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D. R. Buttsworth and T. V. Jones
Department of Engineering Science,
University of Oxford,
Oxford, UK

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Technical Note

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Department of Engineering Science,
University of Oxford,
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NOMENCLATURE

A	arbitrary function of s
c	specific heat of substrate material
h	convective heat transfer coefficient
k	conductivity of substrate material
m	function of s, R, α and σ
q	convective heat flux at the surface
r	radial coordinate
R	radius of curvature of the surface; radius of a sphere or cylinder
s	Laplace variable
t	time (usually measured from the start of heat transfer)
T	temperature
T_i	initial temperature of the substrate
T_r	recovery temperature of the gas
x	depth measured from the substrate surface
α	thermal diffusivity of substrate material, $k/\rho c$
ρ	density of substrate material
θ	temperature referenced to the initial substrate temperature, $T - T_i$
σ	solution index: 0 for flat plate; 1 for cylindrical; 2 for spherical solution
τ	dummy variable for time-integration

Subscripts

fp	quantity obtained using a flat plate analysis
s	surface of the substrate

Superscript

$(\bar{\ })$	denotes the Laplace transformation

1.0 INTRODUCTION

Convective heat transfer data is frequently obtained from transient surface temperature measurements. Thin film resistance gauges, thermocouples, and thermochromic liquid crystals, are used in various situations to measure the surface temperature history. By assuming that uniform semi-infinite flat plate conditions apply, it is possible to express the instantaneous surface heat flux as an analytical function of the transient surface temperature[1]. Various approaches can be used to account for the presence of multi-layered substrates and finite thickness substrate effects (Schultz and Jones[1]; Doorly and Oldfield[2]; Guo *et al*[3]), however, the effects of surface curvature are usually neglected.

If the heat transfer data is obtained on the premise that flat plate conditions apply, then errors will be introduced if the surface is actually curved. Intuitively, the magnitude of such errors will depend on how far the heat penetrates into the substrate relative to the radius of curvature of the surface. Maulard[4] derived expressions to evaluate the accuracy of the flat plate assumption in cases where the surface under consideration is curved. However, his theoretical results do not provide a convenient means of accurately accounting for surface curvature effects in the routine analysis of experimental data.

The current work presents simple analytical curvature corrections for heat transfer results inferred on the assumption that flat plate conditions prevailed during the experiment. Conditions of arbitrary surface heat flux are easily accommodated with the present analysis. The accuracy of the first-order correction analysis is demonstrated by comparing results from the approximate curvature analysis with exact results for a variety of configurations under constant convective heat transfer coefficient conditions. The practical utility of the radial heat conduction modelling is evident from a number of studies in which it has already been employed (Buttsworth and Jones[5]; Hoffs *et al*[6]; Fletcher[7]). Nevertheless, data from a recent heat transfer probe experiment is also considered as a further practical demonstration of the present analysis.

2.0 RADIAL HEAT CONDUCTION

Referring to Fig. 1, the one-dimensional, heat conduction problem of interest in the present investigation may be written,

$$\frac{\partial^2 T}{\partial r^2} + \frac{\sigma}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \ldots (1)$$

subject to the boundary and initial conditions,

$$k\left(\frac{\partial T}{\partial r}\right)_{r=R} = q(t) \qquad \ldots (2)$$

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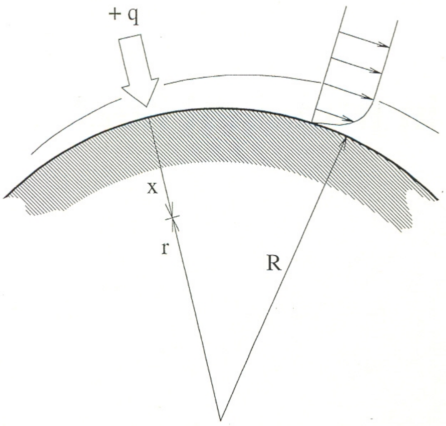

