UNIVERSITY OF SOUTHERN QUEENSLAND

METHODS FOR ANALYSIS OF FUNCTIONALS ON GAUSSIAN SELF SIMILAR PROCESSES

A Thesis submitted by

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Abstract

Many engineering and scientific applications necessitate the estimation of statistics of various functionals on stochastic processes. In Chapter 2, Norros et al's Girsanov theorem for fBm is reviewed and extended to allow for non-unit volatility. We then prove that using method of images to solve the Fokker-Plank/Kolmogorov equation with a Dirac delta initial condition and a Dirichlet boundary condition to evaluate the first passage density, does not work in the case of fBm.

Chapter 3 provides generalisation of both the theorem of Ramer which finds a formula for the Radon-Nikodym derivative of a transformed Gaussian measure and of the Girsanov theorem. A $\mathcal{P}$-measurable derivative of a $\mathcal{P}$-measurable function is defined and then shown to coincide with the stochastic derivative, under certain assumptions, which in turn coincides with the Malliavin derivative when both are defined. In Chapter 4 *consistent quasi-invariant stochastic flows* are defined. When such a flow transforms a certain functional consistently a simple formula exists for the density of that functional. This is then used to derive the last exit distribution of Brownian motion.

In Chapter 5 a link between the probability density function of an approximation of the supremum of fBm with drift and the Generalised Gamma distribution is established. Finally the self-similarity induced on the distributions of the sup and the first passage functionals on fBm with linear drift are shown to imply the existence of transport equations on the family of these densities as the drift varies.

Thesis certification page

This Thesis is entirely the work of Hirdyesh Bhatia except where otherwise acknowledged. The work is original and has not previously been submitted for any other award, except where acknowledged.

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Chapter 1

Introduction

The aim of this research is to develop methods for analysis of probability distributions of the functionals on Gaussian Self similar processes amenable. However, for the reader's convenience, in this chapter we review the main properties of various mathematical objects related to this research.

We begin by providing motivation behind the use of fractional Brownian motion and how it used for modeling Anomalous diffusion. The functionals being studied in this research are then defined and the reasons which motivate their study are provided. We then review the Fokker-Plank/Kolmogorov equations theory related to Markovian processes and how it is used to study the probability distribution of the first passage functional.

The development of measure theory by Lebesgue (Lebesgue 1904) enabled probability theory to be given rigorous foundations, which was achieved by Kolmogorov (Kolmogorov 1950). Understanding probability as a measure has had a profound and ongoing influence on both theoretical and applied probability.

Radon-Nikodym derivatives tell how it is possible to change from one probability measure to another. We next review a few such results, defined over Abstract Wiener space in order to provide enough context and contrast for $\mathcal{P}$-measurable derivatives, which are defined in section 3.2.2. A generalisation of the Girsanov theorem for fBm from (Norros et al. 1999), which makes it usable for non-standard fBm is also provided.

As the reproducing kernel Hilbert space associated with fBm considered as a vector space of functions (and not taking account of its Hilbert space structure and its norm) has a representation as the fractional space $I_{0+}^{H+\frac{1}{2}}\left(L^{[0,T]}\right)$ of certain class of functions, some standard results from fractional calculus are recalled to be used in the verification of a second generalisation of the Girsanov theorem for fBm later.

As part of this research, in Section 4.3 a quasi invariant flow with additional constraints is used to find a new proof of the probability distribution of the last exit of a Brownian motion process. We present a brief literature review on this subject next, which is followed by a review of the theory of probability densities of functionals on Brownian motion.

1.1 Standard and Anomalous diffusion processes

The concept of diffusion is widely used in physical sciences, economics and finance. However, in each case, the object (e.g., atom, price, etc.) that is undergoing diffusion is spreading out from a point or location at which there is a higher concentration of that object. The mathematical modeling of diffusion has a long history with many different formulations including, models based on conservation laws, random walks and central limit theorem, Brownian motion and stochastic differential equations, and models based on Chapman-Kolmogorov and Fokker-Planck equations. A fundamental result common to the different approaches is that the mean square displacement of a diffusing particle scales linearly with time. However there have been numerous experimental measurements in which the mean square displacement of diffusing particles scales as a non linear law in time.

These processes are known to exhibit anomalous diffusion. Unlike typical diffusion, anomalous diffusion is described by a power law (Ben-Avraham and Havlin 2000). When an anomalous-type diffusion process is discovered, the challenge faced by the scientific community is to understand the underlying mechanism which causes it. There are a number of frameworks which give rise to anomalous diffusion which are being studied within the physics community. These include Continuous Time Random Walk (CTRW) (Masoliver et al. 2003), diffusion in disordered media (Havlin and Ben-

Avraham 1987) and fractional Brownian motion.

1.2 Fractional Brownian motion

The following definition is taken from (Lamperti 2012)

Definition 1.2.1. *A stochastic process $\{X(t) : t \in T\}$ is defined as a collection of random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$, where Ω is a sample space, $\mathcal{F}$ is a σ-algebra, and P is a probability measure, and the random variables, indexed by some set T, all take values in the same mathematical space S, which must be measurable with respect to some σ-algebra Σ.*

Fractional Brownian motion was first introduced by Kolmogorov in 1940 in (Kolmogorov 1940), within a Hilbert space framework, where it was called the Wiener Helix. It was further studied by Yaglom in (Yaglom 1955). The name fractional Brownian motion has been coined by Mandelbrot and Van Ness, who in 1968 provided in (Mandelbrot and Ness 1968) a stochastic integral representation of this process in terms of a standard Brownian motion. The following definition of fractional Brownian motion can be found in (Biagini et al. 2008*a*).

Definition 1.2.2. *Let Hurst index H be a constant belonging to $(0,1)$. A standard fractional Brownian motion (fBm) $\left(B^H(t)\right)_{t\geq 0}$ is a continuous and centered Gaussian process, which starts at zero, has expectation zero for all $t \in [0,T]$ with the covariance function*

$$\mathbb{E}\left[B^H(t)\, B^H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |\, t - s\,|^{2H}\right).$$

For $H = \frac{1}{2}$, fBm is then a standard Brownian motion process. By Definition 1.2.2 we obtain that a standard (fBm) $\left(B^H(t)\right)_{t\geq 0}$ process has the following properties:

- $B^H(0) = 0$ and $\mathbb{E}\left[B^H(t)\right] = 0 \quad \forall \quad t \geq 0$.
- B^H has homogeneous increments, i.e., $B^H(t+s) - B^H(s)$ has the same law of $B^H(t)$ for $s,t \geq 0$.

- B^H is a Gaussian process and $\mathbb{E}\left[B^H(t)^2\right] = t^{2H}, t \geq 0 \quad \forall \quad H \in (0,1)$.

The existence of fBm follows from the general existence theorem of centered Gaussian processes with given covariance functions (Rogers and Williams 1994, Nourdin 2012). It is also noteworthy that the situation with continuity of fBm trajectories is more involved, as in we can consider a continuous modification of fBm which exists according to Kolmogorov continuity theorem which guarantees that a stochastic process that satisfies certain constraints on the moments of its increments will have a continuous version (Stroock and Varadhan 2007).

1.3 Functionals

The term *functional* refers to a real-valued function defined on vector space. When the paths of a stochastic process are used to model processes of practical importance, functionals will often represent quantities of economic or social interest, such as costs, resource consumption, service delays, and so on. Therefore it is of great interest to be able to determine the probability distribution of functionals. Currently there is no general method for tackling the problem of determining the distribution of a given functional defined on an abstract Wiener space.

A large class of functionals is defined in terms of a *boundary* which takes the same general form as a path, i.e. it is a real-valued function of time. Given a specific boundary, e.g. $b(t)$, the *first-passage* functional before time $T > 0$ is defined as:

Definition 1.3.1.

$$T_b(\psi) \doteq \inf\left[\{t : (\psi(t) < b(t) \wedge b(0) < \psi(0)) \vee (\psi(t) > b(t) \wedge b(0) > \psi(0))\} \cup \{T\}\right].$$

depending on the initial condition under which the stochastic process started. The past supremum and the past infimum will denoted as $\sup_{t \leq T} \psi_t$ and $\inf_{t \leq T} \psi_t$. These functionals are mutually related and have been extensively studied, both in general and in special cases, as highlighted in the sequel. The last-exit functional relative to a continuous boundary function b for a fixed $t > 0$, is defined as

Definition 1.3.2.

$$\tau_b(\psi) \doteq \inf[\{\sup\{t : \psi(t) \geq b(t)\}\} \cup \{T\}] .$$

The methods of this research are, in principle, applicable to all of them. We focus especially, however, on the first passage, supremum and the last-exit functionals, where the boundary is linear.

1.3.1 First Passage, Infimum and Supremum connections

Let T_b be the first time that the process $\psi(t)$ touches the barrier b, such that $\psi(0) > b(0)$ i.e. the first passage time to a lower barrier. The distributions of the first passage time and the minimum are strongly connected, indeed the event $\{T_b > T\}$ is the same as the event $\{\inf_{t \leq T} \psi(t) > b(t)\}$. We can thus write down the distribution of T_b

$$P\left(\inf_{t \leq T}[\psi(t) - b(t)] \leq 0\right) = P(T_b \leq T) . \tag{1.1}$$

Similarly the law of the past supremum $\sup_{t \leq T} \psi(t)$ of a continuous stochastic process before a deterministic time $T > 0$ also presents some major interest in stochastic modeling, such as queuing and risk theories, as it is related to the law of the first passage time T_b above any level b when $b(0) > \psi(0)$, through the relation

$$P\left(\sup_{t \leq T}[\psi(t) - b(t)] \geq 0\right) = P(T_b \leq T) . \tag{1.2}$$

1.3.2 An example of Applications

Since the traffic for core and metropolitan Internet links is an aggregations of flows from many users and many of these users transmit their data independently, it can be assumed to follow a Gaussian process by the central limit theorem (Zukerman et al. 2003). Also, the long range dependence (LRD) characteristics of the Internet traffic has been well established (Leland et al. 1994, Arlitt and Williamson 1997, Williams et al. 2005). Thus, a Gaussian LRD process, the fractional Bownian motion (fBm), has been considered as the model of choice for heavily multiplexed internet traffic and its accuracy has been validated by various publications including (Norros 1995, Chen et al. 2013). Although a queue fed by fBm input has been considered as a fundamentally important model for Internet queueing performance analysis (Norros 1995, W. Willinger and Wilson 1997, Dijkerman and Mazumdar 1994, Hüsler and Piterbarg 1999, Duffield and O'Connell 1995, Chen et al. 2013, 2015), no explicit accurate results for the mean, variance, third central moment and skewness for the occupancy, Q, of an fBm queue, are available.

1.3.2.1 Applications of the supremum functional

Let $X(t)$ denote an arithmetic fractional Brownian process with drift μ, with SDE

$$\mathrm{d}X(t) = \mu \mathrm{d}t + \sigma \mathrm{d}B^H(t), \qquad X(0) = 0$$

where B^H is an fBm with $H \in [0,1]$. There are various motivations for studying the probability distribution of $\sup_{t\in[0,T]} X(t)$. Fractional Brownian motion has been widely accepted as an accurate model of Internet traffic in parts of networks where there is significant aggregation (Leland et al. 1994, Norros 1995, W. Willinger and Wilson 1997, Taqqu et al. 1997). Other applications include studying pursuit problems (Bramson and Griffeath 1991, Li and Shao 2001), in study of extremes and level sets (Azaïs and Wschebor 2009). Hüsler and Piterbarg (Hüsler and Piterbarg 1999, Theorem 1, Equation (9)) (with $\alpha = 2H$, $\beta = 1$) have shown that

$$\mathbb{P}\left(\sup_{t>0} X(t) > x\right) \sim C x^{\frac{2H^2-3H+1}{H}} e^{\left(-\frac{x^{2-2H}(1-H)^{2H-2}|\mu|^{2H}}{2H^{2H}\sigma^2}\right)} \tag{1.3}$$

for some $C > 0$ (which is given explicitly in (Hüsler and Piterbarg 1999)), in the sense that the ratio of the two sides of (1.3) tends to 1 as $x \to \infty$. A variety of results related to tail asymptotics, extreme value theorems, laws of iterated logarithm etc are related to a set of certain constants known as *Pickands' constants*(Pickands 1969*b*,*a*). In particular, the formula for C given in (Hüsler and Piterbarg 1999) is expressed in terms of a Pickands' constant.

According to (Mandjes 2007, Proposition 5.6.2), which is credited to be from (Debicki and Rolski 1999, Theorem 4.3) if Q denotes the stationary contents of an fBm queue,

$$\frac{P(Q > x)}{x^{2H-3+1/H} \exp\left(-\frac{1}{2}\left(\frac{x}{1-H}\right)^{2-2H}\left(\frac{\mu}{H}\right)^{2H}\right)} \to \frac{\alpha(H)}{\sqrt{2\pi}\beta(H)}$$

as $x \to \infty$, in which

$$\alpha(H) = \frac{\mathcal{H}_{2H}\sqrt{\pi}}{2^{(1-H)/2H}\sqrt{H}} \left(\frac{H}{\mu(1-H)}\right)^{H-1} \left(\frac{1}{1-H}\right)^{(2-H)/H}$$

and

$$\beta(H) = \left(\frac{\mu(1-H)}{H}\right)^{H} \frac{1}{1-H}$$

where $\mathcal{H}_x$ denotes the Pickands' constant. Even in the case of $\mu = 0$, only bounds are known, to the best of our knowledge, for the tail probability of the supremum of fractional Brownian motion (Molchan 1999, Aurzada et al. 2011).

1.3.2.2 Applications of the first passage functional

There are also several other motivation for the study of the first passage functional other than its own importance, such as its relation to Burgers equation with random initial data (She et al. 1992, Bertoin 1998). Other applications include studying pursuit problems (Bramson and Griffeath 1991, Li and Shao 2001), in study of extremes and level sets (Azaïs and Wschebor 2009). Also in the field of Gaussian processes a variety of results related to tail asymptotics, extreme value theorems, laws of iterated logarithm etc are related to a set of certain constants known as Pickands' constants(Qualls and Watanabe 1972, Dębicki 2002, 2006, Dębicki and Kisowski 2008, Arendarczyk and Dębicki 2012, Albin 1994, Darling 1983).

Unfortunately little is known about the first passage probability of a fractional Brownian motion process. Martingale methods and Fokker-Plank boundary value problem approaches do not appear to have been successful in the case $H \neq 0.5$ so far.

Michna (Michna 1999) provided a method of simulation of ruin probability over infinite horizon for fractional Brownian motion with $H > \frac{1}{2}$. Guérin et.el (Guérin et al. 2016) have recently introduced an analytical approach to calculate, in the limit of a large confining volume, the mean first-passage time of a Gaussian non-Markovian random walker to a target, though this still just leads to approximation in case of fBm. The hitting time of a level has also been studied by Decreusefond and Nualart in (Decreusefond and Nualart 2008). They obtained an upper bound for the Laplace transform of the hitting time.

The first passage time density of fractional Brownian motion confined to a two-dimensional open wedge domain with absorbing boundaries has also been shown to satisfy

$$\mathbb{P}_{\Omega}(t) \approx t^{-1+\pi\frac{(2H-2)}{2\Omega}}$$

in the limit as $t \to \infty$, where Ω is the wedge angle, in (Jeon et al. 2011). Prakasa Rao (Rao 2013) has obtain some maximal inequalities for a centered fractional Brownian motion with $H \in \left(\frac{1}{2}, 1\right)$ and with a polynomial drift $g(\cdot)$ by studying the asymptotic behaviour of the tail distribution function

$$\mathbb{P}\left(\sup_{0<t<T}\left(B_t^H + g(t)\right) > a\right)$$

as $T \to \infty$ for fixed a and as $a \to \infty$ for fixed T. Molchan (Molchan 1999), has shown

$$T^{H-1}\exp\left[-\beta\sqrt{\log T}\right] \leq \mathbb{P}(\tau_1 > T) \leq T^{H-1}\exp\left[\beta\sqrt{\log T}\right]$$

for some constant β as T goes to infinity. These bounds have been improved by Aurzada (Aurzada et al. 2011) to

$$T^{H-1}(\log T)^{-\alpha_1} \leq \mathbb{P}(\tau_1 > T) \leq T^{H-1}(\log T)^{\alpha_2} \tag{1.4}$$

for some constants $\alpha_1 > \frac{1}{2H}$ and $\alpha_2 > \frac{2}{H} - 1$, for large enough T. In the physics literature, these results are often used in the sense $\approx T^{H-1}$, disregarding the other factors.

For further background information and links to existing literature, we refer the reader to (Li et al. 2004).

As part of this research, for the first time, a link has been discovered between the probability distribution of Supremum of fBm with drift (fBm queue size distribution) and the Generalized Gamma distribution (Stacy 1962) (a special case of the Amoroso distribution (Amoroso 1925)) which leads to very accurate closed-form approximations for these important statistics. The approximations have also been validated by simulation results in (Chen et al. 2015). Some simplified expressions are also provided for these results in certain cases when the Hurst parameter takes certain special values.

1.4 Fokker-Plank/Kolmogorov equations

The theory of Fokker-Plank/Kolmogorov equations, which are second order elliptic and parabolic equations for measures, goes back to Kolmogorov's works (Kolmogoroff 1931) along with a number of earlier works in the physics literature by Fokker, Smoluchowski, Planck and Chapman (Fokker 1914, Planck 1917). In the recent years several important monographs have also appeared (Bogachev et al. 2015, Risken 1996). One of the main objects here is the elliptic operator of the form

$$L_{A,b}f = \mathrm{Tr}\left(AD^2 f\right) + \langle b, \nabla f\rangle, \quad f \in {C_0}^{\infty}(\Omega),$$

where $A = \left(a_{ij}\right)$ is a mapping on a domain $\Omega \subset \mathbb{R}^d$ with values in the space of nonnegative symmetric linear operators on $\mathbb{R}^d$ and $b = (b_i)$ is a vector field on Ω. In coordinate form, $L_{A,b}$ is given by the expression

$$L_{A,b}f = \sum_{i,j=1}^{d} a_{ij}\partial_{x_i}\partial_{x_j} f + \sum_{i=1}^{d} b_i \partial_{x_i} f.$$

From this operator we can define the adjoint ${L_{A,b}}^*$ via the duality relation

$$\int_{\Omega} \left(L_{A,b}f\right) d\mu = \int_{\Omega} f d\left({L_{A,b}}^*\mu\right) \quad \forall f \in {C_0}^{\infty}(\Omega),$$

and $\mu \in M\left(\mathbb{R}^d\right)$, the space of locally finite (possibly signed) Borel measures. The weak elliptic equation is associated with operator $L_{A,b}$ as

$$L_{A,b}{}^*\mu = 0, \tag{1.5}$$

for Borel measures on Ω, furthermore we say μ solves (1.5) if

$$\int_{\Omega} L_{A,b} f \mathrm{d}\mu = 0 \quad \forall f \in {C_0}^{\infty}(\Omega), \tag{1.6}$$

where we assume that $b_i, a_{ij} \in {L_{loc}}^1(\mu)$, the class of locally integrable functions. It is noteworthy that this is the same as saying that (1.5) holds in the sense of distributions, where we recall that any $\mu \in M\left(\mathbb{R}^d\right)$ gives rise to a distribution in the natural way.

Similarly, one can consider parabolic operators and parabolic Fokker-Planck Kolmogorov equations for measures on $\Omega \times (0,T)$ of the type

$$\partial_t \mu = L_{A,b}{}^*\mu, \tag{1.7}$$

also in the sense of distributions (with A,b possibly time-dependent). Hence, the study of these equations reduces to studying partial differential equations in the distributional setting, as shown in (Evans 2010, Gilbarg and Trudinger 2015, Friedman 2008, Renardy and Rogers 2006). However it is crucial that a priori Fokker-Planck-Kolmogorov equations are equations for measures, not for rough distributions (Bogachev et al. 2015). This becomes relevant when the coefficients are singular or degenerate and, in particular, in the infinite-dimensional case, where no Lebesgue measure exists. The theory of these equations for measures is now a rapidly growing area with connections to many other areas of mathematics such as real analysis, partial differential equations and stochastic analysis.

1.4.1 Probabilistic Motivation

For exposition of the semi-group of operators generated by the transition function of a Markov process, we recommend the reader to (Kallenberg 2006, Revuz and Yor 2013). For further details concerning the semi-group theory from a more functional analysis point of view, please see (Phillips and Hille 1957, Yosida 1995). Knowledge of the infinitesimal generator enables one to derive important characteristics of the initial process; the classification of Markov processes amounts to the description of their corresponding infinitesimal generators (Sarymsakov 1954, Gikhman and Skorokhod 2015). For semi-groups of transformations associated to parabolic partial differential equations, see (Feller 1952).

