A performance index for topology and shape optimization of plate bending problems with displacement constraints

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Abstract This paper presents a performance index for topology and shape optimization of plate bending problems subject to displacement constraints. The performance index is developed based on the scaling design approach. This performance index is used in the Performance-Based Optimization (PBO) method for plates in bending to keep track of the performance history when inefficient material is gradually removed from the design and to identify optimal topologies and shapes from the optimization process. Several examples are provided to illustrate the effectiveness of the proposed performance index for topology and shape optimization of bending plates with single and multiple displacement constraints under various loading conditions. The topology optimization and shape optimization are undertaken for the same plate in bending, and the results are evaluated by using the performance index. The proposed performance index is also employed to compare the efficiency of topologies and shapes produced by different optimization methods. It is demonstrated that the performance index developed is an effective indicator of material efficiency for bending plates. From the manufacturing and efficient point of view, the shape optimization technique is recommended for the optimization of plates in bending.

Keywords: efficiency, performance index, plate, ranking, topology and shape optimization

1 Introduction

The topology and shape optimization of continuum structures has attracted considerable attention in recent years. In the topology optimization of continuum structures, holes in the interior of the design are allowed to be created. On the other hand, in the shape optimization of continuum structures, changes can only be made to the boundaries of the design. A survey on structural shape optimization has been given by Haftka and Grandhi (1986), in which the boundary variation method has been extensively used. In shape optimization using the boundary variation method, the finite element mesh is changing during the optimization process and remeshing the model is often required. To avoid these, Bendsøe and Kikuchi (1988) have proposed a Homogenization method for the topology optimization of continuum structures using a fixed initial design domain. In this method, the material density is treated as design variables and the objective is to minimize the mean compliance under volume constraints. The Homogenization method has been used to find the optimal topologies and shapes of plane stressed problems by Suzuki and Kikuchi (1991) and of plates in bending by Tenek and Hagiwara (1993). The Solid Isotropic Microstructure with Penalty (SIMP) method (Zhou and Rozvany 1991; Rozvany *et al.* 1992) for intermediate densities is efficient in producing solid-empty type topologies in generalized shape optimization.

Recently, a simple approach to the topology and shape optimization namely the Evolutionary Structural Optimization (ESO) method has been developed by Xie and Steven (1993). The ESO method is based on the simple concept of systematically removing inefficient material from the structure after each finite element analysis, so that the quality of the resulting design is gradually improved. The element removal criteria is established by the sensitivity analysis. This method has been extended to frequency optimization of continuum structures by Xie and

Steven (1994, 1996, 1997). The frequency of a structure can be shifted towards a desired direction by removing part of the material from the design based on the sensitivity analysis. Chu *et al.* (1996) has applied the ESO method to the topology and shape optimization of continuum structures with stiffness and displacement constraints. An extension of the ESO method to plate buckling resistance optimization has also been given by Manickarajah *et al.* (1998). Although Chu *et al.* has considered the objective and displacement constraints in the ESO method, it is difficult to identify the optimal topology and shape from the evolutionary path due to the lack of a performance index for evaluating the material efficiency.

Structural optimization is an effective tool of improving the quality of the design, but the quality of the result is limited by the methods used. It has been found that using different optimization methods usually results in different topologies and shapes even for the same problem considered. Extensive research has been devoted to the development of structural optimization methods in the past few decades. Unfortunately, little work has been undertaken to evaluate the efficiency of the results and optimization methods, except that Burgess (1998a, 1998b) has extended the method outlined by Ashby (1992) to derive a set of performance indices for optimizing trusses and beams and for measuring the efficiency of structural layouts produced by different optimization methods. However, these performance indices are only valid for simple discrete structures and cannot be used to evaluate the quality of the topologies and shapes of continuum structures.

Querin (1997) has presented a performance index, which does not consider any type of constraint and cannot objectively evaluate the efficiency of structural layouts for any type of structure. Xie and Steven (1997) have evaluated the quality of material layouts by comparing the volume of a new design with that of the optimized initial design domain, which is obtained by reducing its thickness to satisfy the displacement limit. The indicator of material efficiency proposed by Zhao *et al.* (1998) does not take account of any stress and displacement constraint. Hence, it is only valid for plane stress structures under a single point load and not applicable to plates in bending. Chu *et al.* (1998) has employed the objective weight to find the optimal thickness distribution of a bending plate with displacement constraints. The objective weight is obtained by scaling the design with respect to the displacement limit. Although no performance index has been proposed in their paper, it provides a true understanding of the nature of the optimal material layouts.

