ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
*REIMS (France), 23-26 April 2024*

# THE USE OF STRAIGHTEDGE, COMPASS, AND FIXED SHAPES TO ENHANCE THE CONTENT KNOWLEDGE OF MATHEMATICS TEACHERS IN THEIR PROFESSIONAL TRAINING IN GEOMETRY EDUCATION

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*The paper discusses a special approach to professional learning with a focus on building geometrical reasoning and argumentation through the extensive use of straightedge, compass and fixed shapes together with some special constructions. The novelty of this method is in combining the elements of content knowledge with specially designed rich geometry tasks, where the use of instruments takes the dominant role until the end of the solution for each task.*

## INTRODUCTION AND THEORETICAL BACKGROUND

Despite the wide range of professional learning models related to geometry education that are available in literature (Jacobs et al, 2020; Smith et al, 2005) and known from practical implications (Seago et al, 2010; White et al, 2012), an appeal for new resources and educational tools that can be developed and used for professional training and development in this area remains high. The proposed model of teachers' professional development aims to answer these questions and outline further perspectives bringing attention to a special method of teacher training. The method is based on the use of special constructions from different topics of school geometry affiliated with meaningful tasks that require the active use of drawing instruments while making reasoning and argumentation. It is considered from the perspective of activity theory, where mathematical ideas are regarded as historical artifacts of human culture (Vygotsky, 1987). Artifacts may also include drawing instruments such as straightedge, compass or fixed shape(s) that may be used individually or collaboratively within a group to solve mathematical problems. The novelty of this method is in combining the elements of content knowledge with specially designed rich geometry tasks, where the use of instruments takes the dominant role until the end of the solution for each task. The active use of straightedge and compass for proof and problem-solving activities in geometry was never under a question mark in geometry education. However, its importance diminished within the last several decades. As a result of that, school graduates in Australia, even those who complete 'Specialist Mathematics' and 'Mathematical Methods' – the two most challenging mathematical subjects in Grades 11 and 12, can hardly use a straightedge and compass beyond drawing an initial figure for a geometry task. In other words, these instruments are mainly considered as drawing instruments that do not have any relation to reasoning and argumentation in geometry education. A very similar situation takes place with teachers. The drawing instruments provide unlimited opportunities in school geometry which are not explored often, unfortunately. They help to see appropriate steps towards solutions in a more logical way and, more importantly, help to connect different parts of geometric knowledge together and look at the same piece of content from different perspectives. There is a consensus among mathematics teachers and educators that geometrical facts and statements

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 263-270). ICMI.

that require additional constructions to be added to the initial figure for their proofs are regarded as more challenging for students (Bartolini Bussi et al, 2009; Vondrova and Divisova, 2013; Gridos et al, 2022). As a consequence, such situations are more difficult for teachers to manage in the classroom and achieve desirable learning outcomes for students. This method of training aims to address these issues so teachers can acquire practical skills in using different drawing instruments while making reasoning which is expected to have positive effect on their content knowledge and enrich their professional experience. Specifically, the paper addresses the following research question: How can a program of professional development for mathematics teachers in geometry education integrate geometrical constructions, meaningful tasks and the use of drawing instruments to enhance content knowledge of mathematics teachers and their skills in geometrical reasoning?