Suppose that $\xi = (\xi(x,t))_{t\geq 0}$ is a diffusion process in $\mathbb{R}^d$ governed by the stochastic differential equation

$$\mathrm{d}\xi(x,t) = b(\xi(x,t))\,\mathrm{d}t + \sigma(\xi(x,t))\,\mathrm{d}W_t, \quad \xi_0 = x_0. \tag{1.8}$$

The generator of the transition semigroup $\{T_t\}_{t\geq 0}$ has the form $L_{A,b}$, where $A = \sigma\sigma^*/2$. The matrix A in the operator $L_{A,b}$ is known as the diffusion matrix or diffusion coefficient and the vector field b is called the drift coefficient or just drift. Denote by $P(x,t)(B)$ the probability that ξ moves from point $x \in \mathbb{R}^d$ to a measurable set $B \in \mathbb{B}\left(\mathbb{R}^d\right)$ (the Borel measurable sets in $\mathbb{R}^d$) in time $t \geq 0$. Hence, $P(x,t)(\cdot)$ is a probability measure on $\left(\mathbb{R}^d, \mathbb{B}\right)$ and $(x,t) \mapsto P(x,t)(\cdot)$ is the so-called transition probability function. The transition probabilities of ξ satisfy the corresponding parabolic equation (1.7)(Friedman 2012). If the transition function satisfies certain locality conditions, then we say ξ is a Markovian diffusion, more details are presented in (Øksendal 2003, Stroock 2008). A probability measure μ is called invariant (Bogachev et al. 2001, Von Neumann 1941) for $\{T_t\}_{t\geq 0}$ if the following identity holds:

$$\int_{\mathbb{R}^d} T_t f \mathrm{d}\mu = \int_{\mathbb{R}^d} f \mathrm{d}\mu \quad \forall f \in C_b\left(\mathbb{R}^d\right). \tag{1.9}$$

Any invariant probability measure μ of ξ (if such exists) satisfies (1.5). Measures satisfying (1.5) are called infinitesimally invariant, as this equation has deep connections with invariance with respect to the corresponding semigroups(Bogachev et al. 2015). Also, if there is an invariant probability measure μ, then $\{T_t\}_{t\geq 0}$ extends to $L^1(\mu)$ and is strongly continuous. Denoting L to be the corresponding generator with domain $D(L)$. Then (1.9) is equivalent to the equality

$$\int_{\mathbb{R}^d} Lf \mathrm{d}\mu = 0 \quad \forall f \in D(L).$$

Under some reasonable assumptions on A and b, the generator of the semigroup associated with the diffusion governed by the indicated stochastic equation coincides with $L_{A,b}$ on $C_0^\infty\left(\mathbb{R}^d\right)$, but the invariance of the measure in the sense of (1.9) is not the same as (1.6), and the class $C_0{}^\infty\left(\mathbb{R}^d\right)$ may be much smaller than $D(L)$ (Bogachev et al. 2015, Stroock and Varadhan 2007).

A diffusion SDE, can be used to define a deterministic function of space and time in two fundamentally different ways. First by considering the expected value of some function, as a function of the initial position and time. Secondly by considering the probability of being in a certain state at a given time, given the knowledge of the initial state and time (Øksendal 2003). Thus when studying the first scenario, one explores the mathematical ideas of the backward Kolmogorov equation and the Feynman-Kac formula. When studying the latter viewpoint, which is in fact dual to the first viewpoint, the evolving probability density solves a different PDE, the forward Kolmogorov equation, which is actually the adjoint of the backward Kolmogorov equation.

The link between partial differential equations, boundary value problems and stochastic processes and stochastic analysis has already been well established over the past century. Dynkin's formula (Øksendal 2003) and Feynman-Kac formula (Klebaner 2005) are just of some the prominent results which highlight this interplay for diffusion processes.

1.4.2 Boundary Conditions

It is of interest to consider how and when a diffusion process crosses a barrier. Probabilistically, thinking about barriers means considering exit times. On the PDE side this leads us to consider boundary value problems for the backward and forward Fokker-Plank/Kolmogorov equations.

We now briefly summarize some of the common types of boundary conditions for the Fokker-Plank/Kolmogorov equations. Let us denote a Brownian motion beginning at x_0.

$$\mathrm{d}X_t = \mathrm{d}W_t, \quad X_0 = x_0.$$

We know that the transition density $p(x,t)$ satisfies the Fokker-Plank/Kolmogorov equation

$$\left(-\partial_t + \frac{1}{2}\partial_{xx}{}^2\right) p = 0.$$

1. Natural boundary conditions: This is the condition that $p\,(x,t) \mapsto 0$ as $x \mapsto \infty$ or $x \mapsto -\infty$. With the decay to zero being sufficiently fast to ensure the normalization integral is
$$\int_{-\infty}^{\infty} P\,(x,t)\,\mathrm{d}x = 1.$$

2. Absorbing boundary conditions: Now, suppose that we restrict the process X_t to a region $[L,U]$. We say X_t has an absorbing boundary condition at L, if, when X_t hits L, it stays there forever, which is modeled using the Dirichlet boundary condition $p(L,t) = 0$.

   In case of processes associated with diffusion, absorbing boundary conditions are used to study first passage, supremum and infimum of probability densities corresponding to these processes.

3. Reflecting boundary conditions: Likewise, we say that X_t has a reflecting boundary condition at L, if, when X_t hits L, it is instantaneously reflected back into the region $[L,U]$, which is modeled as the Neumann boundary condition $\partial_x(L,t) = 0$.

1.4.3 Fractional Brownian motion

The Fokker-Plank/Kolmogorov equations are now well known for Fractional Brownian motion dependent processes as shown by subsequent lemma. Although the traditional techniques associated with studying first passage time density via the initial boundary value problems for Markovian processes are not applicable, as is shown in chapter 2. This is, due to the non-Markovian nature of the underlying processes.

The derivation of the Kolmogorov Forward Equation formula for fractional Brownian motion can be found in (Zeng et al. 2012). It should be noted this result was provided without any restrictions on the range of Hurst parameter H. Even though the original proof relies on fractional Itô formula proven for $H \geq \frac{1}{2}$ in (Duncan et al. 2000), the result has since been extended by modifying the fractional white noise approach (Hu and Øksendal 2003) to cover the range $0 < H < 1$ in (Bender 2003). Similar equations have also been considered in (Baudoin and Coutin 2007, Hahn and Umarov 2011).

Lemma 1.4.1. *(Kolmogorov Forward Equation formula for fBm). The Kolmogorov Forward Equation, also known as the Fokker-Plank equation associated with a nonlinear fractional Brownian motion driven SDE of the form*

$$dX(t) = f(X(t),t)\,dt + g(X(t),t)\,dB^H(t) \tag{1.10}$$

where $f(x(t),t)$ and $g(x(t),t)$ are real-valued functions, not equal to 0 possessing the indicated derivatives and $B^H(t)$ is a standard fBm with Hurst parameter H is

$$\begin{aligned}\frac{\partial p(x,t)}{\partial t} &= -\frac{\partial\left(f(x(t),t)\,p(x,t)\right)}{\partial x}\\ &+ \frac{\partial^2\left(g(x(t),t)\,p(x,t)\int_0^t g(x(s),s)\,\phi(s,t)\,ds\right)}{\partial x^2}\end{aligned} \tag{1.11}$$

where

$$\phi(s,t) = H(2H-1)\,|s-t|^{2H-2}.$$

We do not investigate the details of how the integral in the SDE 1.10 is defined. Several alternatives are considered, for example, in (Mishura 2008) and (Biagini et al. 2008*a*).

1.5 Derivatives associated with Stochastic processes

1.5.1 Radon-Nikodym derivatives of transformations

In the theory of stochastic processes, the influence of measure theory can be seen in the use of the term *random function*, instead of stochastic process, as in the books (Yaglom 2004, Lifshits 1995), for example. The concept that the study of stochastic processes is "really" the study of measures on spaces of functions entails difficulties as well as opportunities. In particular, it can be shown (Kuo 1975) that there is no rotationally or translationally invariant measure with bounded values on bounded sets, and positive values on open sets, on any infinite dimensional normed vector space, and hence the concept of *probability density* is no longer applicable. This challenge is potentially addressed by adopting Gaussian measures as the "standard measure" on function spaces.

The Cameron-Martin theorem(Cameron and Martin 1944) provides a partial substitute for the concept of a density in that it provides a "likelihood ratio" between two different Gaussian measures which differ by a shift from the *Cameron-Martin space* (also termed the space of measurable shifts), which is the space of vectors, a shift by which gives rise to a well-defined non-zero Radon-Nikodym derivative.

This result was extended by Girsanov(Girsanov 1960) to enable comparison of arbitrary *Ito* measures, i.e. measures constructed by a stochastic differential equation driven by Brownian motion. Extensions of the Girsanov theorem to processes defined by stochastic differential equations based on fractional Brownian motion, have also been developed in (Norros et al. 1999, Decreusefond 2000, 2003, Mishura 2008, Biagini et al. 2008*b*).

These results may naturally be applied when the two measures being compared are the same, but the mapping (the stochastic differential equation) is non-trivial, and in this case the result compares, in a sense, the same measure at two locations. The Radon-Nikodym derivative formula given by these theorems is a local result in the sense that if the shift was replaced by a different function identical to the shift in a neighbourhood of a certain point, the Radon-Nikodym between the two measures would be the same, at this point. However, since the Radon-Nikodym derivative incorporates in an appropriate way the dilation or concentration effected by the transformation at each location, the Radon-Nikodym derivative can't be isolated from the choice of transformation linking the two measures.

A different extension of the Cameron-Martin theorem, to the situation where the transformation between measures is affine, was considered in (Segal 1958) and (Feldman 1958), and to the general nonlinear case in (Ramer 1974) and (Cruzeiro 1983*b*). The resulting formula in the first three cases takes the form of a product of two terms, one basically the same as the Cameron-Martin formula, and the other term is effectively the Jacobian of the nonlinear transformation.

Since a Jacobian is defined in terms of the derivative of the transformation (in this case the Fréchet derivative is appropriate), Ramer observed that the result can still hold when the traditional definition of Jacobian fails. He replaced the Jacobian by an expression partially induced, by continuity, from the action of the mapping on the Cameron-Martin space of the measure.

Even with this relaxation in the definition of the Jacobian, it appears to be very difficult to find non-trivial examples where Ramer's formula for the Radon-Nikodym derivative can be evaluated, so it is appropriate to consider additional, or alternative ways in which the definition of Jacobian, or derivative of a transformation, can be generalised and made easier to apply.

A Malliavin derivative of a transformation between measure spaces may exist when the traditional Frechét derivative does not. One of the objectives of the Malliavin derivative is to evaluate a derivative which can't be evaluated by conventional means. Note: it is not the derivative of a path that concerns us here, but the derivative of a transformation of paths. The conventional derivative to be used here is the Frechét derivative. But the limits in the Frechét derivative either can't be evaluated, or at least are very difficult to evaluate.

This situation can be tackled in many ways. More assumptions, look for transformations where it can be evaluated, etc. The Malliavin approach is to change the definition of what a derivative is, and how convergence in the definition, to this derivative, takes place.

The $\mathcal{P}$-measurable derivative introduced in Subsection 3.2.2 takes exactly the same form as a Frechét derivative, namely it is still a linear operator on the space of paths, but the limits required in its definition are weaker than conventional limits. This is achieved by using a limit in $\mathcal{P}$-measure in the definition of the derivative.

The concept of limits in $\mathcal{P}$-measure, where $\mathcal{P}$ is some measure or finitely additive set function, has a long history, going back to the start of measure theory, and continuing to this day. In this research $\mathcal{P}$ will always refer to the *standard Gauss measure* (Kuo 1975), which, despite the name, is a finitely additive set function, and not, in general, a measure. The key highlights of the theory of finitely additive set functions ($\mathcal{P}$ say), the $\mathcal{P}$-measurable functions, and integration of $\mathcal{P}$-measurable functions with respect to the measure $\mathcal{P}$, including the appropriate Radon-Nikodym theorem, are presented in (Dunford and Schwartz 1957).

Once the step of using $\mathcal{P}$-measurable functions uniformly in the theory is taken, the Ramer theorem takes exactly the same form as the corresponding Jacobian theorem which would apply if the spaces were all finite-dimensional. The proof of this theorem is also dramatically simpler than in (Ramer 1974).

In (Cruzeiro 1983*b*), the context was not Gaussian measure spaces in general but the specific case of the space $C[0,1]$ with the Wiener measure. In place of a non-linear transformation, Cruzeiro considers a vector field, which defines a flow on the Wiener space. The Radon-Nikodym derivative is now expressed more directly as an integral involving the divergence of the vector field.

In the present research not only the Jacobian, but the formula for the Radon-Nikodym derivative itself is found on the Cameron-Martin space, first, and then the result on the path space is induced by continuity from its value on Cameron-Martin space. This allows us to avoid highly restrictive conditions on the class of non-linear mappings to which the theory can apply.

In particular, it is not assumed that the mapping differs from the identity by a function whose derivative exists and is a Hilbert-Schmidt class operator. Another way to express this generalisation is that we can now define a transformation of paths which uses a different basis for the transformed path than for the original path. This has been found to be a crucial technique in applications.

The idea that it may be more useful to obtain the Radon-Nikodykm derivative *on the Cameron-Martin space* also motivates the authors of (Bagchi and Mazumdar 1993), where the application in mind is to generalise the concept of stochastic process (or random function) to include white noise, which is not a well-defined random function in the conventional sense. The role of measures, in the Cameron-Martin space, is taken by *finitely additive set functions*, and while there is no probability measure for white noise on the whole path space, the identity mapping can be interpreted as a white noise random function defined on the Cameron-Martin space.

There is scope for a theorem which generalises both the Girsanov theorem *and* the Ramer theorem. We present such a generalisation of the Girsanov theorem for fBm in this research, which is not limited to changes in linear drift only in theorem 3.2.3. Although our goal is primarily to make the Ramer theorem easier to apply, by identifying a different type of regularity condition on the function, f, which makes it easy to identify functions to which it can be applied, and easy to evaluate the Radon-Nikodym derivative formula. The key new regularity condition on f which is adopted is $\mathcal{P}$-measurability.

A simple form of the Ramer theorem is shown to apply to the transformation applying on a Hilbert space equipped with an additive set function. Except for needing the theory of $\mathcal{P}$-measurable functions, the proof of this result is quite simple.

Also, it is shown that a Radon-Nikodym derivative between finitely additive measures on the Cameron-Martin space of a Gaussian measure can be uniquely extended to the original space and provides a Radon-Nikodym derivative there. Using this extension theorem, the Ramer theorem on H can be readily used to derive Radon-Nikodym derivatives between a Gaussian measure and its transformation by a $\mathcal{P}$-measurable function between spaces.

In Section 3.1 the abstract Wiener space approach to defining Gaussian measures is reviewed, and in Section 3.2 the theory of *finitely* additive set functions, and, denoting a specific additive set function by $\mathcal{P}$, the theory of $\mathcal{P}$-measurable functions, as developed in (Dunford and Schwartz 1957), is reviewed. Theorem 3.2.4 proves that any $\mathcal{P}$-measurable function on H, the Cameron-Martin space, has a unique P-measurable (where P is the corresponding measure on Ω) extension to the space Ω.

1.5.2 The Girsanov Theorem

Although more general versions of Girsanov formula for fractional Brownian motion are known (Decreusefond et al. 1999, Mishura and Valkeila 2001), in this research we consider Norros et al's version of Girsanov formula for fractional Brownian motion. A proof of the following Girsanov formula for fractional Brownian motion can be found in (Norros et al. 1999), where the term the fundamental martingale is also coined for the process M_t.

Proposition 1.5.1. *(The Girsanov formula for fBm). Let $B^H(t)$ denote fractional Brownian motion with mean 0 and variance t^{2H}, for all $H \in (0,1)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let a be a scalar. Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ via the Radon Nikodym derivative with respect to $\mathbb{P}$*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\mu M_T - \frac{1}{2}\mu^2 \langle M,M\rangle_T\right\} \tag{1.12}$$

where

$$M_T = \frac{1}{2H\Gamma\left(\frac{3}{2}-H\right)\Gamma\left(H+\frac{1}{2}\right)} \int_0^T \left(s\left(T-s\right)\right)^{\frac{1}{2}-H} dB_s{}^H. \tag{1.13}$$

The process M_T is a martingale with independent increments, zero mean and variance function c^2T^{2-2H} where

$$c = \sqrt{\frac{\Gamma\left(\frac{3}{2}-H\right)}{2H\left(2-2H\right)\Gamma\left(H+\frac{1}{2}\right)\Gamma\left(2-2H\right)}}. \tag{1.14}$$

Then the process defined, for all $t \in [0,T]$, by $B^H(t) + \mu t$ is the standard $\mathbb{Q}$-fractional Brownian motion on $[0,T]$. In other words, under probability measure $\mathbb{Q}$, $B^H(t)$ restricted to $t \in [0,T]$ is distributed as an arithmetic fractional Brownian motion with drift μ.

It's noteworthy that using the variance of M_t, (1.12) can also be re-written as

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left\{\mu M_T - \frac{1}{2}\mu^2 c^2 T^{2-2H}\right\}. \tag{1.15}$$

We now present a corollary of theorem 1.5.1 from our current research which makes it easier to apply to a wider range of parameters.

Corollary 1.5.1. *Let $X(t) = bB^H(t)$ be an arithmetic fractional Brownian motion with volatility b, for all $H \in (0,1)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let a be a scalar. Define a new probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ via the Radon Nikodym derivative with respect to $\mathbb{P}$*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\frac{a}{b}M_T - \frac{1}{2}\frac{a^2}{b^2}c^2T^{2-2H}\right\}. \tag{1.16}$$

Then the process defined, for all $t \in [0,T]$, by $Z(t) = X(t) + at$ is an arithmetic $\mathbb{Q}$-fractional Brownian motion process on $[0,T]$ with volatility b. In other words, under probability measure $\mathbb{Q}$, $X(t)$ restricted to $t \in [0,T]$ is distributed as an arithmetic fractional Brownian motion with drift a and volatility b.

Proof. The change of measure (1.15) turns the process $\mathrm{d}B^H(t)$ into the process $\mathrm{d}B^H(t) + \mu\mathrm{d}t$. Therefore, if we use this weight function, on the process $X(t)$ we get $\mathrm{d}X(t) = b\left(\mathrm{d}B^H(t) + \mu\mathrm{d}t\right)$. This becomes the process $Z(t)$ if

$$\mu = \frac{a}{b}.$$

Therefore, the likelihood ratio between $X(t)$ and $Z(t)$ is as claimed. □

1.5.3 Derivatives on Abstract Wiener Space

In measure theory, abstract Wiener space is used to construct a strictly positive and locally finite measure on an infinite-dimensional vector space. A more formal presentation of this concept can be found in section 3.1. The following two definitions and the subsequent proposition can be found in (Di Nunno et al. 2009). Let X be a Banach space that is, a complete, normed vector space over R. The set of all bounded linear functionals is called the dual of X and is denoted by X^*. Let U be an open subset of a Banach space X and let f be a function from U into R.

Definition 1.5.1. *We say that f has a directional derivative (or Gateaux derivative) $D_y f(x)$ at $x \in U$ in the direction $y \in X$ if*

$$D_y f(x) := \frac{d}{d\varepsilon}\left[f(x + \varepsilon y)\right]_{\varepsilon=0} \in \mathrm{R} \tag{1.17}$$

exists.

Definition 1.5.2. *We say that f is Fréchet-differentiable at $x \in U$, if there exists a bounded linear map $A : X \mapsto \mathrm{R}$, that is, $A \in X^*$, such that*

$$\lim_{h \to 0; h \in X} \frac{|f(x+h) - f(x) - A(h)|}{\|h\|} = 0. \tag{1.18}$$

We write

$$f'(x) = A \in X^*, \tag{1.19}$$

for the Fréchet derivative of f at x.

Proposition 1.5.2. *If f is Fréchet differentiable at $x \in U \subset X$, then f has a directional derivative at x in all directions $y \in X$ and*

$$D_y f(x) = \langle f'(x), y \rangle \in \mathbb{R}. \tag{1.20}$$

Conversely, if f has a directional derivative at all $x \in U$ in all directions $y \in X$ and the linear map

$$y \mapsto D_y f(x), \quad y \in X \tag{1.21}$$

is continuous $\forall x \in U$, then there exists an element $\nabla f(x)$ in X^* such that

$$D_y f(x) = \langle \nabla f(x), y \rangle. \tag{1.22}$$

If this map $x \mapsto \nabla f(x) \in X^*$ is continuous on U, then f is Fréchet differentiable and

$$f'(x) = \nabla f(x). \tag{1.23}$$

1.5.3.1 Gross-Sobolev derivative

Let $W = C_0([0,1], \mathbb{R})$ be the classical Wiener space equipped with Wiener measure μ. In order to do Sobolev type analysis on W, so that it can be applied to random variables encountered in applications, a differentiation operator has been constructed. As Fréchet derivative has been found to be unsatisfactory (Üstünel 2006), since many frequently encountered Wiener functionals (Wiener integrals or the solutions of stochastic differential equations with smooth coefficients) are not even continuous with respect to the Fréchet norm of W. Therefor, what is required is to define a derivative on the $L^p(\mu)$-spaces of random variables. In order to be able to do so the following property is necessary: if $F, G \in L^p(\mu)$, and if $F = G$ μ- a.s., it is natural to ask that their derivatives are also equal a.s. For this, the only way is to choose y in a specific subspace of W, namely the Cameron-Martin space H:

$$H = \{h : [0,1] \mapsto \mathbb{R} / h(t) = \int_0^t \hat{h}(s)\, ds, |h|_H{}^2 = \int_0^1 |\hat{h}(s)|^2 ds\}. \tag{1.24}$$

We now briefly sketch the construction of this derivative, for more details please see (Üstünel 2010). Let $S(\mathbb{R}^n)$ denote the space of infinitely differentiable, rapidly decreasing functions on $\mathbb{R}^n$. If $F : W \mapsto \mathbb{R}$ is a function of the following type (called cylindrical):

$$F(x) = f(W_{t_1}(x), \cdots, W_{t_n}(x)), \quad f \in S(\mathbb{R}^n), \tag{1.25}$$

we define, for $y \in H$, the directional derivative as

$$D_y F(x) = \frac{d}{d\varepsilon}[F(x + \varepsilon y)]_{\varepsilon=0}. \tag{1.26}$$

Since $W_t(x+y) = W_t(x) + y(t)$, we obtain

$$D_y F(x) = \sum_{i=1}^n \partial_i f(W_{t_1}(x), \cdots, W_{t_n}(x))\, y(t_i), \tag{1.27}$$

in particular

$$D_y W_t(x) = y(t) = \int_0^t \hat{h}(s)\, ds = \int_0^t 1_{[0,t]}(s)\, \hat{h}(s)\, ds. \tag{1.28}$$

If we denote by U_t the element of H defined as $U_t(s) = \int_0^s 1_{[0,t]}(r)\, dr$, we have $D_y W_t(x) = \langle U_t, y\rangle_H$. Looking at the linear map $y \mapsto D_y F(x)$ we see that it defines a random element with values in H^*, since we have identified H with H^*, $D(F)$ is an H-valued random variable. It can be shown that D is a closable operator on any $L^p(\mu)$ $(p > 1)$ and it can be extended to larger classes of Wiener functionals than the cylindrical ones, as defined below.