Performance indices have been developed by Liang *et al.* (1999, 2000) for evaluating the efficiency of topologies and shapes for plane stress continuum structures with stress and displacement constraints. This paper presents a performance index for determining the optimal topologies and shapes of bending plates with displacement constraints from the optimization process and for comparing the efficiency of structural topologies and shapes obtained by using different optimization methods. The formulation of the performance index is given in section 2 and the outline of the ESO method for bending plates with displacement constraints is presented in section 3. In section 4, several examples are provided to demonstrate the capability of the proposed performance index for topology and shape optimization of plates in bending.

2 Formulation of performance index

2.1 The scaling design approach

The scaling design approach can be used to obtain the feasible constrained design after each iteration in an iterative optimization process (Kirch 1982). When the stiffness matrix of the structure is a linear function of the design variables, the design can be scaled to keep the most active stress or displacement constraint to the prescribed limit (Liang *et al.* 1999, 2000). By using this method, the history of the weight reduction of the structure is easily monitored. For a plate in bending, the stiffness matrix of the plate is not a linear function of the design variable such as the thickness of the plate. The scaling factor needs to be derived if this procedure is applied to plates in bending. By scaling the design, the scaled design variable is represented by

$$t_e^s = \varphi t_e \tag{1}$$

in which t_e^s is the scaled thickness of the *e*th element, φ is the scaling factor which is the same for all elements and t_e is the actual thickness of the *e*th element. The material elastic constants of an element are written in matrix form as

$$[\mathbf{D}_e] = \frac{t_e^3}{12}\begin{bmatrix} \dfrac{E}{1-v^2} & \dfrac{Ev}{1-v^2} & 0 \\ \dfrac{Ev}{1-v^2} & \dfrac{E}{1-v^2} & 0 \\ 0 & 0 & \dfrac{E}{2(1+v)} \end{bmatrix} \tag{2}$$

(2) can be denoted as

$$[\mathbf{D}_e] = t_e^3[\mathbf{C}] \tag{3}$$

where E is the elastic modulus and v is the Poisson's ratio. The material elastic constants of an element can be expressed in term of the scaled design variable as

$$[\mathbf{D}_e] = \left(\frac{t_e^s}{\varphi}\right)^3 [\mathbf{C}] = \frac{1}{\varphi^3}[\mathbf{D}_e^s] \tag{4}$$

in which $[\mathbf{D}_e^s]$ is the scaled material elastic constant matrix of an element. The equilibrium equation for the plate can be expressed in the finite element analysis as

$$\frac{1}{\varphi^3}[\mathbf{K}^s]\{\mathbf{u}\} = \{\mathbf{P}\} \tag{5}$$

where $[\mathbf{K}^s]$ is the stiffness matrix of the scaled plate, which is calculated by using the scaled design variable t_e^s, $\{\mathbf{u}\}$ is the actual nodal displacement vector and $\{\mathbf{P}\}$ is the nodal load vector. Using the scaled design variables, the equilibrium equation for the scaled design is denoted as

$$[\mathbf{K}^s]\{\mathbf{u}^s\} = \{\mathbf{P}\} \tag{6}$$

From (5) and (6), the scaled displacement vector can be obtained as

$$\{\mathbf{u}^s\} = \frac{1}{\varphi^3}\{\mathbf{u}\} \tag{7}$$

It can be seen from (1) and (7) that when the thickness of the plate is reduced by a factor φ, the deflections will increase with a factor of $1/\varphi^3$. In order to satisfy the displacement constraint, the actual design needs to be scaled by

$$\varphi = \left(\frac{|u_j|}{u_j^*} \right)^{1/3} \tag{8}$$

where $|u_j|$ is the magnitude of the jth displacement component in the current design and u_j^* is the prescribed displacement limit of the jth displacement.

2.2 Performance index

The topology and shape optimization of bending plates subject to displacement constraints is to seek the optimal material layouts in the plates. The material layouts in a structure is related to the type of constraint imposed on the structure. The performance index being proposed should be a dimensionless number that can measure the efficiency of material layouts in the plate. It should also reflect the objective of minimizing the weight and displacement constraints. The scaling design approach is used herein to derive such a performance index.

