A NEW CASE OF TRUNCATED PHASE EQUATION FOR COUPLED OSCILLATORS

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Abstract. *Generalized nonlinear phase diffusion equation describes oscillators weakly coupled by diffusion. The equation generally contains infinite number of terms and allows a variety of dynamic balances between them. We consider a truncated version of the equation in which nonlinear excitation drives the dynamics. A group of active systems leading to this truncation is modelled by reaction-diffusion equations with effective nonlocal coupling. We formulate the conditions on the parameters resulting in the truncation and discuss numerical experiments showing complex spatio-temporal behaviour.*

1 Introduction

Under certain conditions reaction-diffusion systems can be reduced to oscillators weakly coupled by diffusion. For the phase of oscillations Kuramoto and Tsuzuki [1, 2] derived the equation

$$\begin{aligned}\partial_t\psi = {} & a_1\nabla^2\psi + a_2(\nabla\psi)^2 + \\ & b_1\nabla^4\psi + b_2\nabla^3\psi\nabla\psi + b_3(\nabla^2\psi)^2 + b_4\nabla^2\psi(\nabla\psi)^2 + b_5(\nabla\psi)^4 + \\ & e_1\nabla^6\psi + \dots ,\end{aligned} \tag{1}$$

where a_n, b_n, e_n, ... are constant coefficients. The right-hand side of (1) can be regarded as a power series in small parameter $\nabla^2 \sim (1/L)^2$, where L is the large spatial scale of variation of ψ. Equation (1) can be truncated to finite forms based on different balances between the terms. In this paper we outline a procedure by which a new kind of truncation is obtained for the reaction-diffusion systems with nonlocal coupling.

Tanaka and Kuramoto [3] analysed the system with effective nonlocal coupling in the form

$$\partial_t\mathbf{X} = \mathbf{f}(\mathbf{X}) + \hat{\delta}\nabla^2\mathbf{X} + k\mathbf{g}(S)\,, \tag{2}$$

$$\tau\partial_t S = -S + D\nabla^2 S + h(\mathbf{X})\,. \tag{3}$$

Here $\mathbf{X}$ is the vector representing concentrations of reactants, $\hat{\delta}$ is a diagonal matrix responsible for diffusion, $\mathbf{f}$ and $\mathbf{g}$ are the vector functions, k, τ and D are constants. Implicitly system (2)–(3) involves a bifurcation parameter μ which, when changing from $\mu < 0$ to $\mu > 0$, brings about oscillatory instability. Note that equation (3) is linear; this allows to eliminate S by expressing it in terms of $\mathbf{X}$ and then substituting into (2). Provided that the parameter k is small, $k \sim O(|\mu|)$, this results in the complex Ginzburg-Landau equation with effective nonlocal coupling [3]

$$\partial_t A = \mu\sigma A - \beta|A|^2 A + \delta\nabla^2 A + k\eta' \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')A(\mathbf{r}',t)\,, \tag{4}$$

where A is (loosely speaking) proportional to $\mathbf{X}$; σ, β, δ and η' are parameters. It is assumed that

$$\mathrm{Re}\,\sigma > 0\,. \tag{5}$$

The coupling function G satisfies the normalization condition

$$\int G(\mathbf{r})\,d\mathbf{r} = 1\,. \tag{6}$$

The complex parameter δ is responsible for diffusion and, therefore, has positive real part,

$$\mathrm{Re}\,\delta > 0\,. \tag{7}$$

Using (6), it is convenient to write (4) in the form

$$\partial_t A = \mu\sigma' A - \beta|A|^2 A + \delta\nabla^2 A + k\eta' \int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}',t) - A(\mathbf{r},t)]\,, \tag{8}$$

where

$$\sigma' = \sigma + k\eta'/\mu\,. \tag{9}$$

It is assumed that the system is supercritical, that is

$$\mathrm{Re}\,\sigma' > 0\,. \tag{10}$$

Some parameters in (8) can be eliminated by changing variables. The imaginary part of the coefficient $\mu\sigma'$ vanishes after transforming $A \to A\exp[i\mu\,\mathrm{Im}\,\sigma' t]$ and the diffusion coefficient D as well as $\mathrm{Re}\,\beta$ and $\mu\,\mathrm{Re}\,\sigma'$ become unity by rescaling A, t and $\mathbf{r}$. Eventually (8) is transformed to