Definition 1.5.3. *$F \in Dom_p(D)$ if and only if there exists a sequence $(F_n; n \in \mathbb{N})$ of cylindrical functions such that $F_n \mapsto F$ in $L^p(\mu)$ and (DF_n) is Cauchy in $L^p(\mu, H)$. Then, for any $F \in Dom_p(D)$, we define*

$$D(F) = \lim_{n\to\infty} D(F_n). \tag{1.29}$$

The extended operator D is known as Gross-Sobolev derivative.

1.5.3.2 Stochastic derivative

The definitions and theory in the current and next subsection are much more comprehensively presented in (Di Nunno et al. 2009).

Definition 1.5.4. *Let $F : W \mapsto \mathbb{R}$ be a random variable, choose $g \in L^2([0,T])$, and consider*

$$\gamma(t) = \int_0^t g(s)\, ds \in W. \tag{1.30}$$

Then we can define the directional derivative of F similar to (1.26).

Definition 1.5.5. *Assume that $F : W \mapsto \mathbb{R}$ has a directional derivative in all directions γ of the form $\gamma \in H$ in the strong sense, that is,*

$$D_\gamma F(\omega) := \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \tag{1.31}$$

exists in $L^2(P)$. Assume in addition that there exists $\psi(t,\omega) \in L^2(P \times \lambda)$ such that

$$D_\gamma F(\omega) = \int_0^T \psi(t,\omega)\, g(t)\, dt, \quad \forall \gamma \in H. \tag{1.32}$$

Then we say that F is differentiable and we set

$$D_t F(\omega) := \psi(t,\omega). \tag{1.33}$$

We call $D.F \in L^2(P \times \lambda)$ the stochastic derivative of F. The set of all differentiable random variables is denoted bu $\mathfrak{D}_{1,2}$.

1.5.3.3 Malliavin derivative

One of the main tools of modern stochastic analysis is Malliavin calculus. In a nutshell, this is a theory providing a way of differentiating random variables defined on a Gaussian probability space (typically Wiener space) with respect to the underlying noise. Malliavin calculus, a differential calculus in Wiener space is named after Paul Malliavin. His seminal work (Malliavin 1978), led to a proof that Hörmander's condition implies the existence and smoothness of a density for the solution of a stochastic differential equation. Hörmander's original proof was based on the theory of partial differential equations.

This new approach proved to be extremely successful and soon a number of authors studied variants and simplifications of the original proof (Bismut 1981*b*,*c*, Kusuoka and Stroock 1984, 1985, Norris 1986).

Even now, more than three decades after Malliavin's original work, his techniques prove to be sufficiently flexible to obtain related results for a number of extensions of the original problem, including for example SDEs with jumps (Takeuchi 2002, Ishikawa and Kunita 2006, Cass 2009, Takeuchi 2010), infinite-dimensional systems(Ocone 1988, Baudoin et al. 2005, Mattingly and Pardoux 2006), and SDEs driven by Gaussian processes other than Brownian motion(Baudoin and Hairer 2007, Cass and Friz 2010, Hairer et al. 2011, Cass et al. 2015).

Let $\mathbb{P}$ denote the family of all random variables $F : \Omega \mapsto \mathrm{R}$ of the form

$$F = \phi(\theta_1, \cdots, \theta_n), \tag{1.34}$$

where $\phi(x_1, \cdots, x_n) = \sum_\alpha a_\alpha x^\alpha$, with $x^\alpha = {x_1}^\alpha \cdots {x_n}^\alpha$ and $\alpha = (\alpha_1, \cdots, \alpha_n)$, is a polynomial and $\theta_i = \int_0^T f_i(t)\,\mathrm{d}W(t)$ for some $f_i \in L^2([0,T]), i = 1, \cdots, n$. Such random variables are called Wiener polynomials. It should be noted that $\mathbb{P}$ is dense in $L^2(P)$. Let $F = \phi(\theta_1, \cdots, \theta_n) \in \mathbb{P}$. Then $F \in \mathfrak{D}_{1,2}$ and

$$D_t F = \sum_{i=1}^{n} \frac{\partial \phi}{\partial \theta_i}(\theta_1, \cdots, \theta_n) . f_i(t). \tag{1.35}$$

We now introduce the norm $||\cdot||_{1,2}$ on $\mathfrak{D}_{1,2}$:

$$||F||^2{}_{1,2} := ||F||^2{}_{L^2(P)} + ||D_t F||^2{}_{L^2(P\times\lambda)}, \quad F \in \mathfrak{D}_{1,2}. \tag{1.36}$$

Unfortunately, as it is not clear if $\mathfrak{D}_{1,2}$ is closed under this norm, hence one works instead with the following family.

Definition 1.5.6. *We define $\mathbb{D}_{1,2}$ to be the closure of the family $\mathbb{P}$ with respect to the norm $||\cdot||_{1,2}$.*

Thus $\mathbb{D}_{1,2}$ consists of all $F \in L^2(P)$ such that there exists $F_n \in \mathbb{P}$ with the property that $F_n \to F$ in $L^2(P)$ as $n \to \infty$ and $\{D_t F_n\}_{n=1}^\infty$ is convergent in $L^2(P \times \lambda)$. If this is the case, it is tempting to define

$$D_t F = \lim_{n\to\infty} D_t F_n. \tag{1.37}$$

It can be shown that this defines D_tF uniquely, by considering the difference $H_n = F_n - G_n$, the operator D_t is closable, that is if the sequence $\{H_n\}_{n=1}^{\infty} \subset \mathbb{P}$ is such that $H_n \mapsto 0$ in $L^2(P)$ as $n \mapsto \infty$ and $\{D_tH_n\}_{n=1}^{\infty}$ converges in $L^2(P \times \lambda)$ as $n \mapsto \infty$, then $\lim_{n\to\infty} D_tH_n = 0$.

Definition 1.5.7. *Let $F \in \mathbb{D}_{1,2}$, so that there exists $\{F_n\}_{n=1}^{\infty} \subset \mathbb{P}$ such that $F_n \to F$ in $L^2(P)$ and $\{D_tF_n\}_{n=1}^{\infty}$ is convergent in $L^2(P \times \lambda)$. Then we define*

$$\mathcal{D}_tF = \lim_{n\to\infty} D_tF_n \quad in \quad L^2(P \times \lambda) \tag{1.38}$$

and

$$\mathcal{D}_\gamma F = \int_0^T \mathcal{D}_tF \cdot g(t)\,dt \quad \forall \quad \gamma(t) = \int_0^t g(s)\,ds \in H, \tag{1.39}$$

with $g \in L^2([0,T])$. $\mathcal{D}_tF$ is called the Malliavin derivative of F.

It has also been shown in (Di Nunno et al. 2009) that if $F \in \mathfrak{D}_{1,2} \cap \mathbb{D}_{1,2}$, then the two derivatives coincide. More precisely, suppose that $\{F_n\}_{n=1}^{\infty} \subset \mathbb{P}$ has the properties $F_n \to F$ in $L^2(P)$ and $\{D_tF_n\}_{n=1}^{\infty}$ converges in $L^2(P \times \lambda)$. Then $D_tF = \lim_{n\to\infty} D_tF_n$ in $L^2(P \times \lambda)$. Hence

$$\mathcal{D}_tF = D_tF \quad \forall \quad F \in \mathfrak{D}_{1,2} \cap \mathbb{D}_{1,2}. \tag{1.40}$$

1.6 Fractional calculus on an interval

We briefly recall the main features of the classical theory of fractional calculus by following (Biagini et al. 2008*a*, Zähle 1999). For a complete treatment of this subject, we refer to (Samko et al. 1993, Oldham and Spanier 1974).

Definition 1.6.1. *Let f be a deterministic real-valued function that belongs to $L_1(a,b)$, where (a,b) is a finite interval of* R. *The fractional Riemann Liouville integrals of order $\alpha > 0$ are determined at almost every $x \in (a,b)$ and defined as the*

(i) Left-sided version:

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\,dy.$$

(ii) Right-sided version:

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, dy.$$

For $\alpha = n \in \mathrm{N}$ *one obtains the n-order integrals*

$$I_{a+}^{n} f(x) := \int_a^x \int_a^{x_{n-1}} \cdots \int_a^{x_2} f(x_1)\, dx_1 dx_2 \cdots dx_n$$

and

$$I_{b-}^{n} f(x) := \int_x^b \int_{x_{n-1}}^b \cdots \int_{x_2}^b f(x_1)\, dx_1 dx_2 \cdots dx_n.$$

Definition 1.6.2. *Consider* $\alpha < 1$. *We define fractional Liouville derivatives as*

$$D_{a+}^{\alpha} f := \frac{d}{dx} I_{a+}^{1-\alpha} f$$

and

$$D_{b-}^{\alpha} f := \frac{d}{dx} I_{b-}^{1-\alpha} f$$

if the right-hand sides are well-defined (or determined). For any $f \in L^1(a,b)$ *one obtains*

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f = f, \quad D_{b-}^{\alpha} I_{b-}^{\alpha} f = f.$$

The fractional derivative of the function x^p (Oldham and Spanier 1974, §2.9):

$$\frac{d^q}{dx^q} x^p = \frac{\Gamma(p+1) x^{p-q}}{\Gamma(p-q+1)}. \tag{1.41}$$

1.7 Flows and probability theory

Flows have been extensively studied in dynamical systems and in Ergodic theory (Ambrose 1941, Lind 1975). The original motivation behind the idea of a flow was rooted in classical mechanics, where given a set of all possible states of a given dynamical system in the phase space, the flow map is the law of motion which prescribes that if the system is at some initial state now then how it will evolve to another state after a unit of time. As the theory of dynamical systems splits into subfields which differ by the structure which one imposes on the state space and into Ergodic theory, several kinds of flows have been studied such as continuous flow, measurable flow, special flow etc. The idea of a vector flow, that is, the flow determined by a vector field, occurs in the areas of differential topology, Riemannian geometry and Lie groups. Similarly in recent years the relationship between stochastic differential equations and stochastic flows of diffeomorphisms has also been extensively explored (Watanabe 1983, Malliavin 1978, Kunita and Ghosh 1986, Ikeda and Watanabe 2014, Elworthy 1978, Bismut 1981*a*).

A quasi-invariant measure μ with respect to a transformation T, from a measure space X to itself, is a measure which remains equivalent to the transformed measure, $\mu \circ T^{-1} \approx \mu$. A flow (for example from a vector field) which generates maps transforming the underlying measure into a family of mutually absolutely continuous transformed measures, is referred to as a quasi-invariant flow. The existence of quasi-invariant flow of measurable maps associated to a vector field with Sobolev regularity was first studied by Cruzeiro (Cruzeiro 1983*b*). As there is no analogue of Lebesgue measure in the infinite-dimensional case, standard Gaussian measure, has been used to get the extension of these results to the infinite dimensional Wiener space in (Bogachev and Mayer-Wolf 1999, Peters 1996, Ambrosio and Figalli 2009, Üstünel and Zakai 2013, Fang and Luo 2010).

Flows of quasi-invariant measurable maps corresponding to vector fields associated with transport equations, with different flavors of regularity and divergence conditions have also been studied for ordinary differential equations in (DiPerna and Lions 1989) and on Wiener space in (Fang and Luo 2010) respectively. By generalizing the results of Ambrosio (Ambrosio 2008, Ambrosio and Figalli 2009) on the existence, uniqueness and stability of regular Lagrangian flows of ordinary differential equations to Stratonovich stochastic differential equations with bounded variation drift coefficients, an explicit solution to the corresponding stochastic transport equation in terms of the stochastic flow was constructed in (Li and Luo 2012).

There is also an extensive literature on nonlinear transformations of Gaussian measures(Ramer 1974, Bogachev 1998, Cruzeiro 1983*a*). The transport property of Gaussian measures under a shift and along with the dichotomy between absolute continuity and singularity of the transported measure was established by Cameron-Martin (Cameron and Martin 1944). Much of this literature either treats general nonlinear transformations close to the identity (Ramer 1974, Kuo 1971) or transformations generated by non smooth vector fields, with values in the Cameron-Martin spaces along with an additional exponential integrability assumption of the divergence, of the corresponding vector field (Cruzeiro 1983*b*).

In contrast to studying quasi invariant flows based on vector fields, we study quasi invariant flows with two additional consistency constraints, as explained in section 4.1.

1.8 Some known densities

Before proceeding we provide a short list of known probability density formulas for functionals of Brownian motion. The proofs of the next two propositions can be found in (Primozic 2011).

Proposition 1.8.1. *Let the arithmetic Brownian motion process $X(t)$ be defined by the following standard Brownian motion $W(t)$ driven SDE*

$$dX(t) = adt + bdW(t).$$

with initial value X_0. Let $\tau = \inf(u|X(u) \leq B)$ denote the first passage time for the barrier $X_0 < B$. Then the first passage time τ is distributed as Inverse Gaussian Distribution

$$\tau \sim IG\left(\frac{B - X_0}{a}, \frac{(B - X_0)^2}{b^2}\right), \tag{1.42}$$

and for $t > 0$ the pdf of τ is

$$f(t) = \sqrt{\frac{(B - X_0)^2}{2\pi b^2 t^3}} \exp\left[-\frac{(at - B + X_0)^2}{2b^2 t}\right]. \tag{1.43}$$

Proposition 1.8.2. *Let the process $S(t)$ denote geometric Brownian motion, which is a solution to the following*

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t).$$

where $W(t)$ denotes the standard Brownian motion process. Let the expected first passage time $\rho = \inf(u|S(u) \leq L)$ for the barrier L. If the conditions $L < S(0)$ and $\mu <$

$\frac{\sigma^2}{2}$ *are satisfied, the first passage time* ρ *is distributed as Inverse Gaussian Distribution*

$$\rho \sim IG\left(\frac{\log L - \log S(0)}{\mu - \frac{\sigma^2}{2}}, \frac{(\log L - \log S(0))^2}{\sigma^2}\right),$$

with the probability density function

$$f(t) = \frac{(\log(S(0)) - \log(L))}{\sqrt{2\pi}\sigma x^{3/2}} \exp\left[-\frac{\left(2\log(L) - 2\log(S(0)) + x\left(\sigma^2 - 2\mu\right)\right)^2}{8\sigma^2 x}\right].$$

1.8.1 Supremum and Infimum Results

Proposition 1.8.3. *Let the arithmetic Brownian motion process* $Y(t)$ *be defined by the following Brownian motion driven SDE*

$$dY(t) = \alpha dt + \beta dB(t), \quad t \geq 0.$$

For $y \geq 0$*, the formula for the probability and the density of the maximum of arithmetic Brownian motion are*

$$\begin{aligned} P\left(\sup_{s\leq t} Y(s) \leq y\right) &= \mathcal{N}\left(\frac{y - \alpha t}{\beta\sqrt{t}}\right) - \exp\left[\frac{2\alpha y}{\beta^2}\right]\mathcal{N}\left(\frac{-y - \alpha t}{\beta\sqrt{t}}\right) \\ &= \frac{1}{2}\operatorname{erfc}\left(\frac{-y + \alpha t}{\beta\sqrt{2t}}\right) \\ &- \frac{1}{2}\exp\left[\frac{2\alpha y}{\beta^2}\right]\operatorname{erfc}\left(\frac{y + \alpha t}{\beta\sqrt{2t}}\right). \end{aligned} \tag{1.44}$$

and

$$\begin{aligned} h(y,t) &= \sqrt{\frac{2}{\pi t \beta^2}}\exp\left[-\frac{(y - \alpha t)^2}{2\beta^2 t}\right] - \frac{2\alpha}{\beta^2}\exp\left[\frac{2\alpha y}{\beta^2}\right]\mathcal{N}\left[\frac{-\alpha t - y}{\beta\sqrt{t}}\right] \\ &= \sqrt{\frac{2}{\pi t \beta^2}}\exp\left[-\frac{(y - \alpha t)^2}{2\beta^2 t}\right] - \frac{\alpha}{\beta^2}\exp\left[\frac{2\alpha y}{\beta^2}\right]\operatorname{erfc}\left[\frac{\alpha t + y}{\beta\sqrt{2t}}\right], \end{aligned} \tag{1.45}$$

where the cumulative distribution $\mathcal{N}$ *is the integral of the standard normal distribution and* erfc *denotes the complementary error function.*

These results are presented in (Shreve 2004, Musiela and Rutkowski 1997).

Proposition 1.8.4. *Let the arithmetic Brownian motion process $Y(t)$ be defined as in proposition 1.8.3. For $y \geq 0$, the formula for the probability and the density of the minimum of arithmetic Brownian motion are*

$$\begin{aligned} P\left(\inf_{s \leq t} Y_s \geq y\right) &= \mathcal{N}\left(\frac{-y+\alpha t}{\beta\sqrt{t}}\right) - \exp\left[\frac{2\alpha y}{\beta^2}\right]\mathcal{N}\left(\frac{y+\alpha t}{\beta\sqrt{t}}\right) \\ h(y,t) &= \sqrt{\frac{2}{\pi t \beta^2}}\exp\left[-\frac{(y-\alpha t)^2}{2\beta^2 t}\right] + \frac{\alpha}{\beta^2}\exp\left[\frac{2\alpha y}{\beta^2}\right]\operatorname{erfc}\left[\frac{-\alpha t - y}{\beta\sqrt{2t}}\right]. \end{aligned}$$

This result can be seen in (Conze 1991, Musiela and Rutkowski 1997).

Chapter 2

First Passage Problems

Systems where resource availability approaches a critical threshold are common to many engineering and scientific applications and often require the estimation of first passage time statistics of a stochastic process. In case of Brownian motion, one of the approaches entails modeling such systems using the associated Fokker-Planck equation subject to a Dirac Delta initial condition and an absorbing barrier, generally modeled as a Dirichlet boundary condition.

The method of images is a technique used for solving differential equations, in which the domain of the desired function is extended by an addition of its mirror image with respect to a symmetry hyperplane, to automatically satisfy certain types of boundary conditions. (Cheng et al. 1989, Jackson 1975). This technique presents an attractive strategy to solve an initial boundary value problem composed of a parabolic partial differential equation, with a Dirichlet boundary condition. A successful application of it, to find the distribution of the first passage of Brownian motion can be found in (Molini et al. 2011).

The primary objective of this chapter is to show that the method of images fails to yield the correct first passage density for fBm, when this initial boundary value problem is generalised by replacing Fokker-Plank/Kolmogorov equation with the one for fBm.

The chapter begins with a short literature review on Pickands constant and some additional preliminary results needed for the proof. We further proceed with recalling the equivalent problem formulation in the Brownian motion case and how it can be solved using method of images. The chapter concludes with the main proof.

2.1 Pickands Constant

James Pickands III (Pickands 1969*b*,*a*) found that asymptotic behavior of the probability

$$\mathrm{P}\left[\sup_{t\in[0,T]}[X(t)]>x\right]=H_{\mathcal{B}_{\alpha/2}}Tx^{\frac{2}{\alpha}}\Psi(x)(1+o(1)),\qquad x\to\infty, \tag{2.1}$$

where $X(t)$ is a continuous stationary Gaussian process with expected value $\mathrm{E}X(t)=0$, covariance function $\mathrm{E}(X(t+s)X(s))=1-|t|^{\alpha}+o\left(|t|^{\alpha}\right)$ with $0<\alpha\leq 2$, $\Psi(u)$ is the tail distribution of standard normal law and $H_{\mathcal{B}_{\alpha/2}}$ is a positive and finite constant, which in the literature has come to be known as the Pickands constant and was defined as

$$H_{\mathcal{B}_{\alpha/2}}=\lim_{T\to\infty}\frac{\mathrm{E}\sup_{t\in[0,T]}\left[\exp\left[\sqrt{2}\mathcal{B}_{\alpha/2}(t)-Var\left(\mathcal{B}_{\alpha/2}(t)\right)\right]\right]}{T}. \tag{2.2}$$

The Pickands constants play a very important role in the exact asymptotes of extreme values for Gaussian Processes(Qualls and Watanabe 1972, Dębicki 2002, 2006, Dębicki and Kisowski 2008, Arendarczyk and Dębicki 2012, Albin 1994, Darling 1983) and in variants of Iterated Logarithmic laws for Gaussian processes(Shao 1992, Qualls and Watanabe 1972). Dębicki (Dębicki 2002) has defined the so-called generalized Pickands constants by

$$H_{\eta}=\lim_{T\to\infty}\frac{H_{\eta(T)}}{T}=\lim_{T\to\infty}\frac{E\sup_{t\in[0,T]}\left[\exp\left[\sqrt{2}\eta(t)-\sigma_{\eta}^{2}(t)\right]\right]}{T} \tag{2.3}$$

where $\eta(T)$ is a centered Gaussian process with stationary increments and variance function $\sigma_{\eta}^{2}(t)$. Dongsheng (Dongsheng 2007) has shown under some spectral conditions, the extended class of generalized Pickands constants for a class of centered Gaussian processes with stationary increments are well defined.