The topology and shape optimization of a bending plate with displacement constraints can be expressed as

$$minimize\ W = \sum_{e=1}^{n} w_e(t_e) \tag{9}$$

$$subject\ to\ |u_j| \le u_j^* \tag{10}$$

where w_e is the actual weight of the eth element. For plates in bending, the stiffness matrix of the plate is the cubic root of the thickness of the plate. To obtain the best topology of a bending plate that has the minimum weight, the design is scaled at each iteration in the optimization process so that the constrained displacement always reaches the prescribed limit. By scaling the initial design, the scaled weight of the initial design domain can be expressed by

$$W_0^s = \left(\frac{|u_{0j}|}{u_j^*} \right)^{1/3} W_0 \tag{11}$$

where W_0 is the actual weight of the initial design domain and $|u_{0j}|$ is the magnitude of the jth nodal displacement in the initial design under the applied loads. In a same manner, by scaling the current design, the scaled weight of the current design at the ith iteration can be written as

$$W_i^s = \left(\frac{|u_{ij}|}{u_j^*} \right)^{1/3} W_i \tag{12}$$

where W_i is the actual weight of the current design at the ith iteration and $|u_{ij}|$ is the magnitude of the jth nodal displacement in the current design at the ith iteration under applied loads.

The performance index, which measures the efficiency of material layouts of a bending plate at the ith iteration, is defined by

$$PI = \frac{W_0^s}{W_i^s} = \frac{\left(\frac{|u_{0j}|}{u_j^*}\right)^{1/3} W_0}{\left(\frac{|u_{ij}|}{u_j^*}\right)^{1/3} W_i} = \left(\frac{|u_{0j}|}{|u_{ij}|}\right)^{1/3} \frac{W_0}{W_i} \tag{13}$$

If the material density is uniformly distributed within the plate, the performance index can be expressed using the volumes of the plate as

$$PI = \left(\frac{|u_{0j}|}{|u_{ij}|}\right)^{1/3} \frac{V_0}{V_i} \tag{14}$$

in which V_0 is the volume of the initial design domain and V_i is the volume of the current design at the *i*th iteration.

It can be seen from (14) that the performance index is a dimensionless number which determines the material efficiency. The performance index is reversely proportional to the volume of the current design and is evaluated by the constrained displacements and the volumes at each iteration. Hence, minimizing the weight of a bending plate subject to displacement constraints can be achieved by maximizing the performance index in an optimization process. The displacement limit u_j^* is eliminated from (14), which indicates that the optimal topology for the minimum weight design of a bending plate is independent of the magnitude of the prescribed displacement limits. The optimal topology that corresponds to the maximum value of the performance index can be identified from the performance index history. It should be noted that the performance index is not proposed for a particular structural optimization method. Therefore, it can be incorporated in any structural optimization method to monitor the material efficiency and to determine the optimal topologies and shape of bending plates with displacement constraints. It can also be used to compare the quality of topologies and shapes for plates in bending optimized by different methods.

3 Performance-based optimization

The Performance Based Optimization (PBO) procedure for structures subject to displacement constraints presented by Liang *et al.* (2000) is based on the consideration that the quality of the design can be improved by gradually removing inefficient material from the structure. Which element should be removed from the design is determined by the sensitivity number, which is calculated for each element using the results of the finite element analysis at each iteration. Elements with the lowest sensitivity numbers have little contribution to the stiffness of the structure and can be removed from the structure. The sensitivity number for the *e*th element within the structure under a single displacement constraint is defined by

$$v_e = \left|\{\mathbf{u}_{ej}\}[\mathbf{k}_e]\{\mathbf{u}_e\}\right| \tag{15}$$

where $\{\mathbf{u}_{ej}\}$ is the nodal displacement vector of the *e*th element under the unit load corresponding to the *j*th displacement component, $[\mathbf{k}_e]$ is the stiffness matrix of the *e*th element and $\{\mathbf{u}_e\}$ is the nodal displacement vector of the *e*th element under the applied loads. For structures under multiple displacement constraints, the sensitivity number for the *e*th element is determined by

$$v_e = \sum_{j=1}^{m} \lambda_j \left| \{\mathbf{u}_{ej}\}^T [\mathbf{k}_e] \{\mathbf{u}_e\} \right| \tag{16}$$

where the weighting parameter λ_j is chosen as $\left|u_j\right| / u_j^*$ and m is the total number of constraints.

In order to obtain a sound optimal result, the optimization process must be evolutionary. This means that only a small number of elements that have the lowest sensitivity numbers are eliminated from the design at each iteration. The Element Removal Ratio (*ERR*) is defined as the ratio of the number of elements to be removed to the total number of elements in the initial design domain. For plates in bending under a symmetrical geometry, loading and boundary condition about the two in-plane axises, the extra codes have been added to the PBO algorithm to maintain the symmetry of the resulting topology and shape. Under the added scheme, elements having the same sensitivity numbers as the removed elements are deleted from the structure at each iteration.

