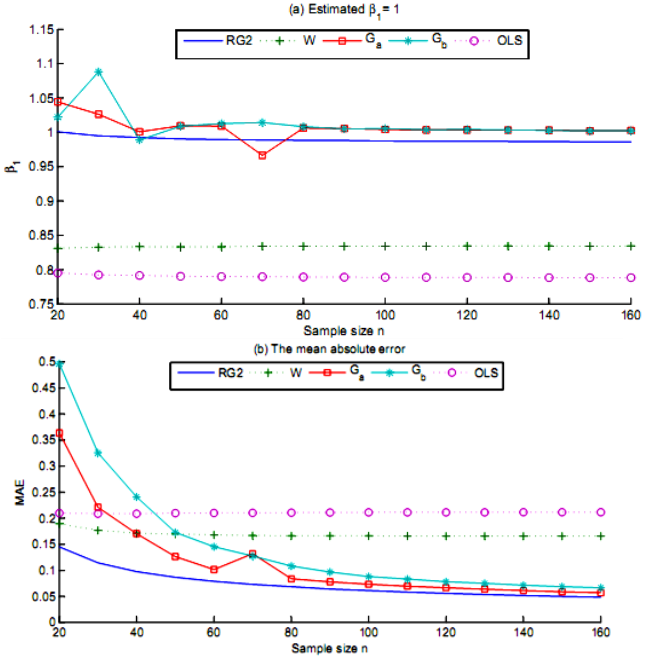

Figure 3: Graph of the estimated slope and MAE for five different estimators when $\lambda < 1$.

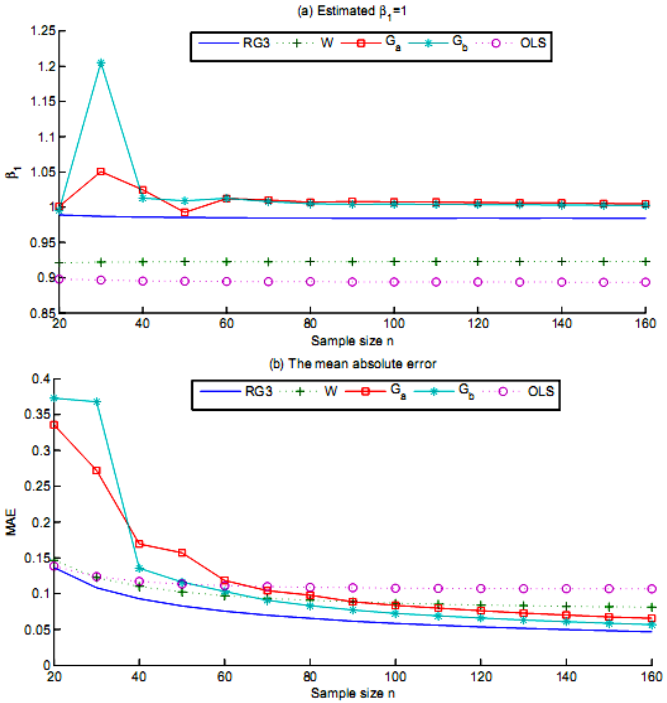

Figure 4: Graph of the estimated slope and MAE for five different estimators when $\lambda < 1$.

In Figures 2a, 3a, and 4a the values of Wald's estimator are away from the true value of β_1, but they appear to be slightly better than those of OLS. The values of the OLS are the lowest and far below the true value of $\beta_1 = 1$. Obviously, the proposed estimator are much closer to the true value of β_1 than the Geary's estimators. Figures 2a, 3a, and 4a show that if the sample size is large then the two estimators of Geary are close to the true value of the slope, but they fluctuate significantly if n is small.

Figures 2b, 3b, and 4b reveal that the MAE of the OLS estimator is the highest. The MAE of Geary's estimators or cumulant method estimators are better and smaller than that of Wald's and OLS. Whereas the MAE of the Wald's estimator appears to be better than that of the OLS, though it is not small. Clearly, the MAE of the RG estimator is better and the smallest compared to the other estimators.

5. Concluding Remarks

The proposed RG method is constructed based on a specific modifications to two grouping method. It works under a fixable and realistic assumption that the ratio of error variances equal, less, or grater than one. The proposed method is straightforward, easy to implement, and handles the ME in both normal and non-normal structural models. The simulated results show that the proposed RG method is appropriate to the normal structural model more than both the ML, and OLS methods even when λ is misspecified. Based on the forgoing discussion it is evident that the proposed RG method significantly increases the efficiency of Wald's grouping method. Moreover, it performs better than the ML method when λ is misspecified, and sample size is small.

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