Pickands original proof was based on the "double sum method", but was rather complicatedly presented as a lemma mixed in with evaluation of upcrossing probabilities for Gaussian stationary processes in (Pickands 1969*b*, Michna 1999). This method was used to find the asymptotic distribution of a Gaussian process with a unique point of maximum variance (2.1)(Piterbarg 1996), and since then has been further extended in (Michna 2009, Dieker 2005).

Unfortunately even for the Fractional Brownian motion case, the exact values of H_η where $\eta = 2H$ are unknown except for two special cases, when Hurst index values are $\frac{1}{2}$ and 1 for which the corresponding H_η values are 1 and $\frac{1}{\sqrt{\pi}}$ respectively. Shao (Shao 1996) and Dębicki (Dębicki and Kisowski 2008) have proved lower and upper bounds.

2.2 Preliminary results

Proposition 2.2.1. *The free-space fundamental solution (Green's function) of the 1D advection-diffusion equation*

$$\frac{\partial \phi}{\partial t} = f(t)\frac{\partial \phi}{\partial x} + g(t)\frac{\partial^2 \phi}{\partial x^2}, \tag{2.4}$$

where the drift $f(t)$ and diffusivity $g(t)$ are functions of t only and not x, with initial condition $\delta(x)$ is given by

$$\phi(x,t) = \frac{1}{\sqrt{4\pi \int_0^t g(s)\, ds}} \exp\left[-\frac{\left[x + \int_0^t f(s)\, \mathrm{d}s\right]^2}{4\int_0^t g(s)\, \mathrm{d}s}\right]$$

provided that $\int f dt$, $\int g dt$ exist.

Proof. Let us make a Galilean transformation to eliminate the first term in the right-hand side

$$z = x + \int_0^t f(s)\, \mathrm{d}s, \qquad \tau = t.$$

After that Eq. (2.4) reads:

$$\frac{\partial \phi}{\partial \tau} = g(\tau)\frac{\partial^2 \phi}{\partial z^2}. \tag{2.5}$$

Denote the Fourier transform with respect to z, for each fixed t of $f(z,t)$ by

$$\hat{\phi}(\xi,\tau) = \mathfrak{F}\{f(z,\tau)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(z)\, \mathrm{e}^{iz\xi}\, \mathrm{d}z.$$

On applying the Fourier transform to both sides of equation (2.5), we get

$$\frac{\partial\hat{\phi}(\xi,\tau)}{\partial\tau} = -g(\tau)\xi^2\hat{\phi}(\xi,\tau).$$

Upon rearranging and integrating we get

$$\hat{\phi}(\xi,\tau) = C(\xi)\exp\left[-\xi^2\int_0^\tau g(s)\,\mathrm{d}s\right].$$

Since

$$\hat{\phi}(\xi,0) = \mathfrak{F}\{\delta(z)\} = \frac{1}{\sqrt{2\pi}},$$

we further obtain

$$\hat{\phi}(\xi,\tau) = \frac{1}{\sqrt{2\pi}}\exp\left[-\xi^2\int_0^\tau g(s)\,\mathrm{d}s\right].$$

The inversion formula now gives the solution as

$$\phi(z,\tau) = \frac{1}{\sqrt{4\pi\int_0^\tau g(s)\,\mathrm{d}s}}\exp\left[-\frac{z^2}{4\int_0^\tau g(s)\,\mathrm{d}s}\right]$$

and in terms of the original variables

$$\phi(x,t) = \frac{1}{\sqrt{4\pi\int_0^t g(s)\,\mathrm{d}s}}\exp\left[-\frac{\left(x+\int_0^t f(s)\,\mathrm{d}s\right)^2}{4\int_0^t g(s)\,\mathrm{d}s}\right].$$

□

We conclude this section with a theorem from Dębicki. For a proof, please see (Dębicki 2006).

Theorem 2.2.1. *$H_{\mathcal{B}_{\alpha/2}}$ is continuous for $\alpha \in (0,2]$, where $\mathcal{B}_{\alpha/2}$ is a fractional Brownian motion with Hurst parameter $\alpha/2$.*

2.3 Initial Boundary value problems for first-passage

2.3.1 Stochastic processes in bounded domain

The main difference of the stochastic process in a bounded domain from the case of stochastic process in $\mathbb{R}^d$ is the influence of the boundary.

2.3.2 Brownian motion case

For a Brownian motion-driven process of the form

$$\mathrm{d}x(t) = \mu(t)\,dt + \sigma \mathrm{d}B(t)\,, \quad x(0) = x_0 \tag{2.6}$$

where B is standard Brownian motion and μ is some time dependent function. The corresponding form of the Fokker Plank equation based on 2.6 is

$$\frac{\partial p(x,t)}{\partial t} = -\mu(t)\frac{\partial p(x,t)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p(x,t)}{\partial x^2} \tag{2.7}$$

where $p(x,t)$ is the conditional density. To obtain the probability density function for the first passage time (Molini et al. 2011), we first solve (2.7) with the initial condition: and boundary value condition:

$$p(x,0) = \delta(x - x_0)\,; \qquad p(\infty,t) = p(k,t) = 0 \qquad (t > 0); \tag{2.8}$$

where $x = x_0$ is the starting point of the diffusive process, containing the initial concentration of the distribution. Upon enforcing the above boundary conditions, $p(x,t)$ reduces to a defective probability density function. For such a system, once $p(x,t)$ is known, the survival probability $S(t)$ is defined as the probability of the process trajectories not absorbed before time t, i.e.

$$S(t) = \int_{-\infty}^{k} p(x,t)\; dx \tag{2.9}$$

and the first passage time density function $f(t)$ is given by (Risken 2012, Cox and Miller 1977)

$$f(t) = -\frac{\mathrm{d}S(t)}{\mathrm{d}t}. \tag{2.10}$$

2.3.3 Failure of method of images for fBm

We now provide an original proof to show that the previous initial boundary value problem with the Dirchlet boundary condition does not yield the correct first passage density when the underlying driving process is fractional Brownian motion.

Theorem 2.3.1. *The Fokker Plank Kolmogorov equation for Fractional Brownian motion driven processes, along with the initial and boundary conditions (2.8), is not a correct formulation to study the first passage density for processes.*

Proof. Suppose that the Fokker Plank Kolmogorov equation for Fractional Brownian motion driven processes, along with the initial and boundary conditions (2.8) is a valid problem formulation to study first passage probability density and consider the stochastic differential equation of the form

$$\mathrm{d}X(t) = \mu t^{2H-1}\mathrm{d}t + \sigma \mathrm{d}B^H(t), \qquad X(0) = x_0 \tag{2.11}$$

where B^H is an fractional Brownian motion with $H \in (0,1]$ with the upper barrier $b > x_0 \geq 0$. The solution of the equation (2.11) is

$$X(t) = x_0 + \frac{\mu}{2H}t^{2H} + \sigma B^H(t).$$

The Kolmogorov Forward equation corresponding to equation (2.11) based on Lemma 1.4.1 is

$$\frac{\partial p(x,t)}{\partial t} = -\mu t^{2H-1}\frac{\partial p(x,t)}{\partial x} + H\sigma^2 t^{2H-1}\frac{\partial^2 p(x,t)}{\partial x^2}. \tag{2.12}$$

In order to obtain the survival probability function, where $x = x_0$ represents the starting point of the diffusive process, containing the initial concentration of the distribution and b is a positive upper barrier, such that $b > x_0$. We must solve (2.12) with the following initial

$$p(x,0) = \delta(x - x_0) \tag{2.13}$$

and boundary value conditions

$$p(\infty,t) = p(b,t) = 0. \qquad (t > 0) \tag{2.14}$$

The free-space fundamental solution (Green's function) of the equation (2.12) is

$$\phi(x,t) \quad = \quad \frac{1}{\sqrt{2\pi}\sigma t^H}\exp\left[-\frac{\left[x - \frac{\mu}{2H}t^{2H}\right]^2}{2\sigma^2 t^{2H}}\right],$$

hence, given initial condition (2.13) the normalized solution for an unrestricted process, starting from x_0 can be obtained as

$$\phi_{x_0}(x,t) = \frac{1}{\sqrt{2\pi}\sigma t^H}\exp\left[-\frac{\left[x - \frac{\mu}{2H}t^{2H} - x_0\right]^2}{2\sigma^2 t^{2H}}\right].$$

To solve this problem with the method of images, the barrier at b is replaced by a mirror source located at a generic point $x = m$, with $m > b$ such that the solutions of equation (2.12) emanating from the original and mirror sources exactly cancel each other at the position of the barrier at each instant of time (Redner 2001). This implies the initial conditions in (2.13) must now be changed to

$$p(x,0) = \delta(x - x_0) - \exp(-\eta)\,\delta(x - m),$$

where η determines the strength of the mirror image source. Due to the linearity of the equation (2.12), a solution of this PDE is provided by

$$p(x,t) = \phi_{x_0}(x,t) - \exp(-\eta)\,\phi_m(x,t), \tag{2.15}$$

where η determines the strength of the mirror image source and $m > b$ is the location of this source. The condition (2.14) requires for $x = b$, $p(x,t) = 0$ for all $t > 0$, which yields

$$\begin{aligned} \frac{\left[x - \frac{\mu}{2H}t^{2H} - x_0\right]^2}{2\sigma^2 t^{2H}} &= \eta + \frac{\left[x - \frac{\mu}{2H}t^{2H} - m\right]^2}{2\sigma^2 t^{2H}} \\ \Leftrightarrow \left[x - \frac{\mu}{2H}t^{2H} - x_0\right]^2 &= 2\eta\sigma^2 t^{2H} + \left[x - \frac{\mu}{2H}t^{2H} - m\right]^2. \end{aligned} \tag{2.16}$$

Upon substituting $x = b$ and $t = 0$, we get

$$[b - x_0]^2 = [b - m]^2$$

upon recalling $m > b$, we see that $m = 2b - x_0$. Upon re-substituting the value of m and $x = b$in equation (2.16), we obtain $\eta = \frac{\mu(x_0 - b)}{H\sigma^2}$. With these choices of m and η, (2.15) gives the solution of the PDE which meets the boundary conditions as

$$\begin{aligned} p(x,t) = {} & \frac{1}{\sqrt{2\pi}\sigma t^H} \exp\left[-\frac{\left[x - x_0 - \frac{\mu}{2H}t^{2H}\right]^2}{2\sigma^2 t^{2H}}\right] \\ & - \frac{1}{\sqrt{2\pi}\sigma t^H} \exp\left[-\frac{\mu(x_0 - b)}{H\sigma^2} - \frac{\left[x + x_0 - \frac{\mu}{2H}t^{2H} - 2b\right]^2}{2\sigma^2 t^{2H}}\right]. \end{aligned} \tag{2.17}$$

Under the condition $b > x_0$, the survival probability $S(t)$ as defined in (2.9) gives

$$\begin{aligned} S(t) &= \int_{-\infty}^{b} p(x,t)\, dx \qquad (2.18) \\ &= \frac{1}{2}\operatorname{erfc}\left[\frac{-2bH + 2x_0H + \mu t^{2H}}{2\sqrt{2}H\sigma t^H}\right] \\ &- \frac{1}{2}\exp\left[\frac{\mu(b - x_0)}{H\sigma^2}\right]\operatorname{erfc}\left[\frac{2bH - 2x_0H + \mu t^{2H}}{2\sqrt{2}H\sigma t^H}\right] \\ &= \mathcal{N}\left[\frac{2bH - 2x_0H - \mu t^{2H}}{2H\sigma t^H}\right] - \exp\left[\frac{\mu(b - x_0)}{H\sigma^2}\right]\mathcal{N}\left[\frac{-2bH + 2x_0H - \mu t^{2H}}{2H\sigma t^H}\right] \end{aligned}$$

where $\operatorname{erfc}(z)$ denotes the complementary error function. By substituting equation (2.19) in equation (2.10), we get the first passage time density for $t>0$ as

$$f_X(b,t)=\sqrt{\frac{2}{\pi}}\frac{H(b-x_0)}{\sigma t^{H+1}}\exp\left[-\frac{\left(2x_0H-2bH+\mu t^{2H}\right)^2}{8H^2\sigma^2t^{2H}}\right]. \tag{2.19}$$

Using the first passage density in equation (2.19), the probability of the supremum can now be obtained as

$$\begin{aligned} P\left(\sup_{s\le t}X_s\le b\right) &= \int_t^{\infty} f_X(s)\,ds \qquad (2.20)\\ &= \sqrt{\frac{2}{\pi}}\frac{H(b-x_0)}{\sigma}\int_t^{\infty}s^{-H-1}\exp\left[-\frac{\left(2x_0H-2bH+\mu s^{2H}\right)^2}{8H^2\sigma^2s^{2H}}\right]ds\\ &= \frac{1}{2}\operatorname{erfc}\left[\frac{2x_0H-2bH+\mu t^{2H}}{2\sqrt{2}H\sigma t^H}\right]\\ &- \frac{1}{2}\exp\left[\frac{\mu(b-x_0)}{H\sigma^2}\right]\operatorname{erfc}\left[\frac{2bH-2x_0H+\mu t^{2H}}{2\sqrt{2}H\sigma t^H}\right]\\ &= \mathcal{N}\left[\frac{-2x_0H+2bH-\mu t^{2H}}{2H\sigma t^H}\right]\\ &- \exp\left[\frac{\mu(b-x_0)}{H\sigma^2}\right]\mathcal{N}\left[\frac{-2bH+2x_0H-\mu t^{2H}}{2H\sigma t^H}\right]. \end{aligned}$$

Upon differentiating we obtain the probability density function of the maximum as

$$\begin{aligned} g_X(b,t) &= \sqrt{\frac{2}{\pi\sigma^2t^{2H}}}\exp\left[-\frac{\left(2H(x_0-b)+\mu t^{2H}\right)^2}{8H^2\sigma^2t^{2H}}\right] \qquad (2.21)\\ &- \frac{\mu}{2H\sigma^2}\exp\left[\frac{\mu(b-x_0)}{H\sigma^2}\right]\operatorname{erfc}\left[\frac{2H(b-x_0)+\mu t^{2H}}{2\sqrt{2}H\sigma t^H}\right]. \end{aligned}$$

It can be easily seen that for $H=1/2$, the stochastic differential equation (2.11) reduces to the following arithmetic Brownian motion stochastic differential equation

$$\mathrm{d}X(t)=\mu\mathrm{d}t+\sigma\mathrm{d}B(t),\quad X(0)=x_0>0,\quad t\ge 0$$

and accordingly the probability density function (2.19) reduces to the well known probability density function of the first passage through the upper barrier b for arithmetic Brownian motion, such that $x_0<b$ as shown in proposition 1.8.1. Similarly for $x_0=0$, the distribution and the density functions of the maximum in equation (2.20) and (2.22) reduce to equations (1.44), (1.45) respectively, but these solutions are not valid for $H\neq\frac{1}{2}$ as shown next. Based on the definition of the Pickands' constant (2.2), let us

consider the following fractional brownian motion driven stochastic differential equation

$$dZ(t) = \sqrt{2}dB^H(t) - 2Ht^{2H-1}dt, \quad Z(0) = z_0 = 0, \quad t \geq 0 \tag{2.22}$$

which has the solution

$$Z(t) = z_0 + \sqrt{2}B^H(t) - t^{2H}. \tag{2.23}$$

It should be noted that for $\sigma = \sqrt{2}$, $\mu = -2H$ and $x_0 = 0$, the SDE in equation (2.11) reduces to (2.22). Using equation(2.22), we get the density function of the supremum of $Z(t)$ as

$$g_Z(t) = \frac{e^{-b}}{2}\,\text{erfc}\left[\frac{b}{2t^H} - \frac{t^H}{2}\right] + \frac{1}{\sqrt{\pi t^{2H}}}\exp\left[-\frac{\left(b+t^{2H}\right)^2}{4t^{2H}}\right] \tag{2.24}$$

We now proceed to compute the expectation

$$\begin{aligned}
E\left[e^{\sup Z(t)}\right] &= \int_0^\infty e^z g_Z\, dz \\
&= \int_0^\infty \frac{1}{2}\,\text{erfc}\left[\frac{b}{2t^H} - \frac{t^H}{2}\right] + \frac{1}{\sqrt{\pi t^{2H}}}\exp\left[-\frac{1}{4}t^{-2H}\left(b-t^{2H}\right)^2\right] db \\
&= \text{erfc}\left[-\frac{t^H}{2}\right] + \frac{t^{2H}}{2}\,\text{erfc}\left[-\frac{t^H}{2}\right] + \frac{t^H}{\sqrt{\pi}}\exp\left[-\frac{t^{2H}}{4}\right].
\end{aligned}$$

We now proceed to compute the limit (2.3)

$$\begin{aligned}
\lim_{t\to\infty}\frac{E\left[e^{\sup Z(t)}\right]}{t} &= \lim_{t\to\infty}\frac{1}{t}\left(\text{erfc}\left[-\frac{t^H}{2}\right] + \frac{t^{2H}}{2}\,\text{erfc}\left[-\frac{t^H}{2}\right] + \frac{t^H}{\sqrt{\pi}}\exp\left[-\frac{t^{2H}}{4}\right]\right) \\
&= \lim_{t\to\infty}\left(Ht^{2H-1}\,\text{erfc}\left[-\frac{t^H}{2}\right] + \frac{Ht^{3H-1}}{2\sqrt{\pi}}\exp\left[-\frac{t^{2H}}{4}\right]\right)
\end{aligned}$$

For $H = \frac{1}{2}$, this reduces to 1, which matches the known value but, for $H > \frac{1}{2}$ we get ∞ which will suggest that Pickands' constant is a discontinuous function of H and yields the desired contradiction with theorem (2.2.1). Hence the initial boundary value problem for finding first passage density corresponding to Brownian motion does not extend to fractional Brownian motion case by a simple replacement of the corresponding Fokker Plank Kolmogorov equation. □

Chapter 3

Radon-Nikodym derivatives associated with transformations

In this chapter the theorem of Ramer which finds a formula for the Radon-Nikodym derivative of a transformed Gaussian measure relative to the untransformed measure, is generalised by relaxing the requirements on the transformation, by stating these requirements entirely in terms of the action of the transformation on the Cameron-Martin space. The Radon-Nikodym derivative itself is first expressed on the Cameron-Martin space which is equipped with a finitely additive set-function, rather than a measure. This requires exploration and development of the concept of $\mathcal{P}$-measurable functions. Its form on the space where the Gaussian process is defined is inferred by continuity from the Cameron-Martin space.

The $\mathcal{P}$-measurable derivative of a $\mathcal{P}$-measurable function is defined and shown to coincide with the stochastic derivative, under certain assumptions which in turn coincides with the Malliavin derivative when they are both defined. The main theorem's results, expresses the Radon-Nikodym derivative as a Jacobian of the $\mathcal{P}$-measurable derivative $\times$ the classical Cameron-Martin theorem. We also provide a generalised version of the Girsanov theorem for fractional Brownian motion which is not restricted to changes of linear drift only and show that it matches the one in (Norros et al. 1999) when $H \geq \frac{1}{2}$.

3.1 Abstract Wiener Space

The theory of Wiener spaces and of the reproducing kernel Hilbert spaces is reviewed following (Kuo 1975), (Bogachev 1998, Section 3.9), (Lifshits 1995, Section 9) and (Addie et al. 2002). Suppose Ω is a Banach space, with norm $\|\cdot\|_\Omega$, $H \subseteq \Omega$ is a Hilbert space, with norm $|\cdot|_H$, H is dense in Ω, in the norm of Ω and the norms satisfy a bound $\|x\|_\Omega < c|x|_H$ for all $x \in H$, for some $c > 0$. The embedding of H in Ω is denoted by $\iota : H \to \Omega$.

Definition 3.1.1. *Let A be Banach space and A^* be its topological dual space. A cylinder set in A takes the form $S = \{a \in A : (\theta_1(a), \ldots \theta_n(a)) \in B\}$, where $\theta_1, \ldots, \theta_n \in A^*$ and B is a Borel set in $\mathbf{R}^n$.*

The collection of all cylinder sets *of A, is called the cylinder algebra associated with A, and denoted by $\mathcal{C}_A$.*

Proposition 3.1.1. *The cylinder algebra $\mathcal{C}_A$ of a Banach space A satisfies :*

- $A \in \mathcal{C}_A$.
- $\forall S \in \mathcal{C}_A \Rightarrow S^c \in \mathcal{C}_A$.
- $\forall S_1, S_2 \in \mathcal{C}_A \Rightarrow S_1 \cap S_2 \in \mathcal{C}_A$.

A measure, P, can be defined on the Banach space, Ω by first defining the value of this measure on the cylinder algebra, $\mathcal{C}$, which is a subalgebra of the Borel sets of Ω. Since $H \subseteq \Omega$, $\Omega^* \subseteq H^* = H$. On the cylinder set, $S = \theta^{-1}(F)$, where $\theta = (\theta_1, \ldots, \theta_n)'$ and $\theta_1, \ldots, \theta_n$ are orthonormal, as elements of H^*, set

$$P(S) = \frac{1}{(2\pi)^{n/2}} \int_F e^{-\frac{1}{2}\sum_{k=1}^n x_k^2} dx_1 \ldots dx_n. \tag{3.1}$$

Any cylinder set can be represented by a set of orthonormal coordinates and the value of $P(S)$ will be the same for each such choice. The additive set-function P can in some cases be extended to a measure on the σ-algebra generated from $\mathcal{C}$ by taking limits of sequences of cylinder sets. In particular, this is true in a Banach space whose norm is *measurable*, as will be defined in Definition 3.1.3 (Kuo 1975, Theorem 4.1).