The performance index developed in this paper can be used in the above PBO method to monitor the performance history of bending plates with displacement constraints when elements of having the lowest sensitivity numbers are gradually deleted from the design. The performance index for each iteration can be calculated using (14) from the results of the finite element analysis by simply recording the jth constrained nodal displacement and the volume of the current design at each iteration. The performance index history is then fully kept track, from which the optimal topology and shape are easily identified. It is noted that the weight of an optimal design is affected by the magnitude of the displacement limits, but it does not affect the optimal topology and shape. One may obtain the optimal topology and shape of a bending plate by using the *PI* formula together with any structural optimization method regardless the magnitude of the displacement limits, and then size the obtained optimal shape to satisfy the displacement constraints.

4 Numerical examples

The proposed performance index complemented the PBO method is used to solve the topology and shape optimization problems of bending plates with single and multiple displacement constraints in this section. Plates under concentrated, area and strip loading conditions are considered. The topology optimization and shape optimization are carried out for the same bending plate to investigate the effects of these two techniques on the optimal design. The efficiency of the structural topology and shape generated by different optimization methods is evaluated using the performance index.

4.1 Clamped plate under concentrated loading

The design domain for a clamped square plate under a concentrated load of 500 N applied to the centre of the plate is shown in Fig. 1. A single displacement constraint is imposed on the loaded point. The design domain is divided into a 50x50 mesh using four-node plate elements. The material properties are: the Young's modulus E=200 GPa, the Poisson's ratio v=0.3 and the thickness of the plate t=5 mm. Four elements around the loaded point are frozen so that this region is not removed during the optimization process. The Element Removal Ratio *ERR*=1% is adopted in the optimization process.

The performance index histories for the topology and shape optimization of the clamped plate are presented in Fig. 2. It can be seen that performance indices are gradually increased while inefficient materials are eliminated from the design in the optimization process. It is interesting of that performance indices for topology and shape optimization are almost identical up to iteration 59. However, the shape optimization provides a slightly higher performance index. The maximum performance indices are 2.09 and 2.13 by topology and shape optimization, respectively. After reaching the maximum performance, further element removal will destroy the structure as shown in Fig. 2. The evolutionary histories of topology and shape optimization for the plate are shown in Fig. 3 and Fig. 4 respectively. It is noted that cavities in the interior of the plate are created by the topology optimization whilst no holes in the interior of the plate are generated by the shape optimization. Based on the consideration of manufacture and structural efficiency, the shape optimization technique should be used in optimizing plates in bending. Table 1 gives a comparison of material volumes required for the initial design and shapes at different iterations shown in Fig. 3 for various displacement limits. It is seen from the table that the material efficiency of the optimal shape does not depend on the magnitude of the displacement limits.

4.2 Plate with multiple displacement constraints

This example illustrates the application of the proposed performance index to bending plates with multiple displacement constraints. Fig. 5 shows the design domain of a simply supported plate under multiple displacement constraints of the same limit imposed on points A, B and C, where three point loads of 10 kN are placed at these points respectively. The design domain is divided into a 60x30 mesh using four-node plate elements. Four elements around each loaded point are frozen. The Young's modulus E=28.6 GPa, Poisson's ratio v=0.2 and the thickness of the plate t=100 mm are assumed. The Element Removal Ratio ERR=1% is used in the shape optimization process.

The performance index history of the plate under multiple displacement constraints is shown in Fig. 6. The displacements at points A and C are equal due to symmetry. The performance index curves presented in Fig. 6 are obtained by using (14) with constrained displacements imposed on points A and B respectively. It is observed that the maximum performance indices calculated using the constrained displacements at points A and B are 2.78 and 2.09 respectively. The optimal shape which corresponds to the maximum performance index at point B is obtained at iteration 65 for this plate. After iteration 65, the central part around point B is cut off from the structure. The remaining structure can still support the loads applied to point A & C. Hence, it is seen from Fig. 6 that the performance index calculated using the displacements at point A is still increased after iteration 65 until the structure is completely destroyed. The weight of the final optimal design should be determined by the constrained critical displacement, which gives a lower performance index. Fig. 7 presents the evolutionary history of shape optimization for this problem.

