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Modulational stability of weakly nonlinear wave-trains in media with small- and large-scale dispersions

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In this paper, we revisit the problem of modulation stability of quasi-monochromatic wave-trains propagating in a media with the double dispersion occurring both at small and large wavenumbers. We start with the shallow-water equations derived by Shrira [Izv., Acad. Sci., USSR, Atmos. Ocean. Phys. (Engl. Transl.) **17**, 55–59 (1981)] which describes both surface and internal long waves in a rotating fluid. The small-scale (Boussinesq-type) dispersion is assumed to be weak, whereas the large-scale (Coriolis-type) dispersion is considered as without any restriction. For unidirectional waves propagating in one direction, only the considered set of equations reduces to the Gardner–Ostrovsky equation which is applicable only within a finite range of wavenumbers. We derive the nonlinear Schrödinger equation (NLSE) which describes the evolution of narrow-band wave-trains and show that within a more general bi-directional equation the wave-trains, similar to that derived from the Ostrovsky equation, are also modulationally stable at relatively small wavenumbers $k < k_c$ and unstable at $k > k_c$, where k_c is some critical wavenumber. The NLSE derived here has a wider range of applicability: it is valid for arbitrarily small wavenumbers. We present the analysis of coefficients of the NLSE for different signs of coefficients of the governing equation and compare them with those derived from the Ostrovsky equation. The analysis shows that for weakly dispersive waves in the range of parameters where the Gardner–Ostrovsky equation is valid, the cubic nonlinearity does not contribute to the nonlinear coefficient of NLSE; therefore, the NLSE can be correctly derived from the Ostrovsky equation. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4937362]

It is a well-known fact that plane surface gravity waves on shallow water are stable with respect to self-modulation, when $k\,h < 1.363$, where k is the wavenumber and h is the water depth. The same remains basically true for gravity-capillary waves, if the surface tension effect is not too strong; however, the criterion of stability in terms of $k\,h$ becomes more complicated and depends on surface tension (Ablowitz and Segur, 1981). The situation becomes even more complicated for waves in a rotating fluid. In 1981, Shrira derived an equation describing long nonlinear waves in a rotating fluid and, neglecting the small-scale (Boussinesq) dispersion, obtained the nonlinear Schrödinger equation (NLSE) for quasi-harmonic waves. According to the analysis performed within the NLSE, surface waves are unstable to self-modulation due to the influence of rotation at very small wavenumbers $k\,h \to 0$. In 2008, Grimshaw and Helfrich found that within the Ostrovsky equation, which is a particular case of Shrira's equation, valid within the certain range of wavenumbers, $k_1 < k < k_2$, the modulation instability occurs for relatively big wavenumbers, when $k_c < k < k_2$, whereas at small wavenumbers $k_1 < k < k_c$, waves are stable. However, their analysis seems incomplete in application to real physical processes, since the cubic nonlinear term is not included into the Ostrovsky equation, whereas it can contribute to the nonlinear coefficient of NLSE. More accurate analysis based on the Gardner–Ostrovsky equation containing both quadratic and cubic nonlinearities was performed by Whitfield and Johnson (2015a; 2015b). The results obtained in the papers by Shrira (1981), on the one hand, and Grimshaw and Helfrich (2008), as well as Whitfield and Johnson (2015a; 2015b), on the other hand, provide a contradictory conclusion about the modulation instability of large scale waves; the resolution of the issue requires more attention. In this work, we revisit the results obtained in all these papers and derive the NLSE applicable to quasi-monochromatic waves both with very small wavenumbers of carrier wave, up to $k \to 0$, and with relatively big wavenumbers, when the influence of small-scale Boussinesq dispersion becomes important. We present the governing equation in the dimensionless form containing the dispersion parameters of either sign, so that the equation is applicable not only to water waves in a rotating fluid but also to a wider class of nonlinear waves of any nature. From the derived NLSE,s we determine then the criterion of modulational stability/instability for all possible signs of dispersion coefficients and show that the results obtained agree well with the findings of Grimshaw and Helfrich (2008) within the range of validity of Ostrovsky equation, but contradict to the results obtained by Shrira (1981) at small k. We also show that the correction to the nonlinear coefficient in the

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NLSE due to the cubic nonlinear term in the Gardner–Ostrovsky equation as derived by Whitfield and Johnson (2015a; 2015b) is actually of the next order of smallness and can be usually neglected, unless the coefficient of the quadratic nonlinearity in the Gardner–Ostrovsky equation is not anomalously small (such situation can occur, e.g., for internal waves in two-layer fluid (Ostrovsky *et al.*, 2015)).

I. INTRODUCTION

It is a matter of a well-known fact that unidirectional trains of shallow-water waves (both surface and internal) are modulationally stable except the case of a very strong surface tension effect (Ablowitz and Segur, 1981). This result formally agrees with what follows from the Korteweg–de Vries (KdV) equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3} = 0, \tag{1}$$

which is applicable to the description of weakly nonlinear long waves. The coefficients α and β may be of either sign depending on the nature of waves. In particular, for the surface gravity-capillary waves the coefficients are (Karpman, 1973)

$$\alpha = \frac{3}{2}\frac{c_0}{h}, \quad \beta = \frac{c_0 h^2}{6}\left(1 - \frac{3\sigma}{\rho g h^2}\right), \tag{2}$$

where $c_0 = (gh)^{1/2}$ is the speed of long linear waves, g is the acceleration due to gravity, h is the fluid depth, σ is the surface tension, and ρ is the fluid density. The dispersion relation between the wave frequency ω and wavenumber k for infinitesimal amplitude waves in the linearised equation (1) is well-known (see, e.g., Ablowitz and Segur (1981); Karpman (1973); and Ostrovsky *et al.* (2015))

$$\omega(k) = c_0 k – \beta k^3. \tag{3}$$

It shows that the dispersion appears at relatively large k (small wavelength λ), when the influence of the second term in the right-hand side is not negligibly small. Such small-scale ("Boussinesq") dispersion is typical for long water waves. The derivation of Eq. (1) is based on the assumption that the dispersion is weak, so that $\beta k^3 \ll c_0 k$ or $k \ll k_1$, where $k_1 = (c_0/|\beta|)^{1/2}$. Depending on the sign of coefficient β, one can distinguish between the negative dispersion, if $\beta > 0$ (the phase speed $V_p \equiv \omega(k)/k = c_0 - \beta k^2$ decreases with k in this case), and positive dispersion, if $\beta < 0$ (the phase speed increases with k).

There are also physical systems containing the large-scale ("Coriolis-type") dispersion which manifests when $k \to 0$ and disappears when $k \to \infty$. In many physical situations the large-scale dispersion is small and of the same order of smallness as the small-scale dispersion, so that the combined dispersion relation can be presented in the form

$$\omega(k) = c_0 k - \beta k^3 + \gamma/k, \tag{4}$$

where γ is some constant which can be of either sign, and $\gamma/k \ll c_0 k$, so that $k \gg k_2$, where $k_2 = (|\gamma|/c_0)^{1/2}$. The corresponding weakly nonlinear evolution equation generalising the KdV equation (1) is known as the Ostrovsky equation (Ostrovsky, 1978; Grimshaw *et al.*, 1998; Grimshaw and Helfrich, 2008; 2012; and Ostrovsky *et al.*, 2015)

$$\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \alpha \eta \frac{\partial \eta}{\partial x} + \beta \frac{\partial^3 \eta}{\partial x^3}\right) = \gamma \eta. \tag{5}$$

This equation is applicable for waves of intermediate spatial scales

$$\sqrt{|\gamma|/c_0} \ll k \ll \sqrt{c_0/|\beta|}, \tag{6}$$

when both small-scale and large-scale dispersions are relatively small.