## THE METHOD OF PROFESSIONAL TRAINING AND DEVELOPMENT

A geometrical construction representing some special property or fact of Euclidean geometry is assigned with a task to complete using a straightedge, or straightedge and compass, or fixed shape(s). A straightedge can be used to construct a line or a line segment through two points. A straightedge cannot be used for measuring any distance. It is also called a ruler (without scaling). A compass can construct a circle or some arc of a circle. However, its main functionality is in measuring distances which cannot be carried out with a straightedge. Fixed shapes represent special geometrical shapes that can be used for drawing and remain unchanged. If a fixed shape contains a straight part of its shape, it can be used for constructing line segments. For example, a fixed shape of a right-angled triangle I discuss later can be used for constructing right angles or line segments up to the length of its hypotenuse. For practical usage, fixed shapes can be made of cardboard or veneer. These instruments play a central role in this method of professional training. A card with a special construction is given to the participants of the training as well as a card with the task to work on. In some cases, a proof (or justification) of the given construction is provided together with its statement, in other cases participants are encouraged to prove some constructions on their own before moving on to work on the tasks. Some cards may contain a set of constructions rather than just a single construction as it happens for some topics, where a number of different constructions can be often considered and used together. For example, many properties of circle geometry are worth to be considered together. The solution to complete the task requires participants to use some of the aforementioned instruments. It also requires to look for the ways of how a special construction assigned to the task can be used as part of a solution for that task. While working on each task, teachers refresh their knowledge of some constructions and make acquaintance with others. Whatever their case is, they need to demonstrate the active use of the prescribed instrument(s) to complete the task successfully. This gives participants the opportunity to be involved in constructing solutions to the given tasks at the time when they are involved in studying some geometrical content and applying it through their practical work. Modules containing constructions and their corresponding tasks within a certain topic are called activities. In the next section I discuss several examples of activities set in an increased order of difficulty. Within each activity I provide a brief analysis of what teachers were asked to do, why that activity was important, how the use of instruments was organized to stimulate teachers' thinking and enhance their content knowledge of the topic. Also, I identify directions for potential questions for teachers to think and implement the tasks in learning environments and present some participants' work and reflections on the selected tasks. The data was collected from task-based interviews with small groups of participants and observations made in the training sessions. The teachers' tasks were analyzed at three levels: competence to solve or make progress in the proposed task, effective (or ineffective) use of the required instruments and skills to see connections between geometrical constructions assigned to the tasks and the tasks.

## THINKING GEOMETRICALLY WITH STRAIGHTEDGE, COMPASS, AND FIXED SHAPES

### Activity 1: A right angle inscribed in a circle

The first activity is based on a stand-alone, simple property of a right angle inscribed in a circle. An instrument for the first activity is a fixed shape which represents a right-angled triangle. One can use this instrument for constructing right angles or line segments of size up to the length of its hypotenuse.

### Construction

A right angle inscribed in a circle is subtended by the diameter of the circle (Figure 1).

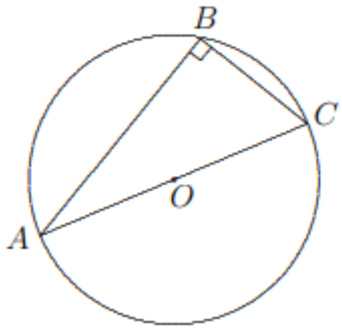


Figure 1: Right angle inscribed in a circle

### Task

A circle on the left side is drawn on paper and a fixed shape of the right-angled triangle on the right side is given as an instrument (Figure 2). The location of the centre of the circle is not known for participants. Using the fixed shape, construct a point which is the centre of the circle.

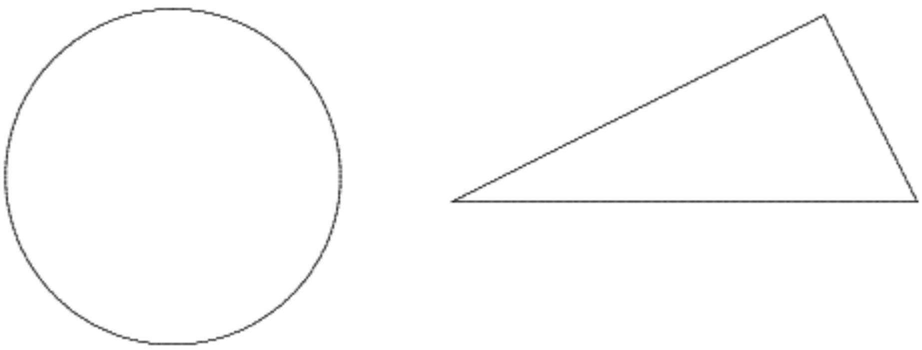

Figure 2: Task for Activity 1

### Analysis

The main idea of this task is to stimulate teachers' awareness that each part of the meaningful task in geometry can be actively used in reasoning and argumentation. A geometrical object can be modified to serve the goals of the current task. The straightforward use of the construction above does not look to be workable here since the hypotenuse of the fixed shape even visually is longer than the diameter of the circle. However, the modified version of the fixed shape where the right angle only is important should work. Indeed, put the right angle vertex of the fixed shape on the circle. Then, the construction assigned with the task can be in use. Indeed, let the sides of the fixed shape intersect the circle at points $A$ and $B$ (Figure 3).