Figure 1. Definition of coordinate system in the present analysis.

$$T(0,t) = T_i \quad \ldots (3)$$

$$T(r,0) = T_i \quad \ldots (4)$$

By referencing the temperature at any point within the substrate to the initial substrate temperature,

$$\theta = T - T_i \quad \ldots (5)$$

and taking Laplace transforms, the unsteady heat diffusion equation (Equation (1)) can be expressed as,

$$\frac{\partial^2\bar\theta}{\partial r^2} + \frac{\sigma}{r}\frac{\partial\bar\theta}{\partial r} - \frac{s}{\alpha}\bar\theta = 0 \quad \ldots (6)$$

If a convex surface is considered, and a change in coordinate is made from r to x as shown in Fig. 1, i.e., $x = R - r$, then Equation (6) becomes,

$$\frac{\partial^2\bar\theta}{\partial x^2} - \frac{\sigma}{r}\frac{\partial\bar\theta}{\partial x} - \frac{s}{\alpha}\bar\theta = 0 \quad \ldots (7)$$

If it is assumed that the heat penetrates only a relatively small distance into the substrate such that $\alpha t/R^2 << 1$, then the approximation, $r \approx R$ can be made for all the values of x which are of interest. Therefore, Equation (7) may be written,

$$\frac{\partial^2\bar\theta}{\partial x^2} - \frac{\sigma}{R}\frac{\partial\bar\theta}{\partial x} - \frac{s}{\alpha}\bar\theta = 0 \quad \ldots (8)$$

If a solution to this equation of the form

$$\bar\theta = Ae^{-mx} \quad \ldots (9)$$

is assumed, then the characteristic equation will be,

$$m^2 + \frac{\sigma}{R}m - \frac{s}{\alpha} = 0 \quad \ldots (10)$$

Only the positive root of the characteristic equation is taken so as to meet the boundary condition, $\theta \rightarrow 0$, for sufficiently large values of x. Thus,

$$m = \sqrt{\frac{s}{\alpha}}\sqrt{1+\frac{\alpha\sigma^2}{4R^2}\frac{1}{s}} - \frac{\sigma}{2R} \quad \ldots (11)$$

By performing a series expansion, the above root can be expressed as,

$$m = \sqrt{\frac{s}{\alpha}}\left(1+\frac{\alpha\sigma^2}{8R^2}\frac{1}{s} - \frac{\alpha^2\sigma^4}{128R^4}\frac{1}{s^2}+\ldots\right) - \frac{\sigma}{2R} \quad \ldots (12)$$

Now the surface boundary condition (Equation (2)) can be written,

$$\bar q = -k\left(\frac{\partial\bar\theta}{\partial x}\right)_{x=0} = km\bar\theta_s \quad \ldots (13)$$

Thus,

$$\bar q = \sqrt{\rho c k}\left(s^{-\frac{1}{2}} + \frac{\alpha\sigma^2}{8R^2}s^{-\frac{3}{2}} - \frac{\alpha^2\sigma^4}{128R^4}s^{-\frac{5}{2}}+\ldots\right)s\bar\theta_s - \frac{k\sigma}{2R}\bar\theta_s \quad \ldots (14)$$

If the inverse Laplace transformation is taken, the above expression becomes,

$$q = \frac{\sqrt{\rho c k}}{\sqrt{\pi}}\int_0^t \left(1+\frac{\alpha\sigma^2(t-\tau)}{4R^2} - \frac{\alpha^2\sigma^4(t-\tau)^2}{96R^4}+\ldots\right)\frac{\mathrm{d}\theta_s}{\mathrm{d}\tau}\frac{1}{\sqrt{t-\tau}}\mathrm{d}\tau - \frac{k\sigma}{2R}\theta_s \quad \ldots (15)$$