Grimshaw and Helfrich (2008; 2012) using Ostrovsky Eq. (5) with $\beta > 0$ and $\gamma > 0$ have shown that the large scale dispersion drastically changes the modulation stability of quasi-harmonic wave-trains. The nonlinear correction to the wave frequency remains negative for all wavenumbers as in the case of KdV equation, but the dispersion coefficient in the NLSE changes its sign at $k = k_c \equiv (\gamma/3\beta)^{1/4}$ when the group velocity $c_g = d\omega/dk$ attains maximum. Therefore, the corresponding NLSE remains modulationally stable for $k < k_c$, but becomes unstable for $k > k_c$. This is in a sharp contrast to the intuition based on the NLSE derived from the KdV equation (1). Indeed, at a first glance one can expect an influence of Coriolis-type dispersion at large scales only, as it vanishes when $k \to \infty$. At this limit, $k \to \infty$, the small-scale dispersion predominates, and wave-trains should be modulationally stable as in the case of KdV equation.

The physical explanation of this apparent contradiction is in the crucial role of a zero harmonic (the "mean flow") generated by the basic wave-train. The nonlinear coefficient in the NLSE usually consists of contributions from both the second and zero harmonics. However, the zero harmonic is beyond the range of applicability of the Ostrovsky equation (see above), which formally requires the zero total "mass" of a perturbation (see, e.g., Grimshaw *et al.* (1998)). Therefore, the zero harmonic cannot contribute to the nonlinear coefficient, and the second harmonic provides the nonlinear coefficient in the NLSE of the opposite sign in comparison with that derived from the KdV equation, where both zero and second harmonics contribute jointly (this will be clearly seen from the analysis presented in the Appendix).

As the Ostrovsky equation is approximate and has a limited range of validity, the issue of modulation stability of wave-trains remained uncertain thus far, because at very small wavenumbers the situation with the modulation stability could be different, and contribution of the zero harmonic into the nonlinear coefficient of NLSE might be important again. Therefore, the problem of modulation stability of wave-trains should be resolved within the framework of more accurate equations in the long-wave limit. Moreover, for the analysis of modulation stability of waves in the real physical systems (e.g., water waves, or plasma waves) the KdV model equation is, obviously, insufficient, because it contains only the quadratic nonlinear term, whereas cubic nonlinear terms usually provide the same order contribution to the nonlinear coefficient of NLSE.

The analysis of modulation stability of long water waves in a rotating fluid was undertaken as earlier as 1981 by Shrira, who derived a set of shallow-water equations and investigated modulation stability of quasi-monochromatic waves, ignoring however the small-scale dispersion. His analysis predicts the modulation instability of wave-trains at very small wavenumbers; this does not match with the result of Grimshaw and Helfrich (2008; 2012). Thus, the problem of modulation stability of water waves requires thorough consideration which motives the current study.

Below the problem of modulation stability of quasi-monochromatic wave-trains is re-examined on the basis of the 2D shallow-water model set of equations derived by Shrira (1981) and augmented by the terms representing the Boussinesq dispersion. We derive the 2D NLSE and study a stability of quasi-monochromatic wave-trains with respect to longitudinal and transverse modulations for various signs of coefficients of the NLSE. The results obtained can be applicable not only to water waves, but in the wider context, including plasma waves, waves in solids, in optical media, etc.

II. THE GOVERNING EQUATIONS AND DISPERSION RELATIONS

We start with the following set of equations applicable (after appropriate scaling) both to surface and internal waves in the Boussinesq approximation (Ostrovsky, 1978; Shrira, 1981; and Grimshaw *et al.*, 1998)

$$\frac{\partial \eta}{\partial t} + \nabla_\perp[(h+\eta)\mathbf{q}] = 0, \tag{7}$$

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla_\perp)\mathbf{q} + [\mathbf{f}\times\mathbf{q}] + \frac{c_0^2}{h}\nabla_\perp\eta + s\,h\frac{\partial^2\nabla_\perp\eta}{\partial t^2} = 0, \tag{8}$$

where η is the perturbation of a free surface in a non-stratified fluid or perturbation of an isopycnal surface (surface of equal density) in a stratified fluid, $\mathbf{q} = (u, \upsilon)$ is the depth averaged fluid velocity with two horizontal components, longitudinal u and transverse υ, $\mathbf{f} = f\mathbf{n}$, where $f = 2\Omega \sin\varphi$ is the Coriolis parameter, Ω is the angular frequency of Earth rotation, φ is the local geographic latitude, $\mathbf{n}$ is the unit vector normal to the Earth surface, and $\nabla_\perp = (\partial/\partial x, \partial/\partial y)$. Other parameters in Eqs. (7), (8) are c_0—the speed of long linear waves, h—the fluid depth, and s—the dispersion parameter. In the case of surface waves the long-wave speed $c_0 = \sqrt{gh}$, and $s = (1 - 3\sigma/\rho\ gh^2)/3$, whereas for internal waves in two-layer fluid $c_0 = \sqrt{\frac{\Delta\rho}{\rho} g \frac{h_1 h_2}{h_1+h_2}}$ and $s = h_1h_2/3h^2$, where h_1 and h_2 are thicknesses of the upper and lower layers correspondingly, and $\Delta\rho = \rho_2 - \rho_1$ is the difference of layer densities (in the Boussinesq approximation both these densities are assumed close to each other ($\rho_1 \approx \rho_2 = \rho$), while the product $\Delta\rho\ g$ is assumed to be finite).

In one-dimensional case when all variables depend on one spatial coordinate x this set of equations can be reduced to one bi-directional equation for the transverse velocity υ (Shrira, 1981). If we put

$$\eta = \frac{h}{f}\frac{\partial \upsilon}{\partial x}, \quad u = -\frac{1}{f+\upsilon_x}\frac{\partial \upsilon}{\partial t}, \tag{9}$$

where indices t and x here and below stand for the corresponding derivatives, then we obtain

$$\upsilon_{tt} - c_0^2\upsilon_{xx} + f^2\upsilon - s\,h^2\upsilon_{ttxx} = \left(\frac{\upsilon_t\upsilon_x}{f+\upsilon_x}\right)_t + \frac{f}{2}\left[\frac{(\upsilon_t)^2}{(f+\upsilon_x)^2}\right]_x. \tag{10}$$

For perturbations of relatively small amplitude, the nonlinear terms in the right-hand side can be simplified, and Eq. (10) can be reduced to

$$\upsilon_{tt} - c_0^2\upsilon_{xx} + f^2\upsilon - s\,h^2\upsilon_{ttxx} \approx \frac{1}{f}(\upsilon_{tt}\upsilon_x + 2\upsilon_t\upsilon_{xt}) - \frac{1}{f^2}\left[\upsilon_{tt}(\upsilon_x)^2 + 4\upsilon_t\upsilon_x\upsilon_{xt} + (\upsilon_t)^2\upsilon_{xx}\right]. \tag{11}$$