5 Concluding remarks

A performance index for topology and shape optimization of bending plates with displacement constraints has been proposed in this paper using the scaling design approach. This performance index is determined by the constrained displacements and the volumes of the plate at each iteration. The Performance-Based Optimization (PBO) method for topology and shape optimization of bending plates has been described, and an extra scheme of maintaining the symmetry of the optimized topologies and shapes with a symmetrical initial condition has been added to the PBO algorithm. The proposed performance index complemented the PBO method has been employed to undertake the topology and shape optimization of bending plates with single and multiple displacement constraints under the concentrated, area and strip loading.

It is shown that the proposed performance index can be incorporated in any structural optimization method such as the PBO approach to monitor the evolutionary performance history, from which the optimal topology and shape of bending plates with displacement constraints can be easily identified. In addition, the quality of the topologies and shapes of bending plates, which are produced by different structural optimization methods, can be objectively evaluated by using the performance index. For a plate under multiple displacement constraints of the same limit, the weight of the final optimal design is governed by the critical displacement, which provides a lower maximum performance index. From the manufacturing and efficient points of view, the shape optimization technique should be used to optimize plates in bending.

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Table 1 Material volumes required for the design at different iteration for various displacement limits

u_j^* (mm)	V_0^s (10^5mm^3)	V_{20}^s (10^5mm^3)	V_{40}^s (10^5mm^3)	V_{opt}^s (10^5mm^3)	$PI_{\max}$
0.5	5.85	4.70	3.69	2.75	2.13
0.75	5.11	4.11	3.22	2.4	2.13
1.0	4.65	3.73	2.93	2.18	2.13

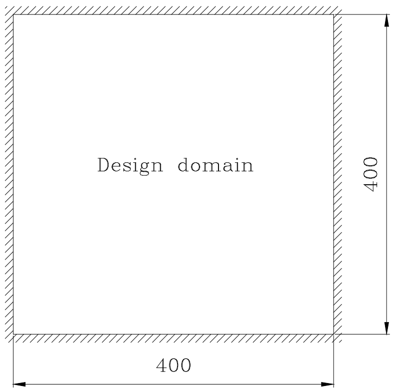


Fig. 1. Design domain for the clamped plate under concentrated loading

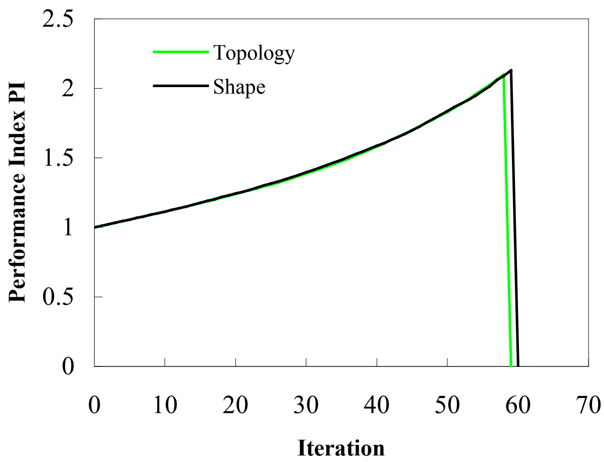
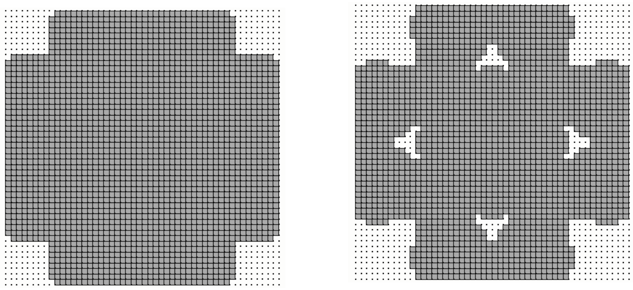
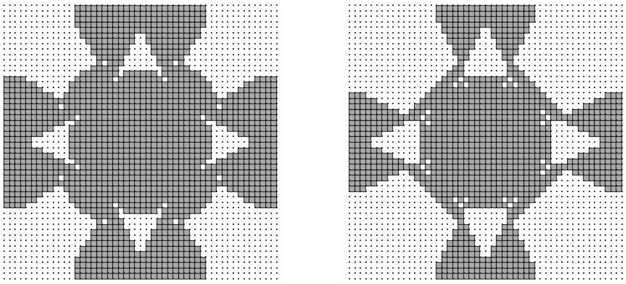


Fig. 2. Performance index history of the clamped plate under concentrated loading