Since the combined value of the bracketed terms, apart from the first, will at most be $O(\alpha t/R^2)$, Equation (15) can be simplified to,

$$q = \frac{\sqrt{\rho c k}}{\sqrt{\pi}}\int_0^t \frac{\mathrm{d}\theta_s}{\mathrm{d}\tau}\frac{1}{\sqrt{t-\tau}}\mathrm{d}\tau - \frac{k\sigma}{2R}\theta_s \quad \ldots (16)$$

or equivalently,

$$q = \frac{\sqrt{\rho c k}}{\sqrt{\pi}}\int_0^t \frac{\mathrm{d}T_s}{\mathrm{d}\tau}\frac{1}{\sqrt{t-\tau}}\mathrm{d}\tau - \frac{k\sigma}{2R}\left(T_s - T_i\right) \quad \ldots (17)$$

because it is presently assumed that $\alpha t/R^2 << 1$.

The first term in Equation (17) corresponds to the result that would be obtained if curvature effects were not present (i.e., $\sigma = 0$ or equivalently, $R \rightarrow \infty$). Thus, if curvature effects are present, the above result indicates that it is relatively straight forward to correct heat flux data obtained on the basis of a semi-infinite flat plate analysis. That is, in general, the expression

$$q = q_{fp} - \frac{k\sigma}{2R}\left(T_s - T_i\right) \quad \ldots (18)$$

can be used to determine the transient heat flux on a curved surface.

When the convective heat transfer coefficient is constant, additional relationships are easily derived to correct for surface curvature effects. In general, the convective heat flux may be written,

$$q = h\left(T_r - T_s\right) \quad \ldots (19)$$

Again, by referencing the surface temperature of the material to its initial value (Equation (5)), and treating the flow recovery temperature in a similar manner,

$$\theta_r = T_r - T_i \quad \ldots (20)$$

the Laplace transformation of Equation (19) can be written,

$$\bar q = s^{-1}h\theta_r - h\bar\theta_s \quad \ldots (21)$$

(assuming that h is constant and that there is a step in the flow recovery temperature of magnitude θ_r at $t = 0$). Now, the Laplace transformation of Equation (16) is,

$$\bar q = \sqrt{\rho c k}\sqrt{s}\bar\theta_s - \frac{k\sigma}{2R}\bar\theta_s \quad \ldots (22)$$

Thus, by combining Equations (21) and (22), the change in the surface temperature can be written,

$$\bar{\theta}_s = \frac{h\theta_r}{s^{\frac{3}{2}}\sqrt{\rho c k} + s\left(h - \frac{k\sigma}{2R}\right)} \quad \ldots (23)$$

The corresponding result for a flat plate can be found by putting $\sigma = 0$ in Equation (23), such that,

$$\bar{\theta}_s = \frac{h_{fp}\theta_{r_{fp}}}{s^{\frac{3}{2}}\sqrt{\rho c k} + s h_{fp}} \quad \ldots (24)$$

Equations (23) and (24) are equivalent if,

$$h = h_{fp} + \frac{k\sigma}{2R} \quad \ldots (25)$$

$$\theta_r = \frac{\theta_{r_{fp}}}{\left(1 + \frac{k\sigma}{2Rh_{fp}}\right)} \quad \ldots (26)$$

Thus, in experiments where the convective heat transfer coefficient is constant and the surface has a convex curvature, the actual value of h will be larger than the value inferred from a flat plate analysis by approximately $k\sigma/2R$. Likewise, the actual gas recovery temperature (relative to the initial surface temperature) will be smaller than the apparent value (based on a flat plate analysis) by a factor of approximately $(1 + k\sigma/2Rh_{fp})^{-1}$.

In the preceding derivations, a convex geometry was considered (e.g. Fig. 1). It should be noted however, that concave surfaces can be treated in a similar manner. The governing equations for a concave surface will be identical in form to the previous results, however, the coordinate change will be given by, $x = r - R$. Thus, for a concave surface, the expression corresponding to Equation (8) will be,

$$\frac{\partial^2\bar{\theta}}{\partial x^2} + \frac{\sigma}{R}\frac{\partial\bar{\theta}}{\partial x} - \frac{s}{\alpha}\bar{\theta} = 0 \quad \ldots (27)$$

Therefore, in the concave surface expressions corresponding to Equations (18), (25) and (26), terms involving $k\sigma/2R$ will have the opposite sign to the corresponding terms in the convex expressions.