By introducing new variables $\tau = = ft/\sqrt{|\Gamma|}$, $\xi = (x/h)(|\mathrm{B}/s|)^{1/2}$, $w = \upsilon\ (|\mathrm{B}/s|)^{1/2}/(\mathrm{A}hf)$, where $\mathrm{A} = \mathrm{B} = \Gamma = 1$, one can present Eq. (11) in the dimensionless form

$$\begin{aligned} w_{\tau\tau} - C^2 w_{\xi\xi} + \Gamma w - \mathrm{B} w_{\tau\tau\xi\xi} &= \mathrm{A}(w_{\tau\tau}w_\xi + 2w_\tau w_{\tau\xi}) - \Phi[w_{\tau\tau}(w_\xi)^2 \\ &\quad + 4w_\tau w_\xi w_{\tau\xi} + (w_\tau)^2 w_{\xi\xi}], \end{aligned} \tag{12}$$

where $C^2 = (c_0/hf)^2(|\mathrm{B}\Gamma/s|)$ stands for the normalised characteristic wave speed, and $\Phi = \mathrm{A}^2$ (we prefer to keep letter Φ rather than A^2 to track the contribution of nonlinear terms in the final equation). For further estimates we put $C = 34.3$ assuming that $h = 1000$ m, $g = 9.8$ m/s ($c_0 = 99$ m/s), $f = 5\times10^{-3}$ s^{-1}, $s = 1/3$, and $\mathrm{B} = \Gamma = 1$.

In what follows we will not restrict ourselves to the particular choice of dimensionless coefficients $\mathrm{A} = \mathrm{B} = \Gamma = 1$ as above, but will assume that these coefficients can be of either sign with the moduli equal to unity. Then Eq. (12) or its unidirectional analogue, the Ostrovsky equation (see below) can be considered in a much wider context. In particular, the coefficient B can be negative for magnetosonic waves in a rotating plasma (Obregon and Stepanyants, 1998) or for capillary waves in a liquid fluid layer (Karpman, 1973) (see Eq. (2)); the coefficient Γ is usually positive, but can be negative in a rotating stratified ocean with shear flows (Alias *et al.*, 2014a; 2014b).

For unidirectional waves propagating, for instance, to the right only, one can derive from Eq. (12) the following Gardner–Ostrovsky equation (Ostrovsky, 1978; Grimshaw *et al.*, 1998; and Ostrovsky *et al.*, 2015)

$$\frac{\partial}{\partial \xi}\left(\frac{\partial w}{\partial t} + C\frac{\partial w}{\partial \xi} + a\zeta\frac{\partial \zeta}{\partial \xi} - a_1\zeta^2\frac{\partial \zeta}{\partial \xi} + b\frac{\partial^3\zeta}{\partial \xi^3}\right) = r\zeta, \tag{13}$$

where $\zeta = \partial w/\partial \xi$ (in terms of Eq. (9) this is just the normalised variable η: $\zeta = (\eta/\mathrm{A}fh)(\mathrm{B}\ \Gamma/s)^{1/2}]$, $a = 3\mathrm{A}C/2$, $a_1 = 3\Phi C$, $b = \mathrm{B}C/2$, and $r = \Gamma/2C$.

For perturbations of infinitesimal amplitudes, the nonlinear terms on the right-hand side of Eq. (12) can be omitted, then the following dispersion relation of the linearised equation can be obtained for $\upsilon \sim \exp[i(\omega t - kx)]$:

$$\omega^2 = \frac{C^2k^2 + \Gamma}{1 + \mathrm{B}k^2}. \qquad (14)$$

In the intermediate range of wavenumbers where $\sqrt{|\Gamma|}/C \ll k \ll 1/\sqrt{|\mathrm{B}|}$, the dispersion relation can be approximated by the first three terms of the Laurent series (cf. Eq. (4))

$$\omega = Ck - bk^3 + \frac{r}{k} \equiv Ck - \frac{\mathrm{B}C}{2}k^3 + \frac{\Gamma}{2Ck}. \qquad (15)$$

In particular, when $r = 0$ ($\Gamma = 0$) we obtain from Eq. (15) the dispersion relation (3) in the dimensionless variables. Plots of the dispersion relations (14) and (15) are shown in Fig. 1 for different values of parameters B and Γ and $C = 34.3$. As one can see from these plots, the dispersion relation (15) approximates well the dispersion relation (14) in the intermediate range of wave numbers indicated above. Therefore, the non-physical singularity which appears at $k = 0$ in Eq. (15) with $|\Gamma| = 1$ is actually beyond the range of its applicability and is just an artefact of approximate character of the dispersion relation. The more accurate Eq. (14) does not have such singularity and is valid up to $k = 0$ inclusive.

From dispersion relations (14) and (15), one can readily obtain the group velocities $V_g \equiv d\omega/dk$

$$V_g = \frac{k(C^2 - \mathrm{B}\Gamma)}{(1 + \mathrm{B}k^2)^{3/2}\sqrt{C^2k^2 + \Gamma}}; \quad V_{gO} = C - 3bk^2 - \frac{r}{k^2}, \qquad (16)$$

where V_g pertains to Eq. (14) and V_{gO} pertains to Eq. (15).

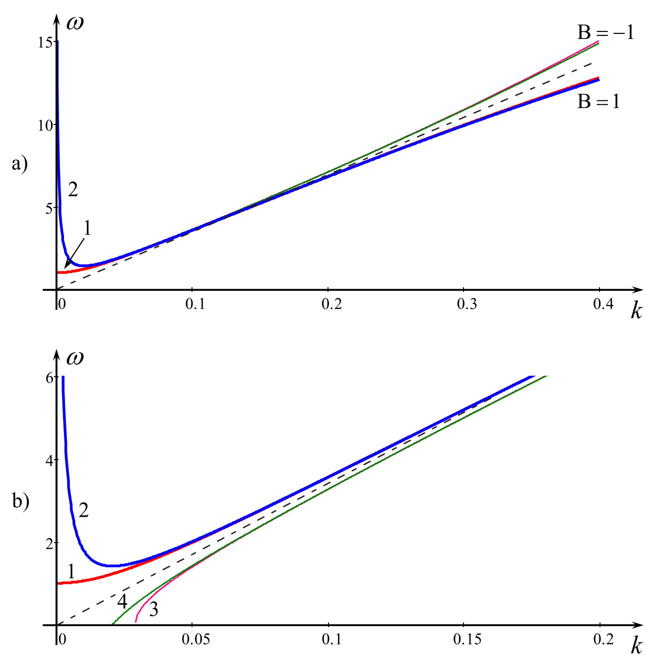

FIG. 1. Dispersion curves as per Eqs. (14) and (15) for different coefficients B and Γ. Lines 1 in panels (a) and (b) pertain to Eq. (14) with $\Gamma = 1$; lines 2 in panels (a) and (b) pertain to Eq. (15) with the same $\Gamma = 1$ ($r = 1.46 \times 10^{-2}$); in panel (b) line 3 pertains to Eq. (14) and line 4 to Eq. (15) with $\Gamma = -1$ ($q = -1.46 \times 10^{-2}$). Dashed lines in panels (a) and (b) represent the dispersionless dependence $\omega = C\,k$.