$$\begin{aligned}\partial_t A = {} & A - (1+ic_2)|A|^2 A + (\delta_1 + i\delta_2)\nabla^2 A \\ & + K(1+ic_1)\int d\mathbf{r}' G(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}',t) - A(\mathbf{r},t)]\,,\end{aligned} \tag{11}$$

where c_1, c_2, δ_1, δ_2 and K are real constants. Condition (7) is equivalent to

$$\delta_1 > 0\,. \tag{12}$$

The complex amplitude A is connected to the real phase of oscillations, φ, via

$$A = ae^{-i\varphi}\,, \tag{13}$$

where a is the real amplitude. In one-dimensional case

$$G(x) = \frac{1}{2}(\zeta + i\eta)e^{-(\zeta+i\eta)|x|}\,, \tag{14}$$

where

$$\zeta = \left(\frac{1+\sqrt{1+\theta^2}}{2}\right)^{1/2},\ \eta = \left(\frac{-1+\sqrt{1+\theta^2}}{2}\right)^{1/2}. \tag{15}$$

Here $\theta = \omega_0\tau$ is the parameter proportional to the basic frequency of oscillations, ω_0, and characteristic time τ. As for the phase, it is convenient to analyse its departure from $c_2 t$, defined as

$$\varphi = c_2 t + \psi\,. \tag{16}$$

It can be shown that the departure ψ satisfies equation (1). It is derived from (11) via the phase reduction procedure using (13). In this procedure the integral is decomposed in a power series in ∇. Each coefficient in equation (1) turns out to be a combination of the independent parameters δ_1, δ_2, c_1, c_2, K and θ.

It is important to note that the parameter K is limited in magnitude. When transferring from (8) to (11), the factor $K(1+ic_1)$ appears as the combination

$$K(1+ic_1) = \frac{k\eta'/\mu}{\mathrm{Re}\,\sigma'}\,.$$

Taking real part and using (5), (9) and (10), we have

$$K = \frac{\mathrm{Re}(k\eta'/\mu)}{\mathrm{Re}\,\sigma'} = \frac{\mathrm{Re}\,\sigma' - \mathrm{Re}\,\sigma}{\mathrm{Re}\,\sigma'} = 1 - \frac{\mathrm{Re}\,\sigma}{\mathrm{Re}\,\sigma'} < 1\,. \tag{17}$$

All the results above are obtained by Tanaka and Kuramoto [3]. Tanaka also showed [4] that, if the values of the independent parameters are properly chosen, equation (11) reduces to the Nikolaevskii equation [5]

$$\partial_t\psi = a_1\nabla^2\psi + b_1\nabla^4\psi + e_1\nabla^6\psi + a_2(\nabla\psi)^2\,, \tag{18}$$

with $b_1 > 0$ ensuring that the fourth derivative provides excitation. Observe that the excitation term is linear.

In this paper we show that it is possible to truncate (1) to a form where the excitation is *nonlinear*. Previously we designed such kind of equation using phenomenological arguments in order to simulate unstable combustion fronts [6],

$$\partial_t \psi = -\varepsilon(\nabla\psi)^2\nabla^2\psi + b_5(\nabla\psi)^4 + e_1\nabla^6\psi\,. \tag{19}$$

In [7] we briefly discussed (19) in the context of the diffusion-coupled oscillators. In (19) the excitation is provided the first term in the right-hand side. The term can be regarded as the negative diffusion, $-\nabla^2\psi$, with the positive nonlinear coefficient $\varepsilon(\nabla\psi)^2$.

The model (19) becomes a truncation of (1) if the following six conditions are met,

$$\begin{aligned} &a_1 = a_2 = b_1 = 0\,, \\ &b_2 = b_3 = 0\,, \quad b_4 = -\varepsilon\,. \end{aligned} \tag{20}$$

It is easy to show that, given ε is small, the balance between the terms of (19) leads to smallness of ψ and of $\nabla \sim 1/L$. Consequently, the rest of the terms in (1) shown by dots are negligible because they are of higher order either in ψ or ∇ or both.