The cylinder sets, $\mathcal{C}_H$, on H, are also those sets S, which take the same form as S in Definition 3.1.1, but in this context θ is a linear mapping from H rather than from Ω. An additive set-function, $\mathcal{P}$, on $\mathcal{C}_H$, can then be defined by (3.1). This set-function cannot be extended to a σ-additive function (a measure) on the σ-algebra generated from $\mathcal{C}_H$, as is shown in (Kuo 1975, Proposition 4.1).

Definition 3.1.2. *The additive set-function $\mathcal{P}$ defined on $\mathcal{C}_H$ is termed the* standard Gauss measure *on H (Kuo 1975).*

The cylinder sets (both $\mathcal{C}$ and $\mathcal{C}_H$) form an algebra, but not a σ-algebra. Whereas the measure P can be extended to the smallest σ-algebra containing $\mathcal{C}$ (the Borel σ-algebra), this is not the case for $\mathcal{P}$; $\mathcal{P}$ is not a measure. (The established term "standard Gauss measure" is therefore a unusual choice of name for $\mathcal{P}$.)

Definition 3.1.3. *A norm, $\|\cdot\|$, defined on the Hilbert space H, is said to be* measurable *(Kuo 1975, p59) if for any $\varepsilon > 0$, there exists a projection, $\Pi_\varepsilon : H \to H$, onto a finite-dimensional subspace of H such that for every finite-dimensional projection, Π, orthogonal to Π_ε,*

$$\mathcal{P}\left(\{x : \|\Pi(x)\| > \varepsilon\}\right) < \varepsilon. \tag{3.2}$$

For the case of separable Hilbert space H, Sazanov's theorem (Sazonov 1958) gives the necessary and sufficient conditions for a functional $\chi(\psi)$, $\psi \in H$, to be the characteristic functional of some probability distribution of H.

Definition 3.1.4. *A triple (H, Ω, ι) in which $H \subseteq \Omega$, with H a Hilbert space and Ω a Banach space equipped with a measure, P, and $\iota : H \to \Omega$ is the canonical injection (a vector space homomorphism) is said to be an* abstract Wiener space *(Kuo 1975, p68) if H is dense in Ω and the norm $\|\cdot\|_\Omega$, restricted to H, is measurable.*

Measurability of $\|\cdot\|_\Omega$ implies $\|\cdot\|_\Omega$ is bounded by $c|\cdot|_H$ for some constant c (Kuo 1975, Lemma 4.2); in particular, if (H, Ω, ι) is an abstract Wiener space then the measure P may be defined on Ω as described at the start of this section.

Definition 3.1.5. *As well as denoting the inner product in H, we shall use $\langle x, y\rangle$ to denote the action of a continuous linear functional in Ω^* on a vector in Ω. Kuo (Kuo 1975, Lemma 4.7) shows that if $\theta_h : \Omega \to \mathbf{R}$ is the map $x \mapsto < h, x >$, which is well-defined for $h \in \Omega^*$, then $\theta_h \in L_2(\Omega, P)$ and whenever it is defined $|\theta_h|_2 = |h|_H$, and so* *the map $h \mapsto \theta_h$ can be extended by continuity to the whole of H^*. This defines $\langle x, y\rangle$ when $x \in H$ and $y \in \Omega$ in the sense that y here is random, and the map is defined* almost everywhere *with respect the measure P for y on Ω.*

Definition 3.1.6. *A sequence of orthogonal projections, $\{\Pi_k\}_{k=1}^{\infty}$, is said to* converge to the identity *on H if $\Pi_j \circ \Pi_k = \Pi_{\min(j,k)}$, j, $k > 0$ and for all $\psi \in H$, $|\Pi_k(\psi) - \psi|_H \to 0$, as $k \to \infty$.*

Remark 3.1.1. *If Π is a projection onto a finite-dimensional subspace $S \leqslant H$, and if $e_1, \ldots, e_n$ is an orthonormal basis for S,*

$$\Pi(\psi) = \sum_{k=1}^{n} \langle e_k, \psi\rangle_H \, e_k.$$

We can therefore define a projection $\widetilde{\Pi} : \Omega \to S$ by

$$\widetilde{\Pi}(\psi) = \sum_{k=1}^{n} \langle e_k, \psi\rangle_H \, e_k.$$

We shall refer to $\widetilde{\Pi}$ as the extension *of Π to Ω.*

It may seem that some sequences of projections converging to the identity may play a specific role, but the following proposition shows that any such sequence is as good as another.

Proposition 3.1.2. *If $\|\cdot\|$ is a measurable norm and $\{\Pi_k\}$ is a sequence of orthogonal projections converging to the identity in H, then $\forall \varepsilon > 0$, $\exists n_0 > 0$, such that for any projection $\Pi \perp \Pi_{n_0}$*

$$\mathcal{P}\left(\{\psi : \|\Pi(\psi)\| > \varepsilon\}\right) < \varepsilon. \tag{3.3}$$

Proof. Choose $\varepsilon > 0$. Since $\|\cdot\|$ is $\mathcal{P}$-measurable, $\exists \Pi_{\varepsilon/2}$, projecting onto a finite-dimensional subspace S, such that for all $\Pi \perp \Pi_{\varepsilon/2}$, (3.3) holds, with $\varepsilon/2$ in place of ε. Suppose $\|\cdot\| < c|\cdot|_H$. We can find $S_{\varepsilon/2} \subseteq S$ and $n_1 > 0$ such that for $n > n_1$,

$$\mathcal{P}\left(\{\psi : \Pi_{\varepsilon/2}(\psi) \notin S_{\varepsilon/2}\}\right) < \varepsilon/2$$

and

$$\mathcal{P}\left(\{\psi : \left|\Pi_{\varepsilon/2}(\psi) - \Pi_n \circ \Pi_{\varepsilon/2}(\psi)\right| > \varepsilon c^{-1}/2 \wedge \Pi_{\varepsilon/2}(\psi) \in S_{\varepsilon/2}\}\right) < \varepsilon/2.$$

Pick $n_0 = n_1$. □

3.1.1 Radon-Nikodym Derivatives

In this chapter we make use of the Radon-Nikodym derivative of one Gaussian probability measure relative to another, which exists when one measure is *equivalent* to the other in the sense that a set has non-zero measure according to one measure if and only if it has a non-zero measure relative to the other.

Definition 3.1.7. *Suppose* $(H.\Omega,\iota)$ *is an abstract Wiener space. Define* $\sigma_\rho : \Omega \to \Omega$ *as the shift,* $\psi \mapsto \psi + \rho$*, on* Ω*, for* $\rho \in \Omega$*.*

The Cameron-Martin theorem (Cameron and Martin 1944) states that the Radon-Nikodym derivative $\frac{dP\circ\sigma_\rho}{dP}(\psi)$ is well-defined and non-zero if and only if $\rho \in H$ and

$$\frac{dP\circ\sigma_\rho}{dP}(\psi) = \exp\left(-\frac{1}{2}|\rho|_H^2 - \langle\rho,\psi\rangle_H\right), \qquad \rho \in H. \tag{3.4}$$

The set of all Gaussian measures on a vector space is closed under affine transformations of the vector space, and it is logical therefore to consider Radon-Nikodym derivatives of a Gaussian measure relative to its transformation by an affine transformation. Another way to interpret a Radon-Nikodym derivative is as a way to compare of one part of the Gaussian measure with another, although it is important to be aware that the comparison cannot be isolated from influence from the transformation which makes the connection between one place and another.

If $f : \Omega \longrightarrow \Omega$ is a measurable function defined on Ω, and μ is a measure on Ω, $\mu \circ f^{-1}$ is also a measure on Ω and if the measure μ is equivalent to $\mu \circ f^{-1}$, there is a Radon-Nikodym derivative of one measure relative to the other.

3.1.2 Paths

We now consider a special case of abstract Wiener space where Ω is a space of paths, where a path is defined as a continuous function from **R** to **R**.

Denote by v the variance function $v(t) = \mathrm{E}X_t^2$, $t \in (-\infty,\infty)$. Fractional Brownian motion is characterised by the fact that $v(t) = t^{2H}$, for some $0 < H < 1$ and that X_t has stationary increments. Observe that in this case

$$\lim_{t\to\infty}\frac{v(t)}{t^{\alpha}} = 0 \tag{3.5}$$

for some $\alpha < 2$. The covariance function of X is denoted by $\Gamma(s,t) = \mathrm{E}X_sX_t$, which, in the fBm case,

$$\begin{aligned} &= \frac{1}{2}(v(s)+v(t)-v(s-t)) \\ &= \frac{s^{2H}+t^{2H}-|s-t|^{2H}}{2}. \end{aligned} \tag{3.6}$$

Definition 3.1.8. *The* smallest centered extension $U^{[c]}$*, of a set* U *is*

$$U^{[c]} = \{x : x \in U \vee -x \in U\}.$$

The following lemma is known as Anderson's inequality (Lifshits 1995, Theorem 11.9).

Lemma 3.1.1. *If* $U \subseteq \Omega$ *is a Borel set and* $h \in H$*, then* $P(U^{[c]}+h) \le P(U^{[c]})$.

Define a norm on paths as follows:

Definition 3.1.9.

$$\|\psi\| = \sup\left\{\left|\frac{\psi(t)}{1+|t|}\right| : t \in \mathbf{R}\right\} \tag{3.7}$$

and let $\Omega = \{\psi : \mathbf{R} \to \mathbf{R}$ such that $\|\psi\| < \infty\}$. In future this norm will generally be denoted by $\|\psi\|_\Omega$ for greater clarity.

Proposition 3.1.3. *The norm* $\|\cdot\|_\Omega$ *is measurable on* H.

Proof. This proof follows the outline of the proof in (Kuo 1975, p91) which applies to the standard abstract Wiener space, $(i, C'[0,1], C([0,1]))$. For each $n > 0$, define

$$\|\psi\|_n = \sup\left\{\left|\frac{\psi(t_j)}{1+|t_j|}\right| : t_j = \frac{j}{2^n}, j = -n2^n, 1-n2^n, \ldots, n2^n\right\}. \tag{3.8}$$

Clearly $\|\cdot\|_n$ is a measurable semi-norm in H and

$$\lim_{n\to\infty}\|\psi\|_n = \|\psi\|_\Omega \tag{3.9}$$

for each $\psi \in H$. Now let us apply Theorem 4 of (Gross 1962), which concludes that, subject to three conditions, the limit norm, $\|\cdot\|_\Omega$ is measurable.

Condition (1) of (Gross 1962, Theorem 4) is that

$$\lim_{P \to I} \|P\psi\|_n = h_n \tag{3.10}$$

exists, where P denotes an orthogonal projection in the space H. It is clear that this condition holds. Condition (2) of (Gross 1962, Theorem 4) is that $\lim_{n\to\infty} h_n = h$ exists, for all ψ, which is also clearly true in the present instance. Define $A_\varepsilon = \{\psi : \|\psi\|_\Omega \leq \varepsilon\}$. Condition (3) of Theorem 4 of (Gross 1962) is that

$$P(A_\varepsilon) > 0 \tag{3.11}$$

for all $\varepsilon > 0$. Since H is separable and dense in Ω, for any $\varepsilon > 0$, there exists a countable set $D \subseteq H$ such that $\Omega = \bigcup_{D_\varepsilon} \bigcup_{\psi \in D} (A_\varepsilon - \psi)$, where $D_\varepsilon = \{\psi_D : \psi_D = \psi \in A_\varepsilon, \}$ hence for some $\psi \in D$, $P(A_\varepsilon - \psi) > 0$. Hence, by Lemma 3.1.1, (3.11) holds for arbitrary $\varepsilon > 0$.

Since all three conditions of (Gross 1962, Theorem 4) hold, we conclude that $\|\cdot\|_\Omega$ is measurable. □

Proposition 3.1.4. *The space H may be identified with the closure of the linear space generated by the paths $f_s : t \mapsto \Gamma(s,t)$ where Γ is the autocovariance of the fBm process. The inner product of this space is characterised by the equation*

$$\langle \psi, \Gamma(t,\cdot) \rangle = \psi(t). \tag{3.12}$$

for all $\psi \in H$, and the norm of f_s as defined above is therefore $\sqrt{\Gamma(s,s)}$.

Proof. Proposition 3.1.3 showed that $\|\cdot\|_\Omega$ is a measurable norm. We have to show that the (3.12) choice for H leads to the paths in Ω to have covariance Γ. In order for any system of coordinates, $\theta_1, \cdots, \theta_n$ to be orthonormal, it must take the form of the linear operator (matrix):$G^{-1/2}$ where $G = (\Gamma(t_i,t_j))_{i=1,\ldots,n,j=1,\ldots,n}$, and $t_1,\ldots,t_n$ is a set of n times. Let us denote the mapping from θ coordinates to x coordinates, by Θ. Thus, the mapping from x coordinates to θ coordinates is $\Theta^{-1} = G^{1/2}$.

The x-coordinates have covariance I, in the probability measure assigned to them by (3.1). It therefore follows that the covariance of $X_{t_1},\ldots,X_{t_n}$, ie the covariance of

$\Theta^{-1}(X_{t_1},...,X_{t_n})'$, is

$$(E(X_{t_i}X_{t_j}))_{i,j=1,...,n} = G^{1/2}(E(x_i x_j))_{i,j=1,...,n}G^{1/2} = G^{1/2}IG^{1/2} = G,$$

as required. □

The inner product definition generalizes to the *reproducing kernel property*:

$$\langle \Gamma(t,\cdot), f\rangle_H = f(t), \quad f \in H. \tag{3.13}$$

H is a subset of Ω since, by the Cauchy-Schwarz inequality,

$$\|f\|_\Omega \leq \|f\|_H \cdot \sup_{t\in\Re} \frac{\|\Gamma(t,\cdot)\|_H}{1+|t|},$$

where the last supremum is finite by (3.5). Thus, the Hilbert topology of H is finer than that induced by Ω. On the other hand, one can show that the space H is an everywhere dense subset of the support of P (see (Bahadur and Zabell 1979)). It follows that each $\psi \in H$ can be extended to a linear functional on Ω and (3.13) generalises to

$$\langle \Gamma(t,\cdot), f\rangle_H = f(t), \quad f \in \Omega. \tag{3.14}$$

Applying this equation a second time, with s in place of t, and equating covariances of the left-hand sides with covariances of the right-hand sides we find

$$\mathrm{Cov}\left(\langle \Gamma(t,\cdot), X\rangle_H, \langle \Gamma(t,\cdot), X\rangle_H\right) = \mathrm{Cov}\left(X(t), X(s)\right) = \Gamma(t,s).$$

and therefore, since the space H is generated by the elements of the form $\Gamma(t,\cdot)$,

$$\mathrm{Cov}\left(\langle \psi_1, f\rangle_H, \langle \psi_2, f\rangle_H\right) = \langle \psi_1, \psi_2\rangle_H. \tag{3.15}$$

The space H coincides with the space of *measurable shifts* (Lifshits 1995, Kuo 1975), also referred to as the *Cameron-Martin space* of the the process X, i.e. the space of paths with the property that translation by this path transforms the probability measure into one which has finite, non-zero Radon-Nikodym derivative almost everywhere relative to the original measure(Bogachev and Mayer-Wolf 1999).

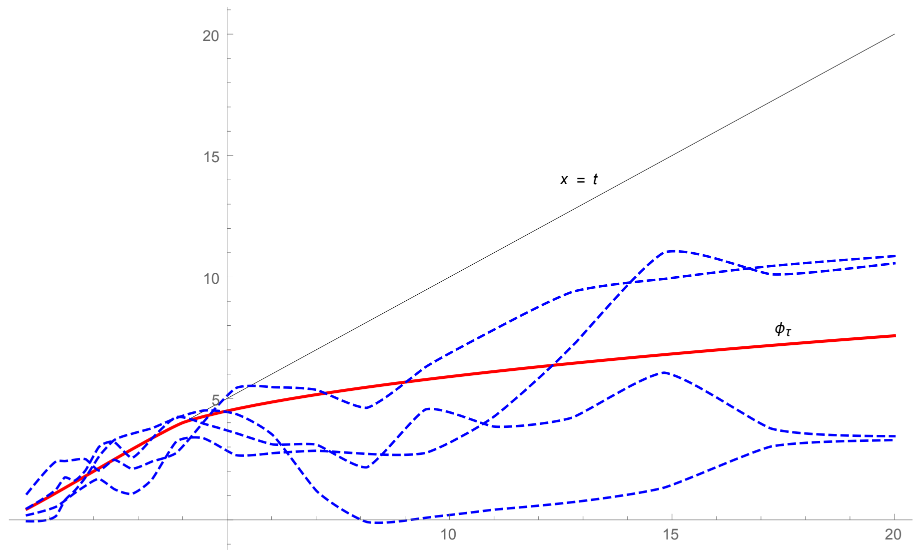

Figure 3.1: The most likely path passing through $x = t$ at $t = 4$ and some random paths which are close to having this property. In this example, $H = 0.72$.

3.1.3 Most likely functions

The path $\phi_t \in H(t) = \{\psi \in H : \phi(t) = t\}$ is defined to be the unique path in this set with minimal value of $|\phi|_H$. An example of a most-likely path is shown in Figure 3.1, together with four random paths which have the property that they nearly pass through the line $x = t$ at $t = t$.

Proposition 3.1.5. *These conditions determine that*

$$\phi_t : s \quad \mapsto \quad \left(\frac{t}{\Gamma(t,t)}\right)\Gamma(t,s) \tag{3.16}$$

which, in the fBm case,

$$= \quad \frac{t^{1-2H}}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right). \tag{3.17}$$

Proof. The set $H(t)$ is not a subspace, but $H(t) = H[t] + \phi_t$ in which $H[t]$ is the subspace of H for which the path value at t is zero. The problem of finding the element of $H(t)$ with minimal H-norm can be solved by first finding the element $\psi \in H[t]$ for which $|\psi - \phi_t|_H$ is minimal. If we call this minimizing element $\widehat{\psi}$, the solution of the original problem will be $\widehat{\phi}_t = \widehat{\psi} + \phi_t$.

According to the Projection theorem of Hilbert spaces (Luenberger 1969, §3.3), the element $\widehat{\psi}_t \in H[t]$ for which $|\psi - \phi_t|_H$ is minimal is the unique $\widehat{\psi}_t \in H[t]$ with the property $\phi_t - \widehat{\psi}_t \perp H[t]$. But $\phi_t \perp H[t]$, so $\widehat{\psi}_t = 0$. Hence $\widehat{\phi}_t = \phi_t$. □

Definition 3.1.10. *The path ϕ_t will be referred to as the most likely path passing through t at time t.*

Definition 3.1.11. *The* normalisation *of the path ϕ_t, considered as a vector in H, will denoted by $\widehat{\phi}_t$, i.e. $\widehat{\phi}_t = t^{H-1}\phi_t$, $t \geq 0$.*

Definition 3.1.12. *The* fBm similarity *on H (or Ω) is the mapping, defined for all $\delta > 0$,*

$$\begin{array}{rcll} \kappa_\delta : H & \to & H & \qquad (\Omega \to \Omega) \\ \psi & \mapsto & \psi' & \end{array}$$

where

$$\psi' = \delta^H \psi(t/\delta), \quad t > 0.$$

Remark 3.1.2. *Notice that $\phi_t(t) = t$,*

$$\|\phi_t\|_H = t^{1-H}, \tag{3.18}$$

$$\langle \phi_s, \psi \rangle_H = s^{1-2H}\psi(s), \tag{3.19}$$

$$\langle \phi_s, \phi_t \rangle_H = (ts)^{1-2H}\Gamma(s,t)$$

which, in the fBm case,

$$= \frac{1}{2}(ts)^{1-2H}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right) \tag{3.20}$$

$$\left\langle \widehat{\phi}_s, \widehat{\phi}_t \right\rangle_H = (ts)^{-H}\Gamma(s,t)$$

$$= \frac{1}{2}(ts)^{-H}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right) \tag{3.21}$$

$$\|\phi_s - \phi_t\|^2 = t^{2-2H} + s^{2-2H} - t^{1-2H}s - s^{1-2H}t + (st)^{1-2H}|s-t|^{2H} \tag{3.22}$$

$$= (t-s)\left(t^{1-2H} - s^{1-2H}\right) + (ts)^{1-2H}|t-s|^{2H} \tag{3.23}$$

similarly

$$\begin{aligned}
\langle \phi_s, \phi_t - \phi_s \rangle &= \frac{(ts)^{1-2H}}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right) - s^{2-2H} \\
&= \frac{(ts)^{1-2H}}{2}\left(t^{2H} + s^{2H} - 2st^{2H-1} - |s-t|^{2H}\right) \\
&= O\left(|s-t|^{2H}\right) \qquad (3.24) \\
\langle \widehat{\phi}_s, \widehat{\phi}_t - \widehat{\phi}_s \rangle &= \frac{(ts)^{-H}}{2}\left(t^{2H} + s^{2H} - |s-t|^{2H}\right) - 1 \\
&= \frac{(ts)^{-H}}{2}\left(\left(t^{H} - s^{H}\right)^2 - |s-t|^{2H}\right) \\
&= O\left(|s-t|^{2H}\right) \qquad (3.25)
\end{aligned}$$

and

$$\kappa_\delta(\phi_t) = \delta^{H-1}\phi_{\delta t}; \qquad (3.26)$$

also, $\phi_t \in H$, the Cameron-Martin space, for any $t > 0$.

Definition 3.1.13. *A linear combination of most-likely paths is referred to as a* simple path. *If the locations of the most-likely paths are $t_1, \ldots, t_n$, the simple path will be termed a simple path with corners at $t_1, \ldots, t_n$.*

In the Brownian motion case, for example, a simple path with corners at $t_1, \ldots, t_n$ will take the form of a connected series of straight lines starting at the origin, finishing as a horizontal line, and with the locations of change in slope at $t_1, \ldots, t_n$.