Plots of group velocities are shown in Fig. 2 for $\mathrm{B} = \Gamma = 1$ and $\mathrm{B} = \Gamma = -1$. All other combinations of signs are not shown in the figure to avoid a mess.

Critical points of group velocities (16) (maxima, minima, or inflection points depending on signs of parameters B and Γ) occur at $k = k_c$ and $k = k_{cO}$, respectively, where

$$k_c = \sqrt{\frac{1}{C}\left[\sqrt{\frac{\Gamma}{3\mathrm{B}}\left(1 + \frac{\mathrm{B}\Gamma}{3C}\right)}\right] - \frac{\Gamma}{C}}; \quad k_{cO} = \sqrt{\frac{1}{C}\sqrt{\frac{\Gamma}{3\mathrm{B}}}} \qquad (17)$$

(notice that the critical points are real if $\Gamma/\mathrm{B} > 0$).

III. THE NONLINEAR SHRODINGER EQUATION AND MODULATION INSTABILITY

In this section, we analyse the stability of quasi-monochromatic wave-trains of small amplitude $\psi \ll 1$ within the framework of NLSE written for the dimensionless variable $\zeta = \partial w/\partial \xi$ in the form

$$i\frac{\partial \psi}{\partial \tau} + p\frac{\partial^2 \psi}{\partial \xi^2} + q|\psi|^2\psi = 0, \qquad (18)$$

where $p(k)$ and $q(k)$ are the dispersion and nonlinear coefficients, respectively; they depend on the central wavenumber of the carrier wave and coefficients of the governing equation. The details of the derivation of this equation from Eq. (12) are presented in the Appendix. Notice that the dispersion coefficient $p(k)$ can be readily obtained directly from the dispersion relations (14), $p(k) = (1/2)(\partial^2\omega/\partial k^2)$ (see, e.g., Karpman (1973); Ablowitz and Segur (1981); and Ostrovsky and Potapov (1999)). When the NLSE (18) is derived from the Gardner–Ostrovsky equation (13), then the corresponding dispersion coefficient $p_O(k)$ can be obtained from the dispersion relations (15). Thus, we have

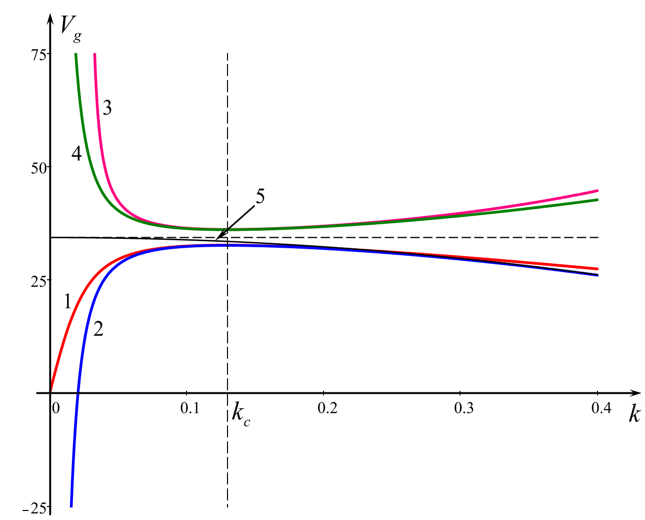

FIG. 2. Group velocities as per Eqs. (16) for different coefficients B and Γ. Line 1 pertains to V_g with $\mathrm{B} = \Gamma = 1$; line 2 pertains to V_{gO} with the same parameters B and Γ; line 3 pertains to V_g with $\mathrm{B} = \Gamma = -1$; line 4 pertains to V_{gO} with the same parameters B and Γ; and line 5 pertains to V_g with $\mathrm{B} = 1$ and $\Gamma = 0$. Dashed horizontal line illustrates the limiting dispersionless case $V_g = C$, and dashed vertical line shows the position of maximum in line 2.

$$p(k) = \frac{C^2 - \mathrm{B}\Gamma}{2}\frac{\Gamma - 2\mathrm{B}\Gamma k^2 - 3\mathrm{B}C^2k^4}{(1+\mathrm{B}k^2)^{5/2}(C^2k^2+\Gamma)^{3/2}}, \tag{19a}$$

$$p_O(k) = -3bk + \frac{r}{k^3} \equiv \frac{C}{2k}\left(-3\mathrm{B}k^2 + \frac{\Gamma}{C^2k^2}\right). \tag{19b}$$

The expression for the coefficient $p_O(k)$ can be obtained from Eq. (19a) as the first terms of Taylor series on small parameters $\mathrm{B}k^2 \ll 1$ and $\Gamma/C^2k^2 \ll 1$. Figure 3 shows the comparison of coefficients $p(k)$ and $p_O(k)$ with each other and with the coefficient $p_K(k) = -3bk \equiv -3\mathrm{B}Ck/2$ that follows from NLSE derived from the KdV equation.

In frame (a) lines 1 and 2 pertain to $p(k)$ and $p_O(k)$ as per Eqs. (19a) and (19b), correspondingly, with $\mathrm{B} = \Gamma = 1$; lines 3 and 4 pertain to the case when $\mathrm{B} = 1$ and $\Gamma = -1$; line 5 shows the coefficient $p_K(k)$. The insertion in frame (a) shows the same lines in the different scale. It is clearly seen that when k increases, lines 2 and 4 asymptotically approach line 5, whereas lines 1 and 3 become indistinguishable and both approach zero.

In frame (b), lines 1 and 2 pertain to $p(k)$ and $p_O(k)$, correspondingly, with $\mathrm{B} = -1$ and $\Gamma = 1$; lines 3 and 4 pertain to the case of $\mathrm{B} = \Gamma = -1$; line 5 shows the coefficient $p_K(k)$. The insertion in frame (b) shows the same lines in the different scale. Again when k increases, lines 2 and 4 asymptotically approach line 5, whereas lines 1 and 3 become indistinguishable and both of them infinitely increase with k.

As one can see, in some cases the dispersion coefficient vanishes at $k = k_{c1,2}$ for Eq. (14) and $k = k_O \approx k_{c1,2}$ for Eq. (15), respectively, where

$$k_{c1,2} = \sqrt{\frac{1}{C}\left[\sqrt{\frac{\Gamma}{3\mathrm{B}}\left(1+\frac{\mathrm{B}\Gamma}{3C}\right)} - \frac{\Gamma}{C}\right]}, \quad k_O = \sqrt{\frac{1}{C}\sqrt{\frac{\Gamma}{3\mathrm{B}}}}, \tag{20}$$

k_{c1} pertains to the case $\mathrm{B} = \Gamma = 1$, whereas k_{c2} pertains to the case $\mathrm{B} = \Gamma = -1$, and $k_{c1} < k_O < k_{c2}$. In such cases the NLSE degenerates and it should be augmented by the terms of the next-order of smallness; then the generalised NLSE can be derived for wavenumbers in the vicinity of such points (see, e.g., Obregon and Stepanyants (1998) and Grimshaw and Helfrich (2008; 2012)).