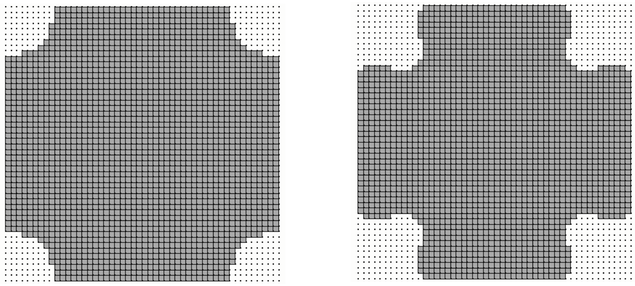


(a) topology at iteration 10 (b) Topology at iteration 20

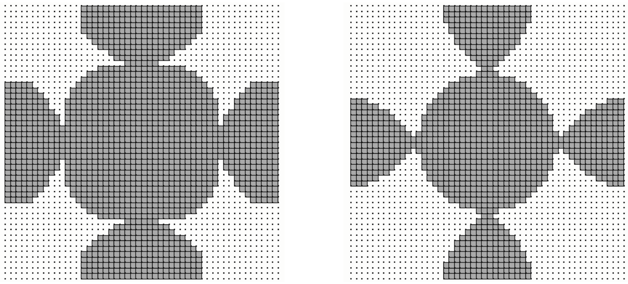


(c) Topology at iteration 40 (d) Optimal topology

Fig. 3. Topology optimization of the clamped plate under concentrated loading



(a) Shape at iteration 10 (b) Shape at iteration 20



(c) Shape at iteration 40 (d) Optimal shape

Fig. 4. Shape optimization of the clamped plate under concentrated loading

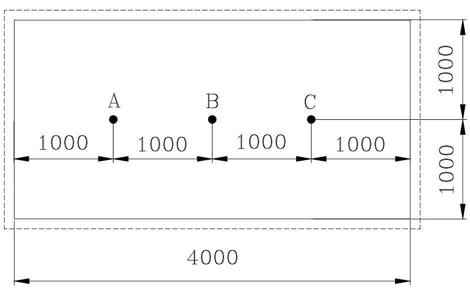


Fig. 5. Design domain for the simply supported plate under multiple displacement constraints

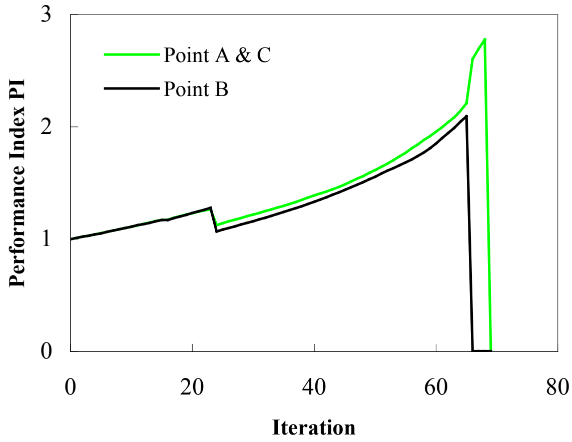
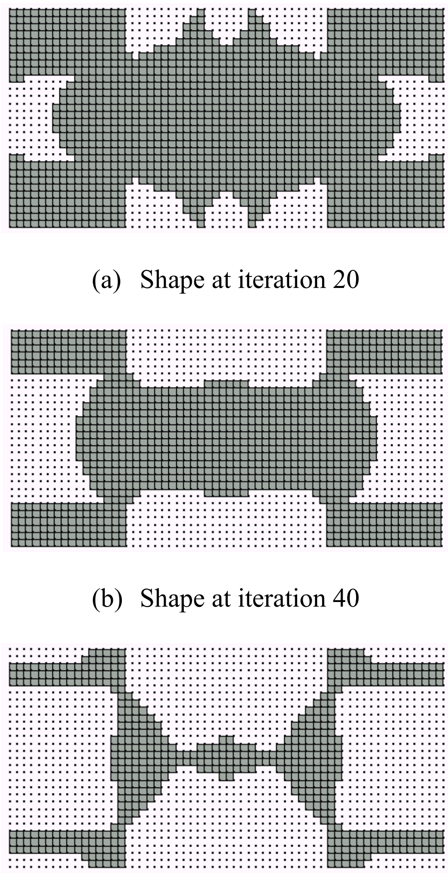


Fig. 6. Performance index history of the simply supported plate under multiple displacement constraints



(a) Shape at iteration 20

(b) Shape at iteration 40

(c) Optimal shape

Fig. 7. Shape optimization of the plate under multiple displacement constraints