3.2 Additive set functions and $\mathcal{P}$-measurability

In this section we review and extend the theory of finitely additive set functions (informally referred to as finitely additive measures), the functions measurable with respect to these set functions, and integration of functions with respect to the set functions (Dunford and Schwartz 1957). The finitely additive set function we have in mind is usually $\mathcal{P}$, as in Definition 3.1.2.

The outer measure corresponding to $\mathcal{P}$, on H, is denoted by $\mathcal{P}^*$ (Dunford and Schwartz 1957):

$$\mathcal{P}^*(A) = \inf\{\mathcal{P}(B) : A \subseteq B, B \in \mathcal{C}_H\}$$

The outer measure is defined on the power set of H. It is not necessarily additive.

Classical measures are required to be σ-additive, however there is a well-developed theory for finitely additive set-functions (Dunford and Schwartz 1957, Volume 1, II.2, p101) and further development and use of the theory of finitely additive set-functions is also proposed in (Bagchi and Mazumdar 1993) for applications in signal processing. A definition of the Radon-Nikodym derivative between additive set-functions is given in (Dunford and Schwartz 1957); a different definition is proposed in (Bagchi and Mazumdar 1993). We use the definition in (Dunford and Schwartz 1957); this definition is directly analogous to the usual definition for measures whereas the one in (Bagchi and Mazumdar 1993) includes a uniformity condition which has not been found to be necessary in the applications under consideration in this research, and therefore has not be adopted.

In the paper of (Kallianpur and Karandikar 1985), quite a different problem area, namely filtering, prediction, and estimation of a stochastic process subject to observations with white noise is studied. Their approach diverges from the one in this research and in (Dunford and Schwartz 1957) at an early point, even though both works use finitely additive measures. (Dunford and Schwartz 1957) use the concept of $\mathcal{P}$-measurabilty to define the functions that can be integrated with respect to the underlying measure, but in (Kallianpur and Karandikar 1985) a new type of measurability is introduced, which although fulfills the same role, ie it defines the functions that can be integrated, the key idea where it differs is to require an additional condition, but not one which depends on $\mathcal{P}$. As a consequence this also requires a different concept of Radon-Nikodym derivative. The reason Kandikar and Kallianpur felt the need to develop a theory somewhat divergent from the existing theory, as presented in (Dunford and Schwartz 1957), may be because of the particular needs of the application area.

Aside from the fact that the application domains are different, the paper of Kandikar and Kallianpur also differs from this research as it focuses on strictly white noise, rather than more general Gaussian processes, including fBm. Also the signal process, which is being estimated, is assumed to have its own distinct probability measure, usually independent from that of the noise process.

A key step in developing theories which use finitely additive measures is to develop a theory of integration (and hence expectation), for some class of functions on the measure space. As Kallianpur and Kandikar acknowledge, this was already achieved, in a reasonably complete way, including a Radon-Nikodym theorem, and this theory is presented rigorously in Dunford and Schwartz. In the research this theory of Dunford and Schwartz is taken as an appropriate basis for further work, and developed where necessary.

In this remainder of this section, the ground is prepared for the extension of Ramer's theorem as follows:

- The theory of finitely additive meaures and $\mathcal{P}$-measurable functions is reviewed.
- It is shown that every $\mathcal{P}$-measurable functional on H induces a certain, uniquely defined, conventionally measurable functional on Ω (Theorem 3.2.4).

3.2.1 $\mathcal{P}$-measurable functions

Definition 3.2.1. *Suppose X is a normed space, with norm $\|\cdot\|_X$. A simple function $(\Omega, \mathcal{C}) \to X$ is a finite linear combination of indicator functions where each indicator function comes from $\mathcal{C}$; if the space X is* **R** *we shall refer to the function as a simple functional. Simple functions and functionals on $(H, \mathcal{C}_H)$ are defined similarly.*

Definition 3.2.2. *If $f : H \to X$ is a function, and $\mathcal{P}$ is an additive set-function on H, the $|\cdot|_{\mathcal{P}}$-norm of f is defined by*

$$|f|_{\mathcal{P}} = \inf\{\varepsilon + \mathcal{P}^*(\{\psi : \|f(\psi)\|_X > \varepsilon\})\}. \tag{3.27}$$

A function $f : H \to X$ is said to be $\mathcal{P}$-measurable if there exist simple functions $\phi_k : (H, \mathcal{C}_H) \to X$, $k = 1, \ldots,$ such that $|f - \phi_k|_{\mathcal{P}} \to 0$ as $k \to \infty$. A set $U \subseteq H$ is termed $\mathcal{P}$-measurable if its indicator functional is $\mathcal{P}$-measurable. The set of $\mathcal{P}$-measurable functions from H to X to X is denoted by $\mathcal{M}(H,X)$, $\mathcal{M}(H,X,\mathcal{P})$, or $\mathcal{M}(H,X,\mathcal{P},\|\cdot\|)$, when it is necessary to indicate the measure, or the norm. Unless otherwise indicated, the $\mathcal{P}$-measurable norm will be adopted on all these spaces. The subspace of linear *functions from H to X is denoted by $\mathcal{L}(H,X)$, etc.*

In the special case $X = H$, this space will be denoted simply by $\mathcal{M}$, and the norm $|\cdot|_{\mathcal{P}}$ will be denoted, also, by $|\cdot|_{\mathcal{M}}$. Since the space in which the norm operates is $\mathcal{M}$, this *notation should not be confusing, and in future we shall encounter a context where having a second notation for this norm will be convenient.*

Similarly, if $f : \Omega \to X$ is a function, and P is an additive set function on $(\Omega, \mathcal{C})$, where $\mathcal{C}$ is an algebra of subsets of Ω, its $|\cdot|_P$-norm is defined by

$$|f|_P = \inf\left\{\varepsilon + P^*\left(\{\psi : \|f(\psi)\|_X > \varepsilon\}\right)\right\}. \tag{3.28}$$

A function $f : \Omega \to X$ is said to be $(\mathcal{C}, P)$-measurable if there exist simple functions $\phi_k : (\Omega, \mathcal{C}) \to H$, $k = 1, \ldots,$ such that $|f - \phi_k|_P \to 0$. The set of P-measurable functions on Ω is denoted by $\mathcal{M}(\Omega, X)$, $\mathcal{M}(\Omega, X, P)$, or $\mathcal{M}(\Omega, X, P, \|\cdot\|)$, when it is necessary to indicate the measure, or the norm.

Proposition 3.2.1. *A set $A \subseteq H$ is $\mathcal{P}$-measurable if and only if for any $\varepsilon > 0$, $\exists C \in \mathcal{C}_H$ with $\mathcal{P}^*(A\Delta C) < \varepsilon$.*

Proof. Suppose $\forall\varepsilon > 0$, $\exists C \in \mathcal{C}_H$ with $\mathcal{P}^*(A\Delta C) < \varepsilon$ and let $\varepsilon \in (0, 1)$. Define $\phi_\varepsilon = \chi_C$. This is a simple function with $|\chi_A - \phi_\varepsilon|_{\mathcal{P}} < 2\varepsilon$; since $\varepsilon > 0$ is arbitrary, χ_A is $\mathcal{P}$-measurable, and hence A is $\mathcal{P}$-measurable.

Now suppose A is $\mathcal{P}$-measurable, i.e. χ_A is $\mathcal{P}$-measurable, and choose $0 < \varepsilon < 0.5$, simple ϕ_ε, with $|\chi_A - \phi_\varepsilon|_{\mathcal{P}} < \varepsilon$. Set $C = \{\psi : |\phi_\varepsilon(\psi)| \geq 0.5\}$. Since χ_C only takes values either 0 or 1, and $\varepsilon < 0.5$, $\{\psi : |\chi_C(\psi) - \chi_A(\psi)| > \varepsilon\} \subseteq \{\psi : |\phi_\varepsilon(\psi) - \chi_A(\psi)| > \varepsilon\}$, from which it follows that $|\chi_A - \chi_C|_{\mathcal{P}} \leq |\phi_\varepsilon - \chi_C|_{\mathcal{P}}$. □

Remark 3.2.1. *The norm $|\cdot|_{\mathcal{P}}$ on $\mathcal{P}$-measurable functionals is not homogeneous. Also, it is not the case that $|f|_{\mathcal{P}} = 0 \Rightarrow f = 0$. Replacing $|\cdot|_{\mathcal{P}}$ by a homogeneous norm is unnecessary since it is not used as the norm of a Banach space.*

Definition 3.2.3. *Adding the sets whose difference from a set in $\mathcal{C}_H$ has outer measure less than ε, for every $\varepsilon > 0$, is termed completing the algebra with respect to the set-function $\mathcal{P}$. If an algebra of sets already contains all these sets it is termed $\mathcal{P}$-complete.*

Remark 3.2.2. *The completion of an algebra is also an algebra.*

The following definition is essentially the same as Definition III.2.9 from (Dunford and Schwartz 1957).

Definition 3.2.4. *A function g is termed* $\mathcal{P}$-simple *if there are n $\mathcal{P}$-measurable sets (i.e. sets in the completion of the algebra on which $\mathcal{P}$ is defined), $G_1, \ldots, G_n$ and n real numbers $g_1, \ldots, g_n$ such that $g(\psi) = g_k$, $\psi \in G_k$, and $= 0$ otherwise.*

Definition 3.2.5. *If $(G, \mathcal{C}_G, \mathcal{P})$ and $(H, \mathcal{C}_H, \mathcal{Q})$ are sets with algebras of subsets $\mathcal{C}_G$ and $\mathcal{C}_H$ and with additive set-functions, $\mathcal{P}$ and $\mathcal{Q}$, their Cartesian product is defined as $(G \times H, \mathcal{C}_G \times \mathcal{C}_H, \mathcal{P} \times \mathcal{Q})$, where for $A \in \mathcal{C}_G$, $B \in \mathcal{C}_H$, $\mathcal{P} \times \mathcal{Q}(A \times B) = \mathcal{P}(A) \times \mathcal{Q}(B)$, $\mathcal{P} \times \mathcal{Q}$ is extended to the algebra generated from these sets by additivity, and to $\mathcal{C}_G \times \mathcal{C}_H$ by $\mathcal{P} \times \mathcal{Q}$ completion.*

The Cartesian product cylinder algebra *$\mathcal{C}_{G\times H}$ is defined to be the smallest algebra which contains the sets of the form $A \times B$, for all $A \in \mathcal{C}_G$ and $B \in \mathcal{C}_H$.*

Remark 3.2.3. *If G and H are Hilbert spaces, with cylinder measures, the cross-product cylinder algebra can also be defined as the cylinder algebra of the direct sum Hilbert space, $G \oplus H$. Note that* **R** *becomes a Hilbert space by adopting the conventional product as the scalar product. This way to define the Cartesian product of two cylinder algebras is equivalent to Definition 3.2.5 whenever it applies.*

Example 3.2.1.. When P is a measure

If P is a measure, the definition of the $|\cdot|_P$ norm definition is still applicable, because it is an additive set-function, and if the whole space has finite P-measure, all measurable functionals are also P-measurable, and conversely. □

Definition 3.2.6. *A sequence $\{\theta_k\}_{k=0}^{\infty}$ of $\mathcal{P}$-measurable functions $H \to X$ is said to* converge in $\mathcal{P}$-measure *to the $\mathcal{P}$-measurable function θ if $|\theta_k - \theta|_{\mathcal{P}} \to 0$. A sequence of measurable functions $\{\theta_k\}_{k=0}^{\infty}$ $\Omega \to X$ is said to* converge in P-measure *to the $\mathcal{P}$-measurable function θ if for any $\varepsilon > 0$, $\exists N > 0$, $\forall k > N$, $P(|\theta_k - \theta|_P > \varepsilon) < \varepsilon$.*

Lemma 3.2.1. *If f is a $\mathcal{P}$-measurable functional, and $\varepsilon > 0$, there exists $A \in \mathcal{C}_H$ such that $\mathcal{P}(H \setminus A) < \varepsilon$ and f is bounded on A.*

Proof. Simple functions are bounded, and for any $\varepsilon > 0$, we can find a simple function ϕ such that $|\phi - f|_{\mathcal{P}} < \varepsilon/2$, so $\mathcal{P}^*(\{\psi : |f(\psi) - \phi(\psi)| > \varepsilon/2\}) < \varepsilon/2$, which implies that there exists $A \in \mathcal{C}_H$ such that $\mathcal{P}(H \setminus A) < \varepsilon$ and f is bounded on A. □

The *integral*, $\int f d\mathcal{P}$, of a $\mathcal{P}$-measurable functional, f, with respect to the additive set function $\mathcal{P}$, is defined in (Dunford and Schwartz 1957, Volume 1, II.2, p101) as the limit of the natural definition of the integral (the sum, over the finite number of sets from $\mathcal{C}_H$ which occur in the definition of the simple functional) which applies when f is simple, applied to each of the simple functionals in a sequence which converges, in $|\cdot|_{\mathcal{P}}$-norm, to f. If U is a $\mathcal{P}$-measurable set, and f is a $\mathcal{P}$-measurable functional, the integral $\int_U f(\psi)\mathcal{P}(d\psi)$ is defined as $\int \chi_U(\psi)f(\psi)\mathcal{P}(d\psi)$.

Lemma 3.2.2. *Suppose $S \subseteq H$ is a finite-dimensional subspace, Π_S is the projection onto S, and $\{\Pi_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional orthogonal projections in H, with Π_n projecting onto S_n, converging strongly to I. For any B a Borel set in $\Pi_S^{-1}(\mathcal{B}(S))$ and any $\varepsilon > 0$, $\exists n > 0$, and an open set $O_n \in \Pi_n^{-1}(\mathcal{T}(S_n))$ such that $P(B \triangle O_n) < \varepsilon$.*

Proof. For simplicity, let us assume S is one-dimensional, spanned by the unit vector v. Generalization of this case will be obvious. Let $\mathcal{G}$ denote the class of Borel sets in S, like B, for which the desired conclusion holds. Consider the set $B_{a,b} = \{u : \langle u,v\rangle_H \in (a,b)\}$, with $a,b \in [-\infty,\infty]$ and $a < b$. Since $\{\Pi_k\}_{k=1}^{\infty}$ converges strongly to the identity, for any $\varepsilon > 0$, $\exists n > 0$, $|\Pi_n(v) - v|_H < \varepsilon$. It is therefore clear that $B_{a,b} \in \mathcal{G}$ for all a, $b \in [-\infty,\infty]$. It is also clear that $\mathcal{G}$ is an algebra, i.e. it contains the finite unions, complements, and intersections of any collection of sets in $\mathcal{G}$.

Now suppose $\{B_k\}_{k=0}^{\infty}$ is an increasing sequence of sets in $\mathcal{G}$ and $B = \bigcup_k B_k$. Choose $K > 0$ such that $P(B \setminus B_K) < \varepsilon/2$ and then choose $n > 0$ such that S_n contains a Borel set $\widetilde{O}_n \in \mathcal{T}(S_n)$ such that if $O_n = \Pi_n^{-1}(\widetilde{O}_n)$, $P(O_n \triangle B_K) < \varepsilon/2$. This choice satisfies the requirement which shows that $B \in \mathcal{G}$. A similar argument shows that a decreasing sequence of sets in $\mathcal{G}$ is also in $\mathcal{G}$. It follows that $\mathcal{G}$ is monotone and therefore contains all the Borel sets in S. □

Proposition 3.2.2. *If f is a $\mathcal{P}$-measurable functional and $\{\Pi_k\}$ is a sequence of projections converging strongly to the identity on H, and $f_k : \psi \mapsto f(\Pi_k(\psi))$, then $f_k \to f$ in $\mathcal{P}$-measure.*

Proof. This follows from the Lemma 3.2.2 directly when f is the characteristic functional of a cylinder set. Additivity of limits implies that it holds for simple functionals, and since any $\mathcal{P}$-measurable functional can be approximated arbitrarily well by simple functionals, the result must therefore hold for arbitrary $\mathcal{P}$-measurable functionals. □

Definition 3.2.7. *Given positive additive set-functions $\mathcal{P}$ and $\mathcal{Q}$ defined on (H,Σ_P), (H,Σ_Q) respectively, we say $\mathcal{P}$ is continuous with respect to $\mathcal{Q}$ (Dunford and Schwartz 1957, II.4.12) and write $\mathcal{P} \le \mathcal{Q}$ if $\Sigma_Q \subseteq \Sigma_P$ and*

$$\lim_{\mathcal{Q}(A)\to 0} \mathcal{P}(A) = 0.$$

If $\mathcal{P}_\alpha$, $\alpha \in \mathcal{A}$ is a collection of additive set functions, defined on $(H,\Sigma_{\mathcal{P}_\alpha})$, $\alpha \in \mathcal{A}$, respectively, we say $\mathcal{P}_\alpha \le \mathcal{Q}$ uniformly *if $\Sigma_Q \subseteq \Sigma_{\mathcal{P}_\alpha}$, $\alpha \in \mathcal{A}$, and for all $\varepsilon > 0$, $\exists\delta > 0$ such that $\mathcal{Q}(A) < \delta \Rightarrow \forall\alpha \in \mathcal{A}, \mathcal{P}_\alpha(A) < \varepsilon$.*

Remark 3.2.4. *In (Dunford and Schwartz 1957) it is assumed that $\Sigma_Q = \Sigma_P$.*

Proposition 3.2.3.

$$\mathcal{P} \le \mathcal{Q} \Longrightarrow (f\ \mathcal{Q}\text{-measurable} \Rightarrow f\ \mathcal{P}\text{-measurable})\,.$$

Proof. Choose $\varepsilon > 0$. We need to find simple g such that $|g-f|_{\mathcal{P}} < \varepsilon$. By $\mathcal{P} \le \mathcal{Q}$ we can find $\delta > 0$ such that $\mathcal{Q}(A) < \delta \Rightarrow \mathcal{P}(A) < \varepsilon$. Choose simple g such that $|g-f|_{\mathcal{Q}} < \min(\delta,\varepsilon)$. This g also meets the requirement $|g-f|_{\mathcal{P}} < \varepsilon$. □

Proposition 3.2.4. *Suppose $f : (G,\mathcal{P}) \to H$ and $T : (G,\mathcal{P}) \to G$ is $\mathcal{P}$-measurable. Then*

$$f\ \mathcal{P}\circ T^{-1}\text{-measurable} \Rightarrow f\circ T\ \mathcal{P}\text{-measurable}.$$

Proof. Suppose f is $\mathcal{P}\circ T^{-1}$-measurable and choose $\varepsilon > 0$ and $\mathcal{P}\circ T^{-1}$-simple g with $|f-g|_{\mathcal{P}\circ T^{-1}} < \varepsilon$. Then $g\circ T$ is $\mathcal{P}$-simple and $|f\circ T - g\circ T|_{\mathcal{P}} = |f-g|_{\mathcal{P}\circ T^{-1}} < \varepsilon$. □

Definition 3.2.8. *Let B_n denote a set of measurable sets which generates the Borel sets of $\mathbf{R}^n$, for each $n > 0$. E.g. the collection of cross-products of open finite intervals, or closed finite intervals, or of semi-infinite closed or open intervals.*

Now let $\mathcal{B}_n$ denote the set of sets of the form $\Pi^{-1}(S)$, $S \in B_n$, for some $n > 0$, for any Π which is an orthogonal projection from H to $\mathbf{R}^n$. The collection of indicator functions of sets in $\mathcal{B}_n$ will also be denoted by $\mathcal{B}_n$.

Example 3.2.2.. A continuous functional which is not $\mathcal{P}$-measurable

In the context of an abstract Wiener space (H,Ω,ι) the functional $\eta : \psi \mapsto |\psi|_H$ is not $\mathcal{P}$-measurable. See (Kuo 1975, Remark after Definition 4.4). The mapping $\psi \mapsto |\psi|_\Omega$ is, however, $\mathcal{P}$-measurable. A norm which is $\mathcal{P}$-measurable is referred to in (Kuo 1975) as *measurable*, and the same terminology is used here, but it can now be understood that this is a shorthand for saying that this norm is $\mathcal{P}$-measurable. □

The following result is a restatement of Lemmas III.2.11 and III.2.12 from (Dunford and Schwartz 1957), in the present context, together with some related results. It plays the role of a *Portmanteau theorem* for $\mathcal{P}$-measurability. The most notable absence from this theorem is a single rule for composition of $\mathcal{P}$-measurable functions. Instead, we have (ii) and (iii). Of these, the latter is the more general, but it still includes an inconvenient condition which is an inevitable consequence of the fact that $\mathcal{P}$-measurability of a function f is a condition relative to the spaces of measurable sets in the domain and range of f and *also* the additive set function $\mathcal{P}$.