Let us analyse now the nonlinear coefficient $q(k)$. Its formal derivation from Eq. (12) yields

$$q(k) = \frac{-3\sqrt{C^2k^2+\Gamma}\left[\Gamma(\mathrm{A}^2 - \Phi) + C^2k^2\mathrm{A}^2 - \mathrm{B}\Phi k^2(5\Gamma + 4C^2k^2)\right]}{(1+\mathrm{B}k^2)^{3/2}[4\mathrm{B}C^2k^4 + \Gamma(1+5\mathrm{B}k^2)]}. \tag{21a}$$

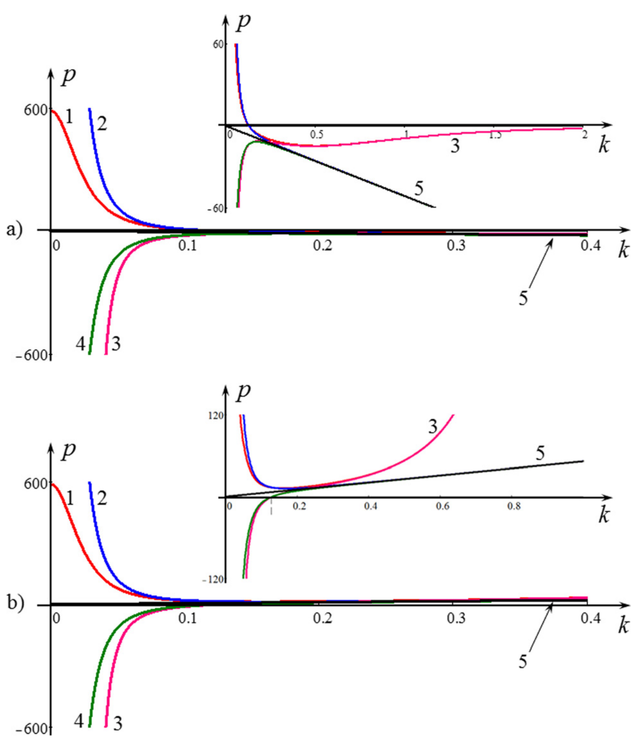

FIG. 3. The dispersion coefficient in the NLSE (18) as a function of wavenumber for different primitive equations.

Bearing in mind that $\Phi = \mathrm{A}^2$ (see after Eq. (12)), one can present Eq. (21a) as

$$q_c(k) = \frac{-3k^2\mathrm{A}^2\sqrt{C^2k^2+\Gamma}\,[C^2(1-4\mathrm{B}k^2) - 5\mathrm{B}\Gamma]}{(1+\mathrm{B}k^2)^{3/2}[4\mathrm{B}C^2k^4 + \Gamma(1+5\mathrm{B}k^2)]}. \tag{21b}$$

This expression can be compared with the case when the cubic nonlinearity in Eq. (12) is omitted ($\Phi = 0$)

$$q_q(k) = \frac{-3\mathrm{A}^2(C^2k^2+\Gamma)^{3/2}}{(1+\mathrm{B}k^2)^{3/2}[4\mathrm{B}C^2k^4 + \Gamma(1+5\mathrm{B}k^2)]}. \tag{21c}$$

As one can see, the coefficients (21b) and (21c) are different, in general, and contribution of the cubic nonlinear term in Eq. (12) into the nonlinear coefficient of NLSE is important. Moreover, in some cases the quadratic nonlinear coefficient can vanish (this occurs, in particular, for internal waves in two-layer fluid with equal layer thicknesses (see, e.g., Grimshaw *et al.* (1998)), then the only cubic nonlinear term contributes into the NLSE coefficient $q(k)$, which reduces to

$$q_{pc}(k) = \frac{3\Phi\sqrt{C^2k^2+\Gamma}[\Gamma + \mathrm{B}k^2(5\Gamma + 4C^2k^2)]}{(1+\mathrm{B}k^2)^{3/2}[4\mathrm{B}C^2k^4 + \Gamma(1+5\mathrm{B}k^2)]}. \tag{21d}$$

If we consider, however, the range of wavenumbers when the Ostrovsky equation is applicable, i.e., $\sqrt{|\Gamma|}/C \ll k$

$\ll 1/\sqrt{|\mathrm{B}|}$ (see above), then we obtain from Eqs. (21b) and (21c) the coefficients of NLSE up to the first-order terms on small parameters ($\mathrm{B}k^2 \sim \Gamma/C^2k^2 \ll 1$)

$$q_{GO}(k) \approx \frac{-3Ck\mathrm{A}^2}{4\mathrm{B}k^2 + \Gamma/C^2k^2}\left[1 + \frac{\Gamma}{C^2k^2}\left(\frac{3}{2} - \frac{\Phi}{\mathrm{A}^2}\right) - \frac{3}{2}\mathrm{B}k^2\left(1 + \frac{8}{3}\frac{\Phi}{\mathrm{A}^2}\right)\right], \tag{22a}$$

if the cubic terms in Eq. (12) are taken into consideration

$$q_O(k) \approx \frac{-3Ck\mathrm{A}^2}{4\mathrm{B}k^2 + \Gamma/C^2k^2}\left[1 + \frac{3}{2}\left(\frac{\Gamma}{C^2k^2} - \mathrm{B}k^2\right)\right], \tag{22b}$$

if the cubic terms in Eq. (12) are neglected ($\Phi = 0$), and

$$q_{pc}(k) \approx 3Ck\Phi\left[1 + \frac{\mathrm{B}\Gamma/C^2 + \Gamma^2/C^4k^4 - 12\mathrm{B}^2k^4}{2(\Gamma/C^2k^2 + 4\mathrm{B}k^2)}\right], \tag{22c}$$

if only the cubic terms in Eq. (12) are taken into consideration and the quadratic terms are ignored.

As one can see, if the quadratic terms do not vanish, then in the lowest order on small parameters the coefficients $q_{GO}(k)$ and $q_O(k)$ do not depend on the cubic terms in Eq. (13) and exactly coincide with the coefficient derived in Grimshaw and Helfrich (2008; 2012) and Whitfield and Johnson (2015a) directly from the Ostrovsky Eq. (13) without cubic nonlinear term ($a_1 = 0$)

$$q_O(k) = q_{GO}(k) = -\frac{2}{3}\frac{a^2k^3}{4bk^4 + r} \equiv \frac{-3Ck\mathrm{A}^2}{4\mathrm{B}k^2 + \Gamma/C^2k^2}. \tag{22d}$$

However, in the next order on small parameters, the corrections depend on the cubic terms in the original Eq. (12).

Notice that in the recent paper by Whitfield and Johnson (2015b) the NLSE was derived from the Gardner–Ostrovsky equation (15) which was formally considered without the restrictions on the range of its validity. The nonlinear coefficient in that paper formally follows from Eq. (22a) if we assume that $\mathrm{B}k^2 \sim \Gamma/C^2k^2 \ll 1$, but $\mathrm{B}k^2\Phi/\mathrm{A}^2 \sim \Gamma\Phi/(\mathrm{A}Ck)^2 \sim 1$.