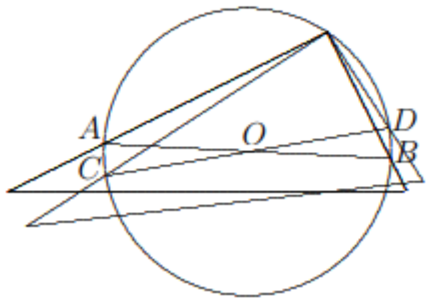


Figure 3: Solution

According to the construction, $AB$ is the diameter. If the right angle vertex of the fixed shape remains at the same point on the circle and the location of the fixed shape is changed by rotation, the sides of the fixed shape intersect the circle at other points, say $C$ and $D$. Note that $CD$ is also a diameter, distinct from $AB$. Hence, using the fixed shape, two diameters $AB$ and $CD$ can be constructed. Both contain the centre of the circle, so the intersection $O$ of $AB$ and $CD$ is the centre of the circle.

This activity helps teachers to advance their knowledge of the simple property of the right angle to a higher level of understanding where they can use that property in the new teaching environment through meaningful tasks that support reasoning and proof building from basic initial ideas. For example, joining points $B$ and $O$ (Figure 1) contributes to potential questions for teachers to think and implement the above task to the concept of the central angle ($\angle BOC$) and its relationship to $\angle BAC$, and to the concept of a tangent line at point $B$. One more potential question to problematize this task relates to constructing the midpoint of $AC$ and $BD$ (Figure 3) using a straightedge and compass. The transcript below shows reflections of two participants of the training about the task above.

Interviewer: Did you find a fixed shape helpful in this task?

Participant A: Yeah, it was. I see the benefits of the fixed shape of a right-angled triangle here. But, after this task, a straightedge and compass seem more meaningful to me than they were before and I can change this task in my class and go in the opposite direction. First, I can ask students to construct a circle using a compass. So, we know the centre. Then, construct a line through the centre using a straightedge. And by connecting each point of intersection the line and the circle have with a point on the circle, we get a right angle constructed.

Participant D: I think so. The solution is clear to me after discussion. A bit tricky though. I am trying to find another way to use the fixed shape here. What if we ask students to construct a circle that this fixed right-angled triangle is inscribed in?

**Activity 2: Corollary of Pythagoras Theorem**

The second activity is based on the famous Pythagoras theorem. Instruments for the second activity are a straightedge and compass. This activity helps participants to look at Pythagoras theorem from a different point of view and develop more understanding of how useful Pythagoras theorem can be in situations where this famous property does not seem at first glance to be part of the solution.

**Construction**

In an acute-angled triangle *ABC* with altitude *AD* (Figure 4) the following property holds

$$AD^2 = AB^2 - BD^2 = AC^2 - CD^2.$$

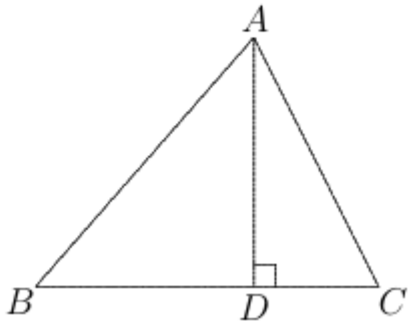


Figure 4: Corollary of Pythagoras theorem

**Task**

An acute-angled triangle *ABC* with altitude *AD* is given (Figure 4 again). Using a straightedge and compass, divide the triangle *ABC* into two triangles that have the equal sums of squares of their sides.