As we mentioned, equation (11) contains the six independent parameters and one may expect that, for some of their combinations, the six conditions (20) can be met. However, our recent research indicates that more independent parameters may be necessary to satisfy (20). Therefore, we add another reactant into (2)–(3),

$$\partial_t \mathbf{X} = \mathbf{f}(\mathbf{X}) + \hat{\delta}\nabla^2\mathbf{X} + k_1\mathbf{g}_1(S_1) + k_2\mathbf{g}_2(S_2)\,, \tag{21}$$

$$\tau_1\partial_t S_1 = -S_1 + D\nabla^2 S_1 + h_1(\mathbf{X})\,, \tag{22}$$

$$\tau_2\partial_t S_2 = -S_2 + D\nabla^2 S_2 + h_2(\mathbf{X})\,, \tag{23}$$

where $k_1 \sim k_2 \sim O(|\mu|)$. The respective nonlocal Ginzburg-Landau equation has the form

$$\begin{aligned} \partial_t A = \mu\sigma A - \beta|A|^2 A + \delta\nabla^2 A \\ +k_1{\eta_1}' \int d\mathbf{r}' G_1(\mathbf{r}-\mathbf{r}')A(\mathbf{r}',t) + k_2{\eta_2}' \int d\mathbf{r}' G_2(\mathbf{r}-\mathbf{r}')A(\mathbf{r}',t)\,, \end{aligned} \tag{24}$$

where each G_n carries its own θ_n, $n = 1, 2$. Now we have the 9 independent parameters at our disposal, namely δ_1, δ_2, c_1, c_2, c_3, K_1, K_2, θ_1 and θ_2. The coupling functions satisfy the normalization conditions

$$\int G_1(\mathbf{r})\,d\mathbf{r} = \int G_2(\mathbf{r})\,d\mathbf{r} = 1\,.$$

Using these conditions we modify (24) to

$$\begin{aligned} \partial_t A = \mu\sigma' A - \beta|A|^2 A + \delta\nabla^2 A \\ +k_1{\eta_1}' \int d\mathbf{r}' G_1(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}',t) - A(\mathbf{r},t)] \\ +k_2{\eta_2}' \int d\mathbf{r}' G_2(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}',t) - A(\mathbf{r},t)]\,, \end{aligned} \tag{25}$$

where

$$\sigma' = \sigma + k_1{\eta_1}'/\mu + k_2{\eta_2}'/\mu\,. \tag{26}$$

Rescaling (25) in the similar way to (8), we obtain

$$\begin{aligned} \partial_t A = A - (1+ic_2)|A|^2 A + (\delta_1 + i\delta_2)\nabla^2 A \\ +K_1(1+ic_1) \int dr' G_1(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}') - A(\mathbf{r})] \\ +K_2(1+ic_3) \int dr' G_2(\mathbf{r}-\mathbf{r}')[A(\mathbf{r}') - A(\mathbf{r})]\,, \end{aligned} \tag{27}$$

where

$$K_1(1+ic_1) = \frac{k_1{\eta_1}'/\mu}{\operatorname{Re}\sigma'}\,, \quad K_2(1+ic_3) = \frac{k_2{\eta_2}'/\mu}{\operatorname{Re}\sigma'}\,. \tag{28}$$

Now we can formulate restrictive conditions on K_1 and K_2 to replace (17). Taking real parts in

(28) and summing up, we get

$$K_1 + K_2 = \frac{\mathrm{Re}(k_1\eta_1\,'/\mu) + \mathrm{Re}(k_2\eta_2\,'/\mu)}{\mathrm{Re}\,\sigma'}$$

and, using (26), (5) and (10),

$$K_1 + K_2 = \frac{\mathrm{Re}\,\sigma' - \mathrm{Re}\,\sigma}{\mathrm{Re}\,\sigma'} = 1 - \frac{\mathrm{Re}\,\sigma}{\mathrm{Re}\,\sigma'} < 1\,. \tag{29}$$

Another restriction to be met is positiveness of the coefficient at $\nabla^6\psi$ in (19). It is needed to ensure that the term is dissipative,

$$e_1 > 0\,. \tag{30}$$

Having the 9 independent parameters we have been able to satisfy the 6 conditions (20) subject to the restrictions (12), (29) and (30). We intend to report full details of this work elsewhere.