Figure 4 illustrates the dependence of the nonlinear coefficient in NLSE on the wavenumber k for different models. The first big difference in the dependences of the coefficients $q(k)$ is seen when $\mathrm{B} = \Gamma = 1$. The coefficient $q_c(k)$ changes its sign at the point $k_0 = (C^2/\mathrm{B} - 5\Gamma)^{1/2}/2C$ (see line 1 in frame (a)), whereas neither $q_q(k)$ nor $q_O(k)$ changes their signs (see lines 2 and 3 in frame (a)). However, at very small $k \ll 1$ the dependences $q_c(k)$, $q_q(k)$, and $q_O(k)$ are close to each other.

In the case $\mathrm{B} = \Gamma = -1$, line 5 for $q_q(k)$ is qualitatively similar to line 4 for $q_c(k)$, but there is an obvious quantitative difference between these curves for relatively big values of $k > 0.2$ (both these lines go to infinity when $k \to 1/\sqrt{(-\mathrm{B})} = 1$). As in the first case, at small $k \ll 1$ the dependences $q_c(k)$, $q_q(k)$, and $q_O(k)$ are close to each other.

When the coefficients B and Γ are of opposite sign, then the nonlinear coefficients change their signs at the point of singularity $k = K_{1,2}$ and $k = K_O$, where

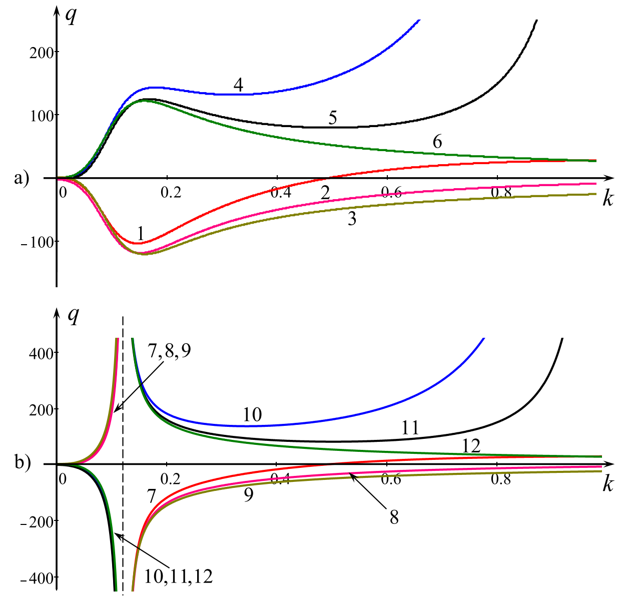

FIG. 4. The nonlinear coefficient in the NLS equation (18) as a function of a wavenumber for different basic equations. In frame (a) lines 1, 2, and 3 are plotted for $\mathrm{B} = \Gamma = 1$, lines 4, 5, and 6 are plotted for $\mathrm{B} = \Gamma = -1$; in frame (b) lines 7, 8, and 9 are plotted for $\mathrm{B} = -\Gamma = 1$, lines 10, 11, and 12 are plotted for $\mathrm{B} = -\Gamma = -1$.

$$K_{1,2} = \sqrt{\frac{1}{2C}\sqrt{-\frac{\Gamma}{\mathrm{B}}\left(1 - \frac{25\mathrm{B}\Gamma}{16C^2}\right)} - \frac{5\Gamma}{8C^2}},$$
$$K_O = \sqrt{\frac{1}{2C}\sqrt{-\frac{\Gamma}{\mathrm{B}}}}, \tag{23}$$

K_1 pertains to the case $\mathrm{B} = -1$, $\Gamma = 1$, whereas K_2 pertains to the case $\mathrm{B} = 1$, $\Gamma = -1$ and K_O pertains to Eq. (22d). These three singular points are very close to each other, although they are ordered in the following way $K_1 < K_O < K_2$.

For the wavenumbers greater than the singular point all corresponding lines, except line 7, do not change their signs any more, but line 7 changes its sign at $k_0 = \frac{1}{2C}\sqrt{C^2/\mathrm{B} - 5\Gamma}$. The points where the coefficients $p(k)$ and $q(k)$ change their signs are very important from the point of view of wave-trains stability with respect to self-modulation; this will be discussed below.

In the case when the NLSE is derived directly from the KdV equation the nonlinear coefficient $q_{KdV}(k) = 9\mathrm{A}^2C/\mathrm{B}k$ depends on the sign of coefficient B and is either positive, when $\mathrm{B} > 0$, or negative, when $\mathrm{B} < 0$. Notice that the expression for this coefficient does not follow from Eq. (22d), if one formally puts $\Gamma = 0$. As explained above, the nonlinear coefficient in the NLSE derived from the KdV equation contains contributions from both the second harmonic and zero harmonic terms. However, as shown in the Appendix, when $\Gamma \neq 0$ the zero harmonic contributes only into the higher-order terms (see also Grimshaw and Helfrich (2008; 2012)).

As well known (see, e.g., Karpman (1973); Ablowitz and Segur (1981); and Ostrovsky and Potapov (1999)), the stability of quasi-monochromatic wave-trains with respect to small modulations is determined in the NLSE (18) by the relative sign of nonlinear and dispersive coefficients. According to the well-known Lighthill criterion (Lighthill, 1965; Ostrovsky and Potapov, 1999; and Zakharov and Ostrovsky, 2009), the stability occurs when $p(k)q(k) < 0$, otherwise, when $p(k)q(k) > 0$, the wave-trains are unstable. In the latter case the "bright" envelope solitons can exist, whereas in the former case only "dark" solitons can exist on the background of a sinusoidal wave (see the references cited above).

To determine the range of wave-train stability/instability for the particular choice of parameters B and Γ and present it in the vivid form, let us define function $F(k) = S\,\mathrm{sign}[p(k)q(k)]$, where $\mathrm{sign}(x) = 1$ if $x > 0$ and $\mathrm{sign}(x) = -1$ if $x < 0$, S is the "amplitude" of function $\mathrm{sign}(x)$ which will help us to distinguish between different lines in Fig. 5. This illustrates the behaviour of function $F(k)$ for the different sets of parameters B and Γ.

Lines 1 in all frames pertain to the NLSE derived from Eq. (12) with the cubic nonlinear term, whereas lines 2 pertain to the NLSE derived from Eq. (12) without cubic nonlinear term ($\Phi = 0$), and lines 3 pertain to the NLSE derived from the Ostrovsky equation (13) without cubic nonlinear term ($a_1 = 0$).

As one can see from this figure, in the case shown in frame (a), the models based on Eq. (12) both with and without cubic nonlinear terms predict the stability of wave trains at $k < k_p \equiv \sqrt{\sqrt{(3C^2 + \mathrm{B}\Gamma)\Gamma/\mathrm{B}} - \Gamma}/C\sqrt{3}$ and instability at $k > k_p$. In the meantime, the model based on Ostrovsky equation (13) without cubic nonlinear term predicts the boundary between the stability and instability at $k = k_{O1} \equiv \sqrt[4]{\Gamma/3\mathrm{B}C^2}$. Note that $k_p \approx k_{O1}(1 - \sqrt{\mathrm{B}\Gamma/12C^2})$, if $\mathrm{B}\Gamma \ll C^2$. Then, the more accurate model (12) with the cubic nonlinear term predicts one more boundary between the stability and instability at $k = k_{01} \equiv \sqrt{C^2/\mathrm{B} - 5\Gamma}/2C$, $k_{01} > k_p$ (see line 1 in frame (a), whereas two other models do not predict stability at high wavenumbers. Note that the case shown in frame (a) pertains to gravity water waves in a rotating fluid.