**Analysis**

The construction is not hidden in the figure for the task. In fact, both, construction and task, refer to the same Figure 4. However, to be connected with the task, this construction requires some preparatory work to be done before it can be used. It can be rewritten as $AB^2 + CD^2 = AC^2 + BD^2$. However, *AB* and *CD* as well as *AC* and *BD* cannot form triangles. Thus, if *CD* can be replaced with a congruent line segment emanating from *B*, say *BE=CD*, then the required property of the equal sums of squares could come to the scene (Figure 5).

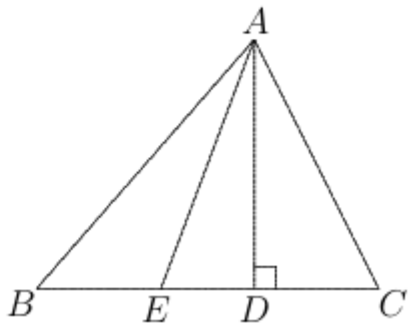


Figure 5: Solution

Distances can be measured with a compass, so one can construct such a point *E* on the side *BC* so that *BE=CD* which implies *BD=CE*. Thus, using a straightedge to draw *AE*, the two triangles, *ABE* and *ACE*, with the required property can be constructed. This activity helps teachers to gain confidence in using a compass for measuring equal distances when required, and constructing new points that have some special properties. It has connections with Activity 4. This task can be made more challenging for participants, if the altitude *AD* is not given in the task. The following transcript demonstrates participants' views in relation to Pythagoras theorem and the use of drawing instruments in the task above.

Participant B: It makes sense to me. I mean that the use of compass and straightedge in tasks like this makes Pythagoras theorem alive and usable in classroom activities.

Participant J: Looks to me that Pythagoras theorem could be somehow used in the task where we constructed the centre of the circle [the task from Activity 1]. Need to think more.

**Activity 3: Symmetry of a rectangle**

The third activity is based on the central symmetry of a rectangle. An instrument for the third activity is a straightedge only. This activity helps participants develop more understanding of how some sophisticated constructions can be carried out with a straightedge only.

**Construction**

Let *ABCD* be a rectangle and *O* be a point where the diagonals *AC* and *BD* intersect. Then, *O* is the centre of symmetry of *ABCD* and any straight line through *O* divides the rectangle *ABCD* into two congruent parts.

**Task**

Let *ABCDEF* be a hexagon with five angles of size 90° and one of 270° (Figure 6).

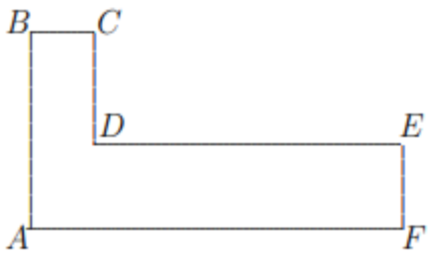


Figure 6: Task for Activity 3

Divide the hexagon *ABCDEF* into two parts of equal area using a straightedge only.

**Analysis**

To see the benefits and connections of the construction above with the task, participants are expected to construct a larger rectangle *ABKF* (Figure 7).

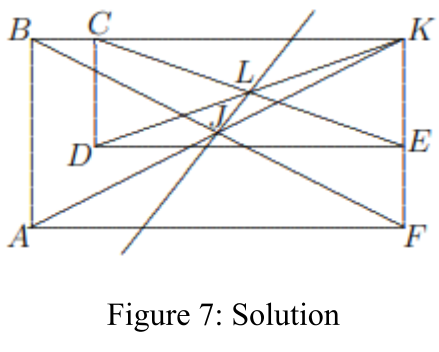


Figure 7: Solution

Then, the diagonals of the rectangles *ABKF* and *CKED* can be constructed, where *J* and *L* are the points of their intersection, respectively. According to the construction above, *J* and *L* are the centres of symmetry of the rectangles *ABKF* and *CKED* respectively, so a straight line through *J* and *L* divides the hexagon *ABCDEF* into two parts of equal area. The next transcript summarizes the difficulties experienced by many participants.