3.0 VALIDATION OF THE ANALYSIS

Results from the approximate radial heat conduction analysis are now compared with exact solutions in order to verify the accuracy of the approximate analytical relationships. A convex cylinder, concave cylinder, convex sphere, and concave sphere were the geometries considered. Approximations to each of these geometries can arise in transient heat transfer testing.

Carslaw and Jaeger[8] present analytical solutions for the transient surface temperature of the above geometries under conditions in which the convective heat transfer coefficient remains constant. Temperature histories were determined using these exact results for a range of Biot numbers (hR/k = 0·1, 1·0, and 10) which might be encountered in aerodynamic and heat transfer testing. Approximate values of the convective heat transfer coefficient were then obtained using Equations (17) and (19). These results are shown in Fig. 2 (normalised by the convective heat transfer coefficient values used in the exact analytical temperature solutions). In general, the maximum time (or the maximum Fourier number, $\alpha t/R^2$) over which the approximate solution remains within 1% of the exact value, varies with both the geometry and the value of hR/k. Some numerical values of these times are given in Table 1. In the spherical cases, it appears that the approximation $r \approx R$, is very well compensated by neglecting higher order terms within the integral that appears in Equation (15).

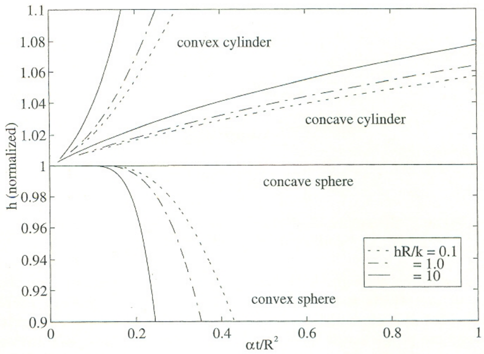

Figure 2. Heat transfer coefficient results from the approximate analysis.

The penetration depth of a thermal wave (defined on the basis of a 1% change in temperature or heat transfer) is approximately given by $4\sqrt{(\alpha t)}$ (e.g. Schultz and Jones[1]). Thus, the approximate results for the convex cylinder geometry are reliable (i.e. are within 1% of the exact solution) virtually up to the time at which the thermal wave penetrates to a depth equalling the radius of curvature. This condition can usually be satisfied in transient heat flux experiments. For the concave cylinder and the spherical geometries, the approximate solution appears to be reliable for even larger times than in the case of a convex cylinder. It is therefore concluded that the present analytical results provide a useful and simple means of accurately accounting for radial heat conduction effects.

4.0 DEMONSTRATION OF THE ANALYSIS

The utility of the heat conduction modelling is now demonstrated with heat transfer measurements obtained using the probe and free jet configuration illustrated in Fig. 3. Heat transfer probe measurements were made using a fused quartz rod having a rounded end with a nominal diameter of 3 mm (see Fig. 3). A platinum thin film heat transfer gauge was located about the stagnation point of this rod. The probe was pre-heated to around 680 K and then traversed through the Mach 4 free jet giving the temperature history shown in Fig. 4(a). Further details of the present experimental arrangement may be found in Buttsworth and Jones[5].

The heat transfer measurements shown in Fig. 4(b) were obtained from an electrical circuit in which the flow of current is directly analogous to the heat flow in a one-dimensional semi-infinite flat plate[9].