Theorem 3.2.1. *If G, H, K, ..., are sets, each equipped with an algebra of sets and an additive set-function $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, (like the Gauss measure),*

(i) if $\{f_k\}$ is a $\mathcal{P}$-convergent sequence of $\mathcal{P}$-measurable transformations $: G \to H$, its limit is $\mathcal{P}$-measurable;

(ii) if $f : G \to H$ is scalar-valued and $\mathcal{P}$-measurable and $g : H \to K$ is scalar-valued and continuous, $g \circ f$ is $\mathcal{P}$-measurable;

(iii) if $f : G \to H$ is $\mathcal{P}$-measurable, $g : H \to K$ is $\mathcal{Q}$-measurable and $\mathcal{P}\circ f^{-1} \leq \mathcal{Q}$, then $g \circ f$ is $\mathcal{P}$-measurable and for all $\varepsilon > 0$, $\exists \delta > 0$ such that $|g|_{\mathcal{Q}} < \delta \Rightarrow |g \circ f|_{\mathcal{P}} < \varepsilon$;

(iv) if $g, g_k : (G,\mathcal{P}) \to H$, $k > 0$, are $\mathcal{P}$-measurable, $g_k \to g$ in $\mathcal{P}$-measure as $k \to \infty$, and $f : H \to K$ is $\mathcal{P}$-measurable $\mathcal{P}\circ f^{-1} \leq \mathcal{P}$, then $g \circ f$ is $\mathcal{P}$-measurable and $g_k \circ f \to g \circ f$ in $\mathcal{P}$-measure;

(v) assuming $\mathcal{P}$ is the standard Gauss measure on $(G, |\cdot|_G)$, if $f, f_k : (G,\mathcal{P}) \to (G, |\cdot|)$, $k > 0$, are $\mathcal{P}$-measurable, $\mathcal{P}\circ f_k^{-1} \leq \mathcal{P}$ uniformly in k, $f_k \to f$ in $\mathcal{P}$-measure as $k \to \infty$, and $g : G \to H$ is $\mathcal{P}$-measurable then $g \circ f$ is $\mathcal{P}$-measurable and $g \circ f_k \to g \circ f$ in $\mathcal{P}$-measure; if $g \in \mathcal{B}_n$ for some $n > 0$, the requirement of uniformity on $\mathcal{P}\circ f_k^{-1} \leq \mathcal{P}$ is not needed;

(vi) *if* $f_1 : G \to H_1$ *and* $f_2 : G \to H_2$ *are* $\mathcal{P}$*-measurable transformations,* $f : \psi \mapsto (f_1(\psi), f_2(\psi))$ *is a* $\mathcal{P}$*-measurable transformation (adopting the Cartesian-product cylinder algebra as the cylinder algebra on* $H_1 \oplus H_2$*);*

(vii) *if* H *is a vector space, with the scalar field* R*, equipped with a finitely additive set function* $\mathcal{P}$*, the operation* $+ : H^2 \to H$ *and the scalar multiplication operation* $\times : R \times H \to H$ *are* $\mathcal{P}$*-measurable transformations;*

(viii) *if* $f_1 : G \to H$ *and* $f_2 : G \to H$ *are* $\mathcal{P}$*-measurable functionals then so are* $f_1 + f_2$ *and (when* H *is scalar)* $f_1 \times f_2$*. Also, if, instead,* $f_2 : G \to K$ *where* K *is the scalar field of* H*,* $f_1 \times f_2$ *is* $\mathcal{P}$*-measurable.*

Proof. (i) See (Dunford and Schwartz 1957, Lemma III.2.11).

(ii) See (Dunford and Schwartz 1957, Lemma III.2.12).

(iii) Choose $\varepsilon > 0$ and simple $\widetilde{g} : H \to K$, such that $|\widetilde{g} - g|_{\mathcal{Q}} < \varepsilon/2$. Suppose

$$\widetilde{g} = \sum_{j=1}^{J} g_j \chi_{B_j},$$

where $B_j \in \mathcal{C}_H$. Since $\mathcal{P} \circ f^{-1} \leq \mathcal{Q}$,
$\exists C_j \in \mathcal{C}_G$, with $\mathcal{P}\left(C_j \Delta f^{-1}(B_j)\right) < \varepsilon/(2J)$. It follows that defining the function

$$\widetilde{h} = \sum_{j=1}^{J} g_j \chi_{C_j}$$

we find $\left|\widetilde{h} - g \circ f\right| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this shows that $g \circ f$ is $\mathcal{P}$-measurable.

Now choose $\varepsilon > 0$ and $\delta_1 > 0$ so that $\mathcal{Q}(A) < \delta_1 \Rightarrow \mathcal{P}(A) < \varepsilon$ and set $\delta = \min(\delta_1, \varepsilon)$. If $|g|_{\mathcal{Q}} < \delta$, $\mathcal{Q}^*(\{\psi : g(\psi) > \varepsilon\}) < \delta_1$, hence

$$\mathcal{P} \circ f^{-1}(\{\psi : g(\psi) > \varepsilon\}) = \mathcal{P}(\{\psi : g \circ f(\psi) > \varepsilon\}) \quad < \quad \varepsilon \tag{3.29}$$

so $|g \circ f|_{\mathcal{P}} < \varepsilon$, as required.

(iv) Apply Theorem 3.2.1 (iii) to show that $g \circ f$ and $g_k \circ f$ are $\mathcal{P}$-measurable, then apply the second part of (iii) to f and $g - g_k$ to show that $g_k \circ f \to g \circ f$ in $\mathcal{P}$-measure.

(v) Suppose $g \in \mathcal{B}_n$,i.e. it is zero except on $V = \Pi^{-1}(W)$, for a multidimensional interval $W \in \mathbf{R}^n$, say, where it takes the value g_0. Then by Proposition 3.2.3, V is also $\mathcal{P} \circ f_k^{-1}$-measurable, and by Proposition 3.2.4, $g \circ f_k$ is $\mathcal{P}$-measurable; in fact, $g \circ f_k$ is the indicator function $\chi_{f_k^{-1}(V)}$ which shows that $f_k^{-1}(V)$ is $\mathcal{P}$-measurable. Likewise, $g \circ f$ is the indicator function of $f^{-1}(V)$. Choose $\varepsilon > 0$. Since $\mathcal{P}$ is the standard Gaussian measure, $\exists \delta > 0$, such that if $U' = \{\psi \in V \wedge \exists \psi' \notin V \wedge |\psi - \psi'| < \delta\}$, $\mathcal{P}(U') < \varepsilon/2$. Now $|f - f_k|_{\mathcal{P}} \to 0$, so choose $K > 0$ so that for $k > K$, $|f - f_k|_{\mathcal{P}} < \varepsilon' = \min(\varepsilon/2, \delta)$. It follows that $\forall k > K \exists U_k \in \mathcal{C}_H$ with $\mathcal{P}(U_k) < \varepsilon'$ such that $\psi \notin U_k \Rightarrow |f(\psi) - f_k(\psi)| < \varepsilon'$. Now for $k > K$ and $\psi \in f_k^{-1}(V) \Delta f^{-1}(V)$ either: (a) $|f(\psi) - f_k(\psi)| \geq \varepsilon'$, (b) $\psi \in V \wedge \exists \psi' \notin V \wedge |\psi - \psi'| < \varepsilon'$, or (c) $\psi \notin V \wedge \exists \psi' \in V \wedge |\psi - \psi'| < \varepsilon'$. Thus, $f_k^{-1}(V) \Delta f^{-1}(V) \subseteq U_k \cup U'$, $k > K$. and therefore $\mathcal{P}^* \left(f_k^{-1}(V) \Delta f^{-1}(V) \right) < \varepsilon$. This shows that $f^{-1}(V)$ is $\mathcal{P}$-measurable and $g \circ f_k \to g \circ f$ in $\mathcal{P}$-measure.

If (v) is true for g of this form it is also true for linear combinations of such g, i.e. simple g, and also, by (iv) (using uniformity of $\mathcal{P} \circ f_k^{-1} \leq \mathcal{P}$), for $\mathcal{P}$-measurable g.

(vi) Based on Definition 3.2.5 $\mathcal{C}_{H \times H}$ contains the set $H \times H$ and $\mathcal{C}_{R \times H}$ contains the set $R \times H$ and both are closed in-regards to complements and finite set unions. Since the inverse image of complement of a set is equal to complement of inverse image of it and the inverse image of finite union of subsets of $H \times H$ or $R \times H$ are equal to union of their inverse images and because the vector space H is closed under vector addition and scalar multiplication which makes the operation $+ : H^2 \to H$ and $\times : R \times H \to H$, bijective set functions, $+^{-1}(\mathcal{C}_{H \times H}) = \times^{-1}(\mathcal{C}_{R \times H}) = \mathcal{C}_H$ and by definition 3.2.2 $+$ and $\times$ are $\mathcal{P}$-measurable transformations.

(vii) Let $\{\phi_k^{\{j\}}\}$ be a sequence of simple functions converging to f_j in $\mathcal{P}$-measure, $j = 1,\ 2$. Set $\phi_k : x \mapsto \left\langle \phi_k^{\{1\}}(x), \phi_k^{\{2\}}(x) \right\rangle$. Then $\{\phi_k\}$ is a sequence of simple functions converging to f in $\mathcal{P}$-measure.

(viii) Let ϕ be the continuous mapping from $H^2 \to H$, defined by $\phi(\zeta_1, \zeta_2) = \zeta_1 + \zeta_2$. Let f_1 and f_2 be $\mathcal{P}$-measurable transformations on G, and let $F(g) = \langle f_1(g), f_2(g) \rangle$, then $F : G \to H^2$ and by (vii) F is a $\mathcal{P}$-measurable transformation. Since ϕ is continuous and $\phi \circ F(g) = f_1(g) + f_2(g)$, hence we conclude $\phi \circ F : G \to H$ is a $\mathcal{P}$-measurable transformation. See also (Dunford and Schwartz 1957, Lemma III.2.8 and Lemma III.2.12).

□

Remark 3.2.5. *Combining Property (vii) with Property (iii), and using the measurable mapping* $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$*, for example, shows that if* $f : H \to \mathbf{R}$ *and* $g : H \to \mathbf{R}$ *are* $\mathcal{P}$*-measurable then the mapping* $\psi \mapsto f(\psi) + g(\psi)$ *is* $\mathcal{P}$*-measurable.*

As with classical measurability, it is likely that virtually all the functions which are relevant in a given context (in the sense that we wish to investigate their properties and apply certain manipulations) are $\mathcal{P}$*-measurable. However, as Example 3.2.2 shows, non-*$\mathcal{P}$*-measurable functions do exist and not all the operations which are applicable to* $\mathcal{P}$*-measurable functions can be employed with them. Consequently, although it is expected that all functions which need to be* $\mathcal{P}$*-measurable will be, it remains necessary to prove this in each case. Theorem 3.2.1 will assist in many cases, but, as the example application in Section 3.3 shows, there remain cases of* $\mathcal{P}$*-measurable functions whose* $\mathcal{P}$*-measurability cannot be demonstrated in this way.*

3.2.2 The $\mathcal{P}$-measurable Derivative

In this subsection we define the derivative of a $\mathcal{P}$-measurable transformation which operates on a space of $\mathcal{P}$-measurable functions. The $\mathcal{P}$-measurable functions will all be assumed to be defined on a Hilbert space H with additive set function $\mathcal{P}$ defined on the cylinder algebra. H is also assumed to be equipped with a measurable norm, denoted by $\|\cdot\|$. The codomain of these functions is assumed to be a normed vector space, X (adopting the same broad concept of norm as applies to the norm on H, i.e we do not assume X is a Banach space.) The two special cases $X = \mathbf{R}$ and $X = H$ are of most interest.

Since a $\mathcal{P}$-measurable function does not have a unique well-defined value at each location of its domain, the limit which usually occurs, as a central feature, in the definition of a derivative can't be a point-wise limit. It is natural, given that functions under consideration are $\mathcal{P}$-measurable, to adopt $\mathcal{P}$-measurable limits instead of point-wise ones. Switching to $\mathcal{P}$-measurable limits is much more than a technical device however.

Since $\mathcal{P}$-measurable limits are defined in terms of underlying finite-dimensional subspaces, the $\mathcal{P}$-measurable limit has similar applicability to the stochastic derivative, as defined in (Di Nunno et al. 2009, A.2). This connection is explored further in Proposition 3.2.7, below. The stochastic derivative is shown in (Di Nunno et al. 2009) to coincide with the Malliavin derivative under certain assumptions. For applicability, we need a derivative which is well-defined under weak assumptions. The assumptions for the stochastic derivative to be well-defined are not as weak as either the Malliavin derivative or the $\mathcal{P}$-measurable derivative. On the other hand, the definition of the stochastic derivative is simpler than the Malliavin derivative, which helps make it clear how it serves the same role as a Frechèt derivative would, if it were defined. Thus, the stochastic derivative also motivates the Malliavin derivative. The $\mathcal{P}$-measurable derivative is more clearly modelled on the Frechét derivative, so motivating it in this way is not necessary. However, the fact that it also coincides with the stochastic derivative makes it clear that it is applicable in the same contexts as the Malliavin derivative.

The next proposition shows that $\mathcal{P}$-measurable transformations are, in a sense, approximately finite-dimensional.

Proposition 3.2.5. *$T : H \to H$ is $\mathcal{P}$-measurable if for any sequence of projections $\{\Pi_k\}$ with $\Pi_k \to I$, strongly, and any sequence of projections $\{\Theta_k\}$ with $\Theta_k \to I$ strongly, $\Theta_k \circ T \circ \Pi_k$ is $\mathcal{P}$-measurable for all $k > 0$, and $|T - \Theta_k \circ T \circ \Pi_k|_{\mathcal{P}} \to 0$.*

Conversely, if $T : H \to H$ is $\mathcal{P}$-measurable, for all *pairs of sequences of projections $\{\Pi_k\}$ with $\Pi_k \to I$ and $\{\Theta_k\}$ with $\Theta_k \to I$ strongly, $|T - \Theta_k \circ T \circ \Pi_k|_{\mathcal{P}} \to 0$.*

Proof. The *if* part follows because any $\mathcal{P}$-measurable limit of $\mathcal{P}$-measurable functions is $\mathcal{P}$-measurable. For the converse, suppose T is $\mathcal{P}$-measurable and suppose $|T - T_\ell|_{\mathcal{P}} < 1/\ell$, $\ell > 0$, where T_ℓ is simple. Suppose, to be specific, that

$$T_\ell = \sum_{j=1}^{M_\ell} h_j \chi_{S_j}.$$

By Lemma 3.2.2, for sufficiently large $k > 0$,

$$|T_\ell - \Theta_k \circ T_\ell \circ \Pi_k|_{\mathcal{P}} < 1/\ell,$$

and hence

$$|T - \Theta_k \circ T \circ \Pi_k|_{\mathcal{P}} < 2/\ell,$$

which concludes the proof.

□

Definition 3.2.9. *Suppose $T:(H,\mathcal{P})\to(X,|\cdot|)$ is $\mathcal{P}$-measurable, $D_\xi T:H\to\mathcal{M}(H,X)$ is defined by*

$$D_\xi T:\psi\mapsto\left(h\mapsto\frac{T(\psi+\xi h)-T(\psi)}{\xi}\right), \tag{3.30}$$

$\xi>0$, $V:H\to\mathcal{L}(H,X)$, and

$$D_\xi T\to V \tag{3.31}$$

in $\mathcal{P}$-measure, as $\xi\to 0$. Then $\mathcal{D}_\psi T\doteq V$ is termed the $\mathcal{P}$-measurable derivative of T.

This definition, is illustrated in Section 3.3. In some cases the following proposition may provide an easier way to find the $\mathcal{P}$-measurable derivative.

Proposition 3.2.6. *Suppose $T_k\to T$ in $\mathcal{P}$-measure and the defining limits for $\mathcal{D}_\psi T_k$ exist* uniformly *over $k>0$.*

Then, if $\lim_{k\to\infty}\mathcal{D}_\psi T_k$ exists (as a $\mathcal{P}$-measurable function with convergence in $\mathcal{P}$-measure), $\mathcal{D}_\psi T$ exists and $\mathcal{D}_\psi T_k\to\mathcal{D}_\psi T$ in $\mathcal{P}$-measure, and conversely, if $\mathcal{D}_\psi T$ exists, $\mathcal{D}_\psi T_k\to\mathcal{D}_\psi T$ in $\mathcal{P}$-measure.

Proof. Uniformity justifies interchanging the order of the limits. More formally, choose $\varepsilon>0$ and $E>0$ such that the defining limits for $\mathcal{D}_\psi T_k$ have been reached with error less than ε for all $\xi<E$, for all $k>0$.

Suppose also $|\mathcal{D}_\psi T_k-V|_{\mathcal{P}}<\varepsilon$ for all $k>K$. Therefore for $\xi<E$

$$\begin{aligned}\left|D_\xi T_k-V\right|_{\mathcal{P}} &< \left|D_\psi T_k-\mathcal{D}_\psi T_k\right|_{\mathcal{P}}+\left|\mathcal{D}_\psi T_k-V\right|_{\mathcal{P}}\\ &< 2\varepsilon.\end{aligned}$$

Choose a specific ξ. Then, for sufficiently large $K_1>0$, for all $k>K_1$,

$$\left|D_\xi T-D_\xi T_k\right|_{\mathcal{P}}<\varepsilon,$$

Hence, for $k>\max(K,K_1)$, $\left|D_\xi T-V\right|_{\mathcal{P}}<3\varepsilon$. Since ε is arbitrary, $\mathcal{D}_\psi T=V$.

For the converse, suppose instead, that $D_\xi T$ exists. In particular, suppose that for all $\xi<E$, the $|\cdot|_{\mathcal{P}}$-norm of the difference between the LHS and RHS at (3.31) is less than ε.

Choose $E_1 > 0$ such that for all $\xi < E_1$, $\left|D_\xi T_k - \mathcal{D}_\psi T_k\right|_{\mathcal{P}} < \varepsilon$. Then, for some $\xi < \min(E, E_1)$, we can choose $K > 0$ such that for $k > K$, $\left|D_\xi T_k - D_\xi T\right|_{\mathcal{P}} < \varepsilon$ and hence $\left|\mathcal{D}_\psi T_k - D_\xi T\right|_{\mathcal{P}} < 2\varepsilon$ and hence $\left|\mathcal{D}_\psi T_k - \mathcal{D}_\psi\right|_{\mathcal{P}} < 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\mathcal{D}_\psi T_k \to \mathcal{D}_\psi T$ in $\mathcal{P}$-measure.

□

The next proposition shows that on $C([0,T])$, the stochastic derivative and the $\mathcal{P}$-measurable derivative are essentially the same, on H, where H in this context is the Cameron-Martin space associated with $C([0,T])$.

Proposition 3.2.7. *If $F : C([0,T]) \to \mathbf{R}$ is $\mathcal{P}$-measurable and has a stochastic derivative, $D_t F$, in the sense defined in (Di Nunno et al. 2009, A.2), then it also has a $\mathcal{P}$-measurable derivative, $\mathcal{D}_\psi TF$, and the mapping $\omega \mapsto \left\{\gamma \mapsto \int_0^T D_t F(\omega)\gamma'(t)dt\right\} = \mathcal{D}_\psi TF$ in $\mathcal{P}$-measure on H.*

Proof. Since $t \mapsto D_t F(\omega) \in L_2([0,T])$, for almost all ω, the mapping

$$\gamma \mapsto \int_0^T D_t F(\omega)\gamma'(t)dt$$

can be regarded as a linear functional on H, and hence, we can define a continuous mapping, $U : H \to \mathcal{L}(H, \mathbf{R})$ by $\omega \mapsto \left\{\gamma \mapsto \int_0^T D_t F(\omega)\gamma'(t)dt\right\}$. This shows that $D_t F$ (the stochastic derivative) and V (the $\mathcal{P}$-measurable derivative) have the same form, i.e. fundamentally both are a mapping from H to H'. (We can ignore the fact that U is defined on the whole of Ω, not just on H.)

It remains to show that whenever the stochastic derivative is defined, the $\mathcal{P}$-measurable derivative is defined, and that they are equal, in $\mathcal{P}$-measure, i.e. $\exists U \Rightarrow U = V$ in $\mathcal{P}$-measure.

By Definition A.10 of (Di Nunno et al. 2009), $D_t F$ is well-defined only if

$$D_\gamma F = \lim_{\xi \to 0} \frac{F(\omega + \xi\gamma) - F(\omega)}{\xi} \qquad (A.12)$$

exists and is in $L_2(P)$ for almost all $\omega \in \Omega$.

Since F is $\mathcal{P}$-measurable, so is $D_\xi F$, for all $\xi > 0$. Suppose (3.31) is not the case, with V in Definition 3.2.9 $= U$. Then $\exists \delta > 0$, $\forall \xi > 0$, $\exists A_{\delta,\xi} \in \mathcal{C}(H)$ with $\mathcal{P}(A_{\delta,\xi}) > \delta$ and

$$\left| \left\{ \gamma \mapsto \frac{F(\omega + \xi\gamma) - F(\omega)}{\xi} \right\} - U(\omega) \right|_{\mathcal{P}} > \delta$$

on $A_{\delta,\xi}$. Hence, for some $B_{\delta,\xi} \in \mathcal{C}(H)$, with $\mathcal{P}(B_{\delta,\xi}) > \delta$,

$$\left| \frac{F(\omega + \xi\gamma) - F(\omega)}{\xi} - U(\omega)\gamma \right| > \delta$$

for all $\gamma \in B_{\delta,\xi}$. Thus,

$$\int \left| \frac{F(\omega + \xi\gamma) - F(\omega)}{\xi} - U(\omega)\gamma \right|^2 P(d\omega) > \delta^3.$$

This contradicts (A.12). □

3.2.3 The Radon-Nikodym Derivative

In the remainder of this section $\mathcal{P}$ and $\mathcal{Q}$ denote positive additive set functions on $(H, \mathcal{C}_H)$, and G, H, and K denote Hilbert spaces equipped with a measurable norm and an algebra of subsets.

Definition 3.2.10. *If $\mathcal{P} \leq \mathcal{Q}$ are additive set functions on an algebra of subsets $\mathcal{C}_H$ of a space H, and $U \in \mathcal{C}_H$, we write*

$$\frac{d\mathcal{P}}{d\mathcal{Q}} = \rho$$

on U if ρ is $\mathcal{Q}$-measurable and for all simple $\mathcal{Q}$-measurable functionals f on H,

$$\int_U f(x)\mathcal{P}(dx) = \int_U f(x)\rho(x)\mathcal{Q}(dx). \tag{3.32}$$

Note: by Proposition 3.2.3, whenever f is $\mathcal{Q}$-measurable it is also $\mathcal{P}$-measurable, so the integral on the left of (3.32) is well defined.