In the case shown in frame (b), the model (12) with the cubic nonlinearity predicts modulation instability at very small $k < k_{02} \equiv \sqrt{-\Gamma}/C$, whereas other models do not predict such instability. Then, there is the range of instability at $k > k_p$ predicted by the models based on Eq. (12) both with and without cubic nonlinear term, as well as by the model based on Ostrovsky equation (13) without cubic nonlinear term at $k > k_{O1}$.

In the case shown in frame (c), the model (12) with the cubic nonlinearity predicts two regions of modulation instability, $0 < k < k_{02}$ and $k_p < k < k_{01}$, and two regions of stability, $k_{02} < k < k_p$ and $k > k_{01}$ (see line 1 in frame (c). In the meantime, other two models predict only one boundary between the stability, $k < k_p$ ($k < k_{O2}$), and instability, $k > k_p$ ($k > k_{O2}$), regions (here $k_{O1} \equiv \sqrt[4]{-\Gamma/4\mathrm{B}C^2}$).

And, at last, in the case shown in frame (d), the models with and without cubic nonlinearity predict the same regions of wave-train stability $k < k_p$ ($k < k_{O2}$), and instability, $k > k_p$ ($k > k_{O2}$).

In the case when the NLSE is derived from the KdV equation the product $p(k)q(k) = -27(\mathrm{A}C)^2/2 < 0$, so that the wave-trains are stable against self-modulations for all k within the range of validity of KdV equation.

IV. DISCUSSION AND CONCLUSION

Thus, in this paper we have shown that the NLSE derived from the uniderctional Gardner–Ostrovsky equation (13) agrees well with the NLSE derived from the more general Shrira equation (12) within the range of validity of the Gardner–Ostrovsky equation. Moreover, within this range the coefficients of NLSE can be correctly obtained even without the cubic nonlinear terms in the governing equation (12). Beyond the range of validity of the Gardner–Ostrovsky equation the existence of ranges of modulation stability/instability is rather non-trivial and depends on the signs of small-scale and large-scale dispersions characterised by dimensionless coefficients B and Γ. In this paper, we have considered all possible signs of nonlinear and dispersive coefficient and presented the complete analysis of the problem for media with double dispersion. Therefore, our analysis is applicable not only to water waves in rotating or non-rotating fluids (e.g., various oceanic waves) but also to plasma waves, waves in solids, optical fibres, and others.

In application to surface and internal waves in a rotating ocean (see Fig. 5, frame (a)) our analysis shows

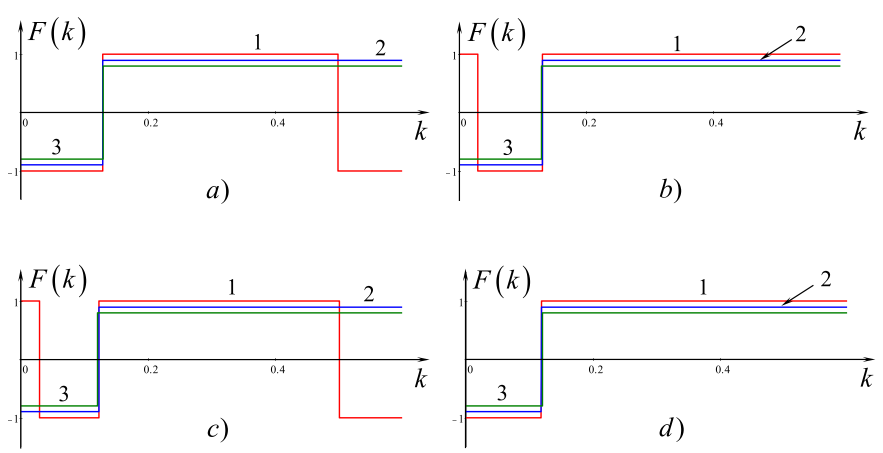

FIG. 5. Ranges of stability (when $F(k) < 0$) and instability (when $F(k) > 0$) of wave-trains against self-modulation. Frame (a) pertains to $\mathrm{B} = \Gamma = 1$; frame (b) pertains to $\mathrm{B} = \Gamma = -1$; frame (c) pertains to $\mathrm{B} = -\Gamma = 1$; and frame (d) pertains to $\mathrm{B} = -\Gamma = -1$. For the meaning of lines see the text.

that wave-trains are stable against self-modulation at small wavenumbers less than some critical value $k < k_p \equiv \sqrt{\sqrt{(3C^2 + \mathrm{B}\Gamma)\Gamma/\mathrm{B}} - \Gamma}/C\sqrt{3} \qquad (k < k_{O1} \equiv \sqrt[4]{\Gamma/3\mathrm{B}C^2})$ which depends on the dispersion coefficients of the governing equations. This is in agreement with the results obtained in Grimshaw and Helfrich (2008; 2012) and Whitfield and Johnson (2015a), where the analysis was performed without cubic nonlinear terms in the limited range of wavenumbers as per Eq. (6).

A similar analysis can be performed in the two-dimensional case when all dependent variables are functions of spatial coordinates x and y, so that Shrira's set of equations (8), (9) cannot be reduced to one, Equation (10). In contrast to the non-rotating fluid, the 2D NLSE appears in this case alone without a complementary equation for the mean-flow component which is suppressed in the first order by the influence of Coriolis-type dispersion. Therefore, instead of the usual Davey–Stewartson set of equation for a non-rotating fluid of finite depth (Ablowitz and Segur, 1981), we obtain just a single NLSE. The dispersion coefficients $p_x = (1/2)(\partial^2\omega/\partial k_x{}^2)$ and $p_y = (1/2)(\partial^2\omega/\partial k_y{}^2)$ of this equation can be easily derived from the dispersion relation (18), where now $k^2 = k_x{}^2 + k_y{}^2$. The new feature of the 2D NLSE is the possibility of the collapse phenomenon when self-modulation and self-focussing instabilities occur simultaneously (Zakharov and Kuznetsov, 2012). This interesting problem is currently under investigation, and results obtained will be published elsewhere.

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APPENDIX: DERIVATION OF NLS EQUATION

The nonlinear Schrödinger equation describing the evolution of wave-trains with the central wavenumber k of a carrier wave can be derived from Eq. (12) following the standard approach (see, e.g., Ablowitz and Haut (2009)), therefore details are reproduced here only briefly. Introduce the fast phase variable $\theta = \omega\,\tau - k\,\xi$, the slow spatial $X = \varepsilon\,\xi$ and temporal $T = \varepsilon\,\tau$ variables, as well as the "super-slow" temporal variable $t = \varepsilon T = \varepsilon^2\tau$, where $\varepsilon \ll 1$ is a small parameter, and present a solution of Eq. (12) in the form of the series on small parameter ε

$$w(\theta,X,T,t) = \varepsilon w_1(\theta,X,T,t) + \varepsilon^2 w_2(\theta,X,T,t) + \varepsilon^3 w_3(\theta,X,T,t) + \ldots \qquad \text{(A1)}$$