Table 1
Maximum values of $\alpha t/R^2$ for a 1% tolerance in the accuracy of the approximate solution

Configuration	hR/k	$\alpha t/R^2$
convex	10	0·040
cylinder	1·0	0·053
	0·1	0·057
concave	10	0·074
cylinder	1·0	0·102
	0·1	0·111
convex	10	0·171
sphere	1·0	0·218
	0·1	0·241

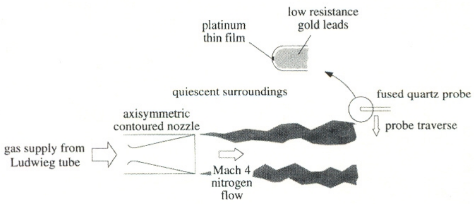

Figure 3. Schematic illustration of the heat transfer probe and the turbulent free jet experiment.

As such, the analogue heat transfer signal does not incorporate the curvature effects associated with the heat conduction process. From Fig. 4(b), it is clear that the apparent heat flux does not return to zero when the probe leaves the lower side of the jet (at around 30 ms on this time scale). The radial heat conduction correction term (the second term on the right-hand-side of Equation (18)) is also shown in Fig. 4(b). (In this correction, k = 1·84 $Wm^{-1}K^{-1}$, which is the conductivity of fused quartz at around 660 K; $\sigma = 2$, since the probe tip is hemispherical; $R = 1{\cdot}5 \times 10^{-3}$ m; T_s is the measured surface temperature history in Fig. 4(a); and T_i was taken as 674 K.) Once the measurements are corrected for radial conduction effects (i.e. once the sum of the two results in Fig. 4(b) has been formed), it can be seen (from Fig. 4(c)) that the heat flux level in the quiescent region on the lower side of the jet returns to approximately zero. In the present example, the relative magnitude of the correction term varies from around 5% at the centreline, to approximately 100% at the lower edge of the turbulent jet.

5.0 CONCLUSION

Radial conduction effects can have a significant influence in transient heat flux experiments if the test surface is curved.

Simple analytical expressions derived in the present note can be used to correct experimental data for radial heat conduction effects.

The accuracy of these approximate analytical expressions varies with the geometry, and the values of both the Biot (hR/k) and Fourier ($\alpha t/R^2$) numbers.

As a general guide, an accuracy of better than 1% can be expected from the approximate expressions provided $4\sqrt{(\alpha t)} < R$.

The approximate analytical expressions are easy to implement and can be applied in the routine analysis of experimental data.

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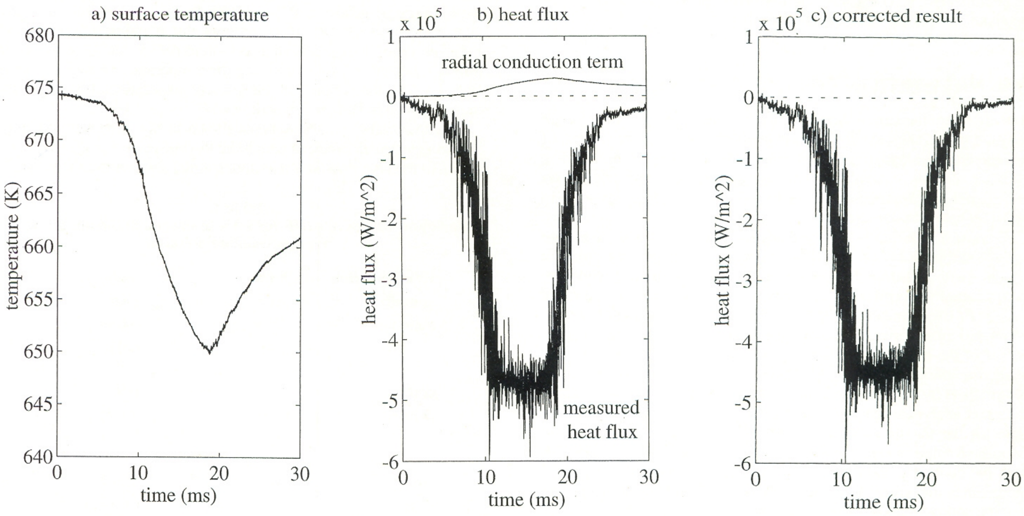

Figure 4. Results from the free jet experiment.