Participant M: I see why I was stuck with this task… All three points that can be constructed with a straightedge are hidden here, and the conclusion is still not obvious after that... A very convincing example to demonstrate how powerful a straightedge can be in good hands.

**Activity 4: A cyclic quadrilateral**

The fourth activity includes a special construction related to cyclic quadrilaterals. Instruments for this activity are a straightedge and compass. This activity helps participants develop more understanding for the case where additional constructions need to be completed.

**Construction**

A cyclic quadrilateral *ABDC* is such that *AB*>*AC* and *D* is the midpoint of the arc *BC* not containing *A*. Then, there exists a point *K* on the side *AB* such that the quadrilateral *ABDC* can be divided into three isosceles triangles, *ACK*, *DKC* and *DBK* (Figure 8).

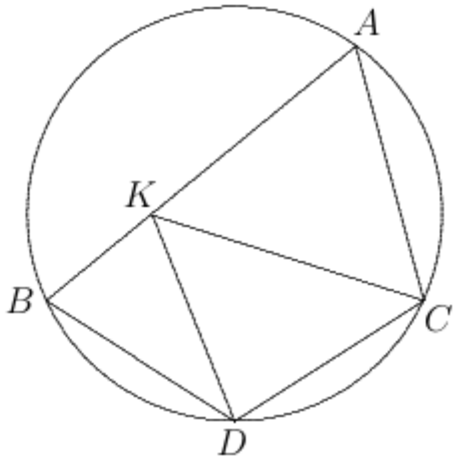


Figure 8: A property of a cyclic quadrilateral

Similarly to Activity 3, participants need to prove this property first, and then work on the task that follows. At first, using a compass, point *K* is constructed on *AB* such that *AK*=*AC*, so participants need to show that the triangles *DKC* and *DBK* are isosceles. Using a straightedge to connect *A* and *D*, and considering congruent triangles *AKD* and *ACD* (SAS) one can conclude that *CD*=*DK* which implies that triangles *DKC* and *DBK* are isosceles.

**Task**

Given triangle ABC, where *AB*>*AC*, and its circumcircle. Using a straightedge and compass, construct the midpoint of the arc *BC* not containing *A*. Both instruments can be used just once.

**Analysis**

The main challenge for participants remains the same as in Activity 3 – to find out where the construction above is hidden in the task. Extending *DK* beyond *K*, let *DK* intersect the circumcircle of *ABC* at point *L*. Taking into account that angles subtended by the same arc are equal, and vertically opposite angles are equal, $\angle ALK = \angle ALD = \angle ABD = \angle KBD = \angle BKD = \angle AKL$ which implies *AL*=*AK*=*AC*. This paves the way to complete the task using a compass to construct a circle of radius *AC* with the centre at *A*, and then a straightedge for a line through the points *K* and *L* (Figure 9).

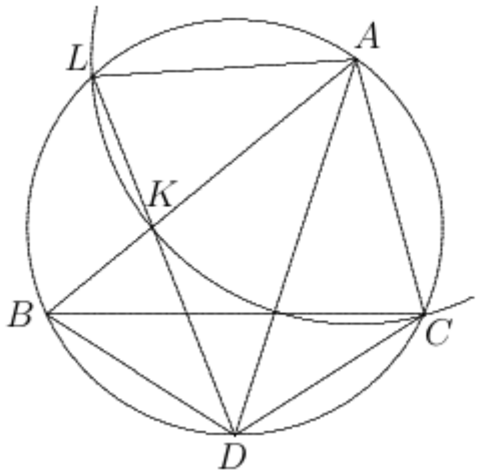


Figure 9: Solution

The following short transcript summarizes the most common responses received from participants.

Interviewer: What useful conclusions did you draw from this task?

Participant C: I think I got it. Isosceles triangles represent a good example of how some constructions with a compass can be interpreted. Using one vertex of a triangle as the centre of a circle and the equal sides as radius, we can construct the isosceles triangle that we need.

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