2 Numerical experiments

In this section we present some numerical simulations with the two-dimensional version of (19). The computations were performed in the square region, $0 < x < 2$, $0 < y < 2$ with $b_4 = -10$, $b_5 = e_1 = 1$. In Cartesian coordinates

$$\begin{aligned}\partial_t\psi =& b_4\,(\partial_x^2\psi + \partial_y^2\psi)\left[(\partial_x\psi)^2 + (\partial_y\psi)^2\right] + \\ & b_5\left[(\partial_x\psi)^4 + 2(\partial_x\psi)^2\,(\partial_y\psi)^2 + (\partial_y\psi)^4\right] + \\ & e_1\left(\partial_x^6\psi + 3\,\partial_x^4\partial_y^2\psi + 3\,\partial_x^2\partial_y^4\psi + \partial_y^6\psi\right)\,.\end{aligned} \tag{31}$$

The boundary conditions were chosen arbitrarily,

$$\begin{aligned}&\partial_x\psi = 0\,, \quad \partial_x^2\psi = 0\,, \quad \partial_x^3\psi = 0 \quad \text{at} \quad x = 0 \;\text{ and }\; x = 2\,,\\ &\partial_y\psi = 0\,, \quad \partial_y^2\psi = 0\,, \quad \partial_y^3\psi = 0 \quad \text{at} \quad y = 0 \;\text{ and }\; y = 2\,.\end{aligned} \tag{32}$$

Equation (31) was discretized in space on a uniform grid with the step 0.025 using 2nd-order accurate central differences and a 7×7 point stencil. Boundary conditions (32) were discretized using a fictitious point approach with 3 additional fictitious layers of discretization points introduced outside of each of the boundaries. Then formally the physical boundary points are treated as interior. The boundary conditions (32) are then approximated using 2nd-order central finite differences and the resulting equations are used to eliminate the fictitious points in favour of the interior points adjacent to the boundary. An explicit 1st-order accurate forward Euler method was used to discretize in time. Comparing the nonstationary term and dissipation, we see that the time step must be as small as $\kappa\Delta x^6$, where Δx is the spatial step and κ is some factor. The experiments showed that κ should not exceed 0.003.

As an initial condition we placed a narrow hump in the corner $(x, y) = (2, 2)$. Snapshots of the solution are presented in Fig. 1. The phase field is curved at all times and exhibits seemingly irregular behaviour. Periods of slow evolution dominated by the dissipation intermit with surges of activity driven by the nonlinear excitation. The computations were performed up to $T \approx 5.9 \times 10^{-5}$. A closer look at the topography reveals more or less discernible step-like structures which propagate in normal direction to the average motion of the field (upwards in the Figure).

It is interesting to compare characteristic time scales: the discretization step, Δt, the typical time scale of the structures, Δt_1, and the total duration of the experiment, T. As we mentioned above, $\Delta t = 0.003\Delta x^6 = 0.003 \cdot 0.025^6 \sim 7 \cdot 10^{-13}$. For the step-like structures roughly $\Delta t_1 \sim \ell^{\,6}/e_1$, where ℓ is their typical length and $e_1 = 1$. Observe from the snapshot at $t = 1.383 \cdot 10^{-5}$, that a step occupies about 5 to 6 grid cells, giving $\ell \sim (5 \text{ to } 6) \cdot 0.025^6$. Thus, $\Delta t_1 \sim \ell^{\,6}/c \sim (0.38 \text{ to } 1.1) \cdot\ 10^{-5}$ and we have

$$\Delta t \sim 10^{-12}\,, \quad \Delta t_1 \sim (0.38 \text{ to } 1.1) \cdot 10^{-5}\,, \quad T \sim 5.9 \cdot 10^{-5}\,.$$