Substitution of this series into Eq. (12) gives

$$\begin{aligned}
&\varepsilon\left[(\omega^2 - C^2k^2)\frac{\partial^2 w_1}{\partial\theta^2} - \mathrm{B}k^2\omega^2\frac{\partial^4 w_1}{\partial\theta^4} + \Gamma w_1\right] + \varepsilon^2\left[(\omega^2 - C^2k^2)\frac{\partial^2 w_2}{\partial\theta^2} - \mathrm{B}k^2\omega^2\frac{\partial^4 w_2}{\partial\theta^4} + \Gamma w_2\right.\\
&\quad\left. + 3\alpha\omega^2k\frac{\partial w_1}{\partial\theta}\frac{\partial^2 w_1}{\partial\theta^2} - 2\omega\frac{\partial^2 w_1}{\partial\theta\partial T} - 2C^2k\frac{\partial^2 w_1}{\partial\theta\partial X} + 2\mathrm{B}\omega k\frac{\partial^3}{\partial\theta^3}\left(\omega\frac{\partial w_1}{\partial X} - k\frac{\partial w_1}{\partial T}\right)\right]\\
&\quad + \varepsilon^3\left\{\left[(\omega^2 - C^2k^2)\frac{\partial^2 w_3}{\partial\theta^2} - \mathrm{B}k^2\omega^2\frac{\partial^4 w_3}{\partial\theta^4} + \Gamma w_3\right] + \frac{\partial^2 w_1}{\partial T^2} - C^2\frac{\partial^2 w_1}{\partial X^2} + 2C^2k\frac{\partial^2 w_2}{\partial\theta\,\partial X}\right.\\
&\quad + \mathrm{B}\frac{\partial^2}{\partial\theta^2}\left[2\omega k\frac{\partial}{\partial\theta}\left(\omega\frac{\partial w_2}{\partial X} - k\frac{\partial w_2}{\partial T}\right) + 4\omega k\frac{\partial^2 w_1}{\partial T\partial X} - \omega^2\frac{\partial^2 w_1}{\partial X^2} - k^2\frac{\partial^2 w_1}{\partial T^2} - 2\omega k^2\frac{\partial^2 w_1}{\partial\theta\partial t}\right]\\
&\quad + \omega\mathrm{A}\left[3\omega k\left(\frac{\partial w_1}{\partial\theta}\frac{\partial^2 w_2}{\partial\theta^2} + \frac{\partial w_2}{\partial\theta}\frac{\partial^2 w_1}{\partial\theta^2}\right) + 2k\left(\frac{\partial w_1}{\partial T}\frac{\partial^2 w_1}{\partial\theta^2} + 2\frac{\partial w_1}{\partial\theta}\frac{\partial^2 w_1}{\partial\theta\,\partial T}\right) - \omega\left(\frac{\partial w_1}{\partial X}\frac{\partial^2 w_1}{\partial\theta^2} + 2\frac{\partial w_1}{\partial\theta}\frac{\partial^2 w_1}{\partial\theta\,\partial X}\right)\right]\\
&\quad\left. + 2\omega\frac{\partial}{\partial\theta}\left(\frac{\partial w_2}{\partial T} + \frac{\partial w_1}{\partial t}\right) + 6\Phi\omega^2k^2\frac{\partial w_1}{\partial\theta}\frac{\partial^2 w_1}{\partial\theta^2}\right\} + o(\varepsilon^3) = 0.
\end{aligned} \qquad \text{(A2)}$$

Let us consider a solution to Eq. (A2) in the form of quasi-monochromatic wave

$$w_1(\theta,X,T,t) = W_1(X,T,t)e^{i\theta} + W_1^*(X,T,t)e^{-i\theta}, \qquad \text{(A3)}$$

$$w_2(\theta,X,T,t) = W_2(X,T,t)e^{2i\theta} + W_2^*(X,T,t)e^{-2i\theta} + W_0(X,T,t) + W_0^*(X,T,t), \qquad \text{(A4)}$$

where star stands for complex conjugate.

Substitute now solutions (A3) and (A4) into Eq. (A2) and collect the terms proportional to $e^{i\theta}$. In the leading order with respect to ε, we obtain the dispersion relation (14). Collecting then the terms proportional to ε^2, we obtain

$$2i\left[-\omega(\mathrm{B}k^2+1)\frac{\partial W_1}{\partial T} + k(\mathrm{B}\omega^2 - C^2)\frac{\partial W_1}{\partial X}\right]e^{i\theta} + [(4C^2k^2 - 4\omega^2 - 16\mathrm{B}\omega^2k^2 + \Gamma)W_2 + 3i\mathrm{A}\omega^2k|W_1|^2]e^{2i\theta} + \Gamma W_0 + \text{c.c.}, \qquad \text{(A5)}$$

where c.c. stands for complex conjugate terms.

Equating to zero the coefficient of $e^{i\theta}$, we obtain the simple wave equation

$$\frac{\partial W_1}{\partial T} + V_g \frac{\partial W_1}{\partial X} = 0, \tag{A6}$$

where V_g is the group speed as per Eq. (16).

Equating then to zero the coefficient of $e^{2i\theta}$, we obtain a relationship between the amplitudes of the first and second harmonics

$$W_2 = \frac{3i\mathrm{A}\omega^2 k|W_1|^2}{16\mathrm{B}\omega^2 k^2 - \Gamma + 4\omega^2 - 4C^2k^2}. \tag{A7}$$

In the same approximation, the term W_0, which is independent of exponent (the mean flow term) vanishes, $W_0 = 0$; this is a specific feature of wave systems with the large-scale dispersion (Shrira, 1981; Obregon and Stepanyants, 1998; Grimshaw and Helfrich, 2008; 2012; and Whitfield and Johnson, 2015a; 2015b).

In the next order on the parameter ε the coefficient of $e^{i\theta}$ gives

$$2i\omega(1 + \mathrm{B}k^2)\frac{\partial W_1}{\partial t} + \left(C^2 + \mathrm{B}V_g\omega k + \mathrm{B}\omega^2\right)\frac{\partial^2 W_1}{\partial X^2} + 6k\omega^2\left(i\mathrm{A}W_2 + \Phi k|W_1|^2\right)W_1 = 0. \tag{A8}$$

Substituting here W_2 from Eq. (A7), we obtain the NLSE in the form

$$i\frac{\partial W_1}{\partial t} + p(k)\frac{\partial^2 W_1}{\partial X^2} + Q(k)|W_1|^2 W_1 = 0, \tag{A9}$$

where $p(k) = \frac{C^2 + \mathrm{B}V_g\omega k + \mathrm{B}\omega^2}{2\omega(1+\mathrm{B}k^2)}$ and $Q(k) = -3k^2\omega \frac{3\mathrm{A}^2\omega^2 - \Phi[4k^2(4\mathrm{B}\omega^2 - C^2) + 4\omega^2 - \Gamma]}{(1+\mathrm{B}k^2)(16\mathrm{B}\omega^2k^2 - \Gamma + 4\omega^2 - 4C^2k^2)}$. Coming back to the original variables τ and ξ and using the relationship between the variables w and $\zeta = \partial w/\partial\xi$, which in the first approximation for quasi-monochromatic wave reads $\zeta = ikw$, we finally obtain Eq. (18) with the coefficients $p(k)$ as per Eq. (19a) and $q(k) = Q(k)/k^2$ as per Eq. (21a).

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