The time Δt_1 exceeds the numerical step Δt by several orders, and the experiment duration T is larger

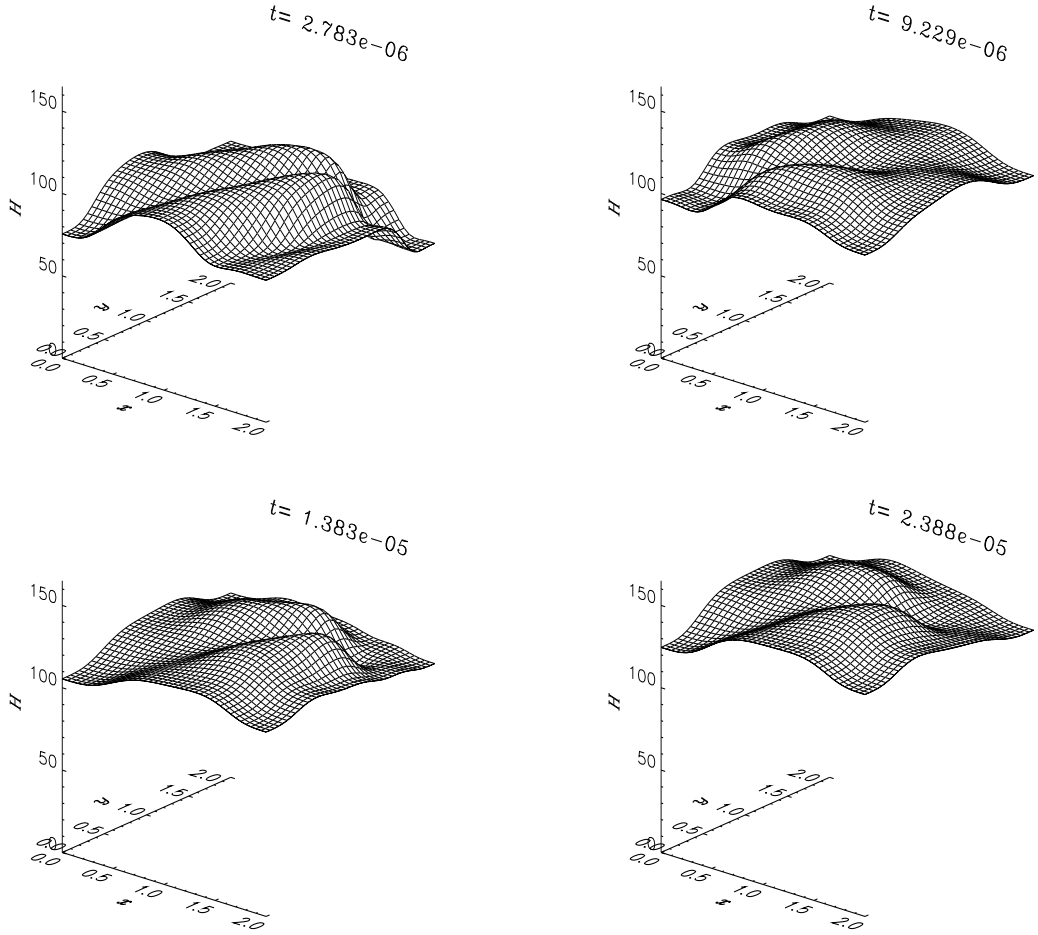

Figure 1: The phase field at different moments

still by approximately an order. Therefore, the numerical time step was sufficiently short to resolve the motion of the structures and the experiment was sufficiently long to embrace this motion. The repeating bursts of the phase activity hint that the dynamics would last infinitely long because of the persistent action of the source.

3 Conclusions

A set of oscillators weakly coupled by diffusion are generally described by the infinite nonlinear phase equation (1). Omitting full technical details in this short paper we outlined a procedure by which we have obtained a truncated version of this equation with *nonlinear* excitation. Numerical experiments reveal seemingly random dynamics.

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