Theorem IV.9.14 from (Dunford and Schwartz 1957) implies that whenever $\mathcal{P} \leq \mathcal{Q}$, $\frac{d\mathcal{P}}{d\mathcal{Q}}$, as in Definition 3.2.10, exists. Dunford and Schwartz attribute this result to Bochner and describe it as a generalization of the Radon-Nikodym theorem, and also, in the index, as the *Radon-Nikodym theorem for bounded additive set functions*.

The definition of Radon-Nikodym derivative given here differs from that in (Bagchi and Mazumdar 1993) in that the defining condition is simply for (3.32) to hold, for all $\mathcal{P}$-measurable functions f, whereas in (Bagchi and Mazumdar 1993) the condition includes an additional requirement of uniform convergence over all sequences of finite-dimensional approximations to this integral. Another difference in the approach of (Bagchi and Mazumdar 1993) is the concept of *physical random variables*, which sometimes serve in place of $\mathcal{P}$-measurable functions. It follows from Lemma 3.2.2 below, that real-valued physical random variables are precisely the same as $\mathcal{P}$-measurable functionals. This is not necessarily true in the more general case of H'-valued physical random variables, however, which are involved in the applications which motivate the work in (Bagchi and Mazumdar 1993).

When (3.32) holds for all *simple* $\mathcal{P}$-measurable functions, f, it holds when f is *any* $\mathcal{P}$-measurable function. Furthermore, it is sufficient for (3.32) to hold for an even more limited class of $\mathcal{P}$-measurable functions to ensure this, as the next proposition shows.

Proposition 3.2.8. *For (3.32) to be true for all $\mathcal{P}$-measurable functions it is sufficient if it is true for all $f \in \mathcal{B}_n$ (as in Definition 3.2.8), for all $n > 0$.*

Proof. Suppose (3.32) is true for all $f \in \mathcal{B}_n$. Select a specific projection $\Pi : H \to R^n$ and let F_n denote the class of Borel sets, $F \subseteq \mathbf{R}^n$ for which (3.32) is true for the indicator function of $\Pi^{-1}(F)$. Thus $B_n \subseteq F_n$. By the linearity of the integral in (3.32), F_n must form an algebra. By the assumption that $\mathcal{P} \circ \Pi^{-1}$, and $Q \circ \Pi^{-1}$ are measures, F_n must be a σ-algebra, and hence includes all Borel sets.

Now suppose g is an arbitrary $\mathcal{P}$-measurable function. There is therefore a simple function, f, which differs from g in $|\cdot|_{\mathcal{P}}$-norm by less than ε, for any $\varepsilon > 0$. Thus (3.32) holds with g in place of f to accuracy ε for any $\varepsilon > 0$, and hence must hold exactly, which also shows that the right-hand integral is well-defined, i.e. $f(x)\rho(x)$ is Q-measurable. □

Remark 3.2.6. *A further refinement of this proposition which could be developed is that it is sufficient for (3.32) holds for all indicator functions of the form stated in the proposition but where Π, instead of being an* arbitrary *orthogonal projection, is an orthogonal projection in a sequence which converges strongly to the identity.*

The case where $Q = \mathcal{P} \circ T^{-1}$ for some $\mathcal{P}$-measurable transformation T is more important than any other, in the rest of this research, so let us take time to interpret the preceding definition in this context.

Proposition 3.2.9. *Given an additive set-function $\mathcal{P}$ on the algebra of sets $(H, \mathcal{C}_H)$, a $\mathcal{P}$-measurable mapping $T : H \to (H, |\cdot|_\Omega)$, and $\mathcal{P}$-measurable $U \subseteq H$, if there exists a $\mathcal{P}$-measurable function ρ with the property that for all simple $\mathcal{P}$-measurable functions, f, of the form set out in Proposition 3.2.8,*

$$\int_U f(T(x))\mathcal{P}(dx) = \int_{T(U)} f(x)\rho(x)\mathcal{P}(dx) \tag{3.33}$$

then $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$ and

$$\frac{d\mathcal{P} \circ T^{-1}}{d\mathcal{P}} = \rho$$

on U.

Proof. The integrals in (3.33) reduce to those in Definition 3.2.10 by means of the substitution rule for integration with respect to an additive set function (Dunford and Schwartz 1957). Using these integrals we can show $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$. □

3.2.4 Ramer's theorem in H

Definition 3.2.11. *The* determinant $\operatorname{Det} T$ *of a linear operator on $T : G \to H$ is defined as* $\lim_{k\to\infty} \operatorname{Det} M_k$*, where M_k is a matrix representation of the operator $\Theta_k \circ T \circ \Pi_k$, which uses an orthonormal bases for both the domain and range, when this limit exists and is consistent for all sequences $\{(\Pi_k, \Theta_k)\}_{k=1}^\infty$ of pairs of finite-dimensional projections, with* $\dim \Pi_k G = \dim \Theta_k H$*, $k \geq 1$, converging to the identity in G and H.*

For example, $\operatorname{Det} I = 1$. In more generality, if T can be represented in the form

$$\begin{pmatrix} A & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

for some finite-dimensional square matrix A, in some pair of orthonormal bases $\{e_k\}$, $\{f_k\}$, then $\operatorname{Det} T = \operatorname{Det} A$.

Definition 3.2.12. *If $S \subseteq H$, $[S]$ denotes the smallest subspace of H which contains S. For example, $[\phi_t : t \geq 1]$ is the smallest subspace containing $\{\phi_t : t \geq 1\}$.*

Remark 3.2.7. *Definition 3.2.11 includes the Fredholm determinant; however, it also includes operators, even when $G = H$, which are not Hilbert-Schmidt, or of the form $I + A$ where A is Hilbert-Schmidt. For example, the fBm isometry, κ_δ, has determinant 1 for all $\delta > 0$, even though when $\delta \neq 1$ it has no eigenvalues, and is not Hilbert-Schmidt, and neither is $I - \kappa_\delta$. For an example where $G \neq H$, consider $\kappa_\delta : [\phi_t : t \geq 1] \to [\phi_t : t \geq \delta]$, for which $\operatorname{Det} \kappa_\delta = 1$.*

Theorem 3.2.2. *If $T : (H, \mathcal{P}) \to H$ is $\mathcal{P}$-measurable, with $\mathcal{P}$-measurable inverse, $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$, and $T : (H, \mathcal{P})$ has a well-defined $\mathcal{P}$-measurable derivative, $\mathcal{D}_\psi T$, with well-defined, $\mathcal{P}$-measurable and non-zero, determinant, $\operatorname{Det} \mathcal{D}_\psi T$, on H, then the Radon-Nikodym derivative induced on $(H, \mathcal{P})$ is*

$$\frac{d\mathcal{P} \circ T}{d\mathcal{P}}(\psi) = \rho(\psi) = \left|\operatorname{Det} \mathcal{D}_\psi T\right| \exp\left(-\frac{1}{2}\,|T\psi - \psi|_H^2 - \langle T\psi - \psi, \psi\rangle\right). \tag{3.34}$$

Proof. Notice that the result is identical to the Jacobian theorem for transformation of a Gaussian density on $\mathbf{R}^n$, except for the dimension of H.

The condition $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$ is required because it is a precondition for the Radon-Nikodym derivative to be defined.

By Proposition 3.2.9, it is sufficient to show (3.33) (with $U = H$) for all $B \in \mathcal{C}_H$. Choose $B \in \mathcal{C}_H$, and $\varepsilon > 0$ and let us show

$$\left|\int \chi_B(T(x))\mathcal{P}(dx) - \int_B \rho(x)\mathcal{P}(dx)\right| < \varepsilon. \tag{3.35}$$

Note that for the left-hand integral in (3.35) to be well-defined it is necessary that $\chi_B \circ T$ is $\mathcal{P}$-measurable, which is justified by Theorem 3.2.1, Part (iii), which also requires the condition $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$.

Let $\{\Pi_n\}$ be a sequence of projections converging to the identity on H. Since T is $\mathcal{P}$-measurable, by Proposition 3.2.5, $\exists n_1 > n_0$ such that $\forall n > n_1$,

$$|\Pi_n \circ T \circ \Pi_n - T|_{\mathcal{P}} < \varepsilon/3.$$

Set $T_n = \Pi_n \circ T \circ \Pi_n$,

$$\widetilde{\rho}_n(\psi) = \operatorname{Det}\left(\mathcal{D}_\psi T_n\right) \exp\left(-\left|\mathcal{D}_\psi T_n \psi - \psi\right|_H^2 - \frac{1}{2}\langle T_n\psi - \psi, \psi\rangle\right),$$

and

$$\rho_n(\psi) = \operatorname{Det}\left(\mathcal{D}_\psi T_n\right) \exp\left(-|T\psi - \psi|_H^2 - \frac{1}{2}\langle T\psi - \psi, \psi\rangle\right).$$

Since $\operatorname{Det}\mathcal{D}_\psi T$ is well-defined, $\operatorname{Det}\mathcal{D}_\psi T_n = \operatorname{Det}\left(\Pi_n \circ \mathcal{D}_\psi T \circ \Pi_n\right) \to \operatorname{Det}\mathcal{D}_\psi T$ in $\mathcal{P}$-measure and so $\exists n_2 > 0$ such that $\forall n > n_2$,

$$|\rho(\psi) - \rho_n(\psi)|_{\mathcal{P}} < \varepsilon/3.$$

Since $T_n \to T$ in $\mathcal{P}$-measure, $\exists n_3 > 0$ such that $|T_n - T|_{\mathcal{P}} < \varepsilon/3$, $\forall n > n_3$, and also $\exists n_4 > n_3$ such that $|\rho_n - \widetilde{\rho}_n|_{\mathcal{P}} < \varepsilon/3$ for all $n > n_4$. Also, by Theorem 3.2.1, Part (iv), $\exists n_5 > 0$ such that $\forall n > n_5$, $|\int \chi_B \circ T(x) - \int \chi_B \circ T_n(x)|_{\mathcal{P}} < \varepsilon/3$. For $n > \max(n_1, n_2, n_4, n_5)$, therefore,

$$\begin{aligned}
&\left|\int \chi_B(T(x))\mathcal{P}(dx) - \int_B \rho(x)\mathcal{P}(dx)\right| = \Bigg|\int \chi_B(T(x))\mathcal{P}(dx) \\
&- \int \chi_B(T_n(x))\mathcal{P}(dx) + \int \chi_B(T_n(x))\mathcal{P}(dx) - \int_B \widetilde{\rho}_n(x)\mathcal{P}(dx) \\
&\quad + \int_B \widetilde{\rho}_n(x)\mathcal{P}(dx) - \int_B \rho_n(x)\mathcal{P}(dx) + \int_B \rho_n(x)\mathcal{P}(dx) - \int_B \rho(x)\mathcal{P}(dx)\Bigg| \\
\leq\ & \varepsilon/3 + 0 + \varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily small, (3.33) holds, with $f = \chi_B$, and since B is any set $\in \mathcal{C}_H$, (3.38) is true. □

Proposition 3.2.10. *Suppose $\left\{\mu_p\right\}_{p\in\mathbf{R}}$, $\mu_p : H \to H$, is a family of $\mathcal{P}$-measurable invertible transformations on H with*

$$\frac{d\mathcal{P}\circ\mu_p}{d\mathcal{P}} = \theta_p, \tag{3.36}$$

$p \in \mathbf{R}$ and $\{\pi_k\}$ is a sequence of $\mathcal{P}$-measurable functionals converging to a $\mathcal{P}$-measurable functional π such that,

(a) setting $\theta_{\pi_k} : \psi \mapsto \theta_{\pi_k(\psi)}(\psi)$ and $\theta_\pi : \psi \mapsto \theta_{\pi(\psi)}(\psi)$, $\theta_{\pi_k} \to \theta_\pi$ in $\mathcal{P}$-measure;

(b) and, setting $\mu_{\pi_k} : \psi \mapsto \mu_{\pi_k(\psi)}(\psi)$ and $\mu_\pi : \psi \mapsto \mu_{\pi(\psi)}(\psi)$, μ_{π_k} is invertible for all $k > 0$, μ_π is invertible, and $\mu_{\pi_k} \to \mu_\pi$ in $\mathcal{P}$-measure.

Then

$$\frac{d\mathcal{P}\circ\mu_\pi}{d\mathcal{P}} = \theta_\pi. \tag{3.37}$$

Proof. First, note that

$$\frac{d\mathcal{P}\circ\mu_{\pi_k}}{d\mathcal{P}} = \theta_{\pi_k}$$

because π_k is a linear combination of functionals which are constant, and we can use (3.36) for each of these constant functionals.

Suppose $V \in \mathcal{B}_n$. Then

$$\int V(\mu_\pi(\psi))\mathcal{P}(d\psi) = \lim_{k\to\infty}\int V(\mu_{\pi_k}(\psi))\mathcal{P}(d\psi),$$

by Theorem 3.2.1 Part (v), noting that $\mathcal{P}\circ\mu_p^{-1} \leq \mathcal{P}$ follows from (3.36),

$$= \lim_{k\to\infty}\int V(\psi)\theta_{\pi_k}(\psi)\mathcal{P}(d\psi)$$

$$= \int V(\psi)\theta_\pi(\psi)\mathcal{P}(d\psi)$$

Since $V \in \mathcal{B}_n$ and n are arbitrary, (3.37) holds. □

3.2.5 Girsanov Form

Results of type which describe how the dynamics of stochastic processes change when the original measure is changed to an equivalent probability measure have been studied by many including Cameron and Martin in the 1940s and by Girsanov in 1960 (Girsanov 1960). The first Girsanov-type theorem for fBm was obtained by (Molchan 1969) in 1960s. More recently Norros et al (Norros et al. 1999) proved such a result for fractional Brownian motion by finding an integral transformation which changes the original centered fBm process to a process with independent increments. Using the recent advances in the white noise machinery as described in (Biagini et al. 2008*a*, Section 3.1) the proof of the Benth-Gjessing version of the Girsanov formula (Holden et al. 1996, Corollary 2.10.5), can be verified and shown to apply in the fractional case. By further generalising the fractional white noise framework in which processes with all indices can be considered under the same probability measure, the Girsanov theorem for fBm valid for every Hurst index $H \in (0,1)$ has also been obtained in (Elliott and Van der Hoek 2003, Section 5).

The Girsanov theorem can be viewed as a special case and, in some respects, a generalisation of the Cameron-Martin theorem. In particular, there is a way of expressing the Girsanov theorem which is attractive to many researchers, which can be applied in the more general context of this research. There is an excellent discussion on the relationship between the Ramer and the Girsanov formulas in (Zakai et al. 1992). Using characterization of quasinilpotent Hilbert-Schmidt operators they show that these two are identical if the Frechèt derivative is nuclear (trace class) and in that case, the Jacobian is just 1.

Theorem 3.2.3. *If $T : (H, \mathcal{P}) \to H$ is $\mathcal{P}$-measurable, with $\mathcal{P}$-measurable inverse, $\mathcal{P} \circ T^{-1} \leq \mathcal{P}$, and $T : (H, \mathcal{P})$ has a well-defined $\mathcal{P}$-measurable derivative, $D_\psi T$, with well-defined, $\mathcal{P}$-measurable and non-zero, determinant,* $\operatorname{Det} \mathcal{D}_\psi T$*, on H, and we define a $\mathcal{P}$-measurable measure $\mathcal{P}_a$ on H by*

$$\frac{d\mathcal{P}_a}{d\mathcal{P}}(\psi) = \left|\operatorname{Det} \mathcal{D}_\psi T\right|^{-1} \exp\left(-\frac{1}{2}|T\psi - \psi|_H^2 + \langle T\psi - \psi, \psi\rangle\right). \tag{3.38}$$

Then the distribution of ψ with respect to $\mathcal{P}_a$ is the same as the distribution of $T(\psi)$ with respect to $\mathcal{P}$.

Remark 3.2.8. *The expression* "the distribution of ψ with respect to $\mathcal{P}_a$" *really means: $\mathcal{P}_a$. Therefore, restating the theorem more formally, it is: $\mathcal{P}_a$* (as defined by (3.38)) is the probability measure of $T(\psi)$.

Proof. Let $B \in \mathcal{C}_H$. Then

$$\mathcal{P}(T(\psi) \in B) \quad = \quad \mathcal{P} \circ T^{-1}(B),$$

which, by Theorem 3.2.2 applied with T^{-1} in place of T,

$$= \quad \mathcal{P}_a(B).$$

□

Let us now verify that the Theorem 3.2.3 reduces to the Norros et el's formula from (Norros et al. 1999) when appropriate substitutions are made. Consider the case, namely $T : \psi \mapsto \psi'$ where

$$\psi'(s) = \psi(s) + ms,$$

for $s \leq t$, for some constant t. In order to be able to apply Theorem 3.2.3 we assume H is the Hilbert space of paths on $[0,t]$ rather than on $[0,\infty)$. Defining $\mathcal{P}_a$ as in Theorem 3.2.3, (3.38) becomes

$$\frac{d\mathcal{P}_a}{d\mathcal{P}}(\psi) \quad = \quad \left|\mathrm{Det}\,\mathcal{D}_\psi T\right|^{-1} \exp\left(-\frac{1}{2}\left|T\psi-\psi\right|_H^2 + \langle T\psi-\psi,\psi\rangle\right).$$

There are three separate terms here which need to be evaluated:

(i) $\left|\mathrm{Det}\,\mathcal{D}_\psi T\right|^{-1}$,

(ii) $|T\psi-\psi|_H^2 = |\mu|_H^2$, where $\mu : s \mapsto ms$, and

(iii) $\langle T\psi-\psi,\psi\rangle = \langle \mu,\psi\rangle_H$.

The $\mathcal{P}$-measurable derivative of T is I, so the first term is 1. To evaluate the second term we need to find $|\mu|_H$. According to (Biagini et al. 2008*b*), $|\mu|_H = |\widetilde{\mu}|_{L_2(0,t)}$ where $\widetilde{\mu}$ is the unique function such that

$$\mu(t) = \int_0^t K_H(t,s)\widetilde{\mu}(s)ds \tag{3.39}$$

in which

$$K_H(t,s) = c_H s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du,$$

where

$$\begin{aligned}
c_H &= \sqrt{\frac{H\,(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}} \\
&= \sqrt{\frac{H\,(2H-1)\,\Gamma\left(\frac{3}{2}-H\right)}{\Gamma(2-2H)\Gamma(H-\frac{1}{2})}} \\
&= \sqrt{\frac{H\,(2H-1)\,\Gamma\left(\frac{3}{2}-H\right)\left(H-\frac{1}{2}\right)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})}} \\
&= \left(H-\frac{1}{2}\right)c_N
\end{aligned}$$

where $c_N = \left(\frac{2H\Gamma(3/2-H)}{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}}$ is the constant c_H of (Norros et al. 1999). Using the fractional integral of order α, denoted by I_{0+}^{α}, and the fractional derivative of order α, denoted by D^{α}_{0+}, if ϕ is differentiable, (3.39), with ϕ in place of μ, implies

$$\begin{aligned}\frac{d\phi}{dt} &= c_H t^{\frac{1}{2}-H}\int_0^t s^{H-\frac{1}{2}}(t-s)^{H-\frac{3}{2}}\widetilde{\phi}(s)ds \\ &= c_H t^{H-\frac{1}{2}}\Gamma(H-\frac{1}{2})I_{0+}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}\widetilde{\phi}(t)\end{aligned}$$

so

$$\widetilde{\phi}(t) = \frac{t^{H-\frac{1}{2}}}{c_N\Gamma(H+\frac{1}{2})}D_{0+}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}\frac{d\phi}{dt}. \tag{3.40}$$

To evaluate this fractional integral when ϕ is a power of t we use (1.41). For $\phi=\mu$, where $\mu(t)=mt$, (3.40) reduces to

$$\begin{aligned}\widetilde{\mu}(t) &= \frac{mt^{\frac{1}{2}-H}}{c_N\Gamma(H+\frac{1}{2})}\frac{\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \\ &= \frac{mt^{\frac{1}{2}-H}}{\Gamma(H+\frac{1}{2})}\frac{\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)}\left(\frac{\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}{2H\Gamma(3/2-H)}\right)^{\frac{1}{2}} \\ &= mt^{\frac{1}{2}-H}\left(\frac{\Gamma(3/2-H)}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}} \\ &= c_1 t^{\frac{1}{2}-H}\end{aligned} \tag{3.41}$$

in which

$$c_1 = m\left(\frac{\Gamma(3/2-H)}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}\right)^{\frac{1}{2}}, \tag{3.42}$$

so

$$|\widetilde{\mu}|^2 = \frac{m^2\Gamma(3/2-H)}{2H(2-2H)\Gamma\left(H+\frac{1}{2}\right)\Gamma(2-2H)}T^{2-2H}.$$

For the third term we must evaluate

$$\langle\mu,\psi\rangle_H = \langle\widetilde{\mu},\widetilde{\psi}\rangle_{L_2(0,t)}$$

which, using (3.40) and (3.41),

$$
\begin{aligned}
&= \frac{c_1}{c_N\Gamma\left(H+\frac{1}{2}\right)}\int_0^T t^{\frac{1}{2}-H}t^{H-\frac{1}{2}}D_{0+}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}D_{0+}^{1}\psi(t)dt \\
&= \frac{c_1}{c_N\Gamma\left(H+\frac{1}{2}\right)}\int_0^T D_{0+}^{H-\frac{1}{2}}t^{\frac{1}{2}-H}D_{0+}^{1}\psi(t)dt \\
&= \frac{c_1}{c_N\Gamma\left(H+\frac{1}{2}\right)}I_{0+}^{\frac{3}{2}-H}t^{\frac{1}{2}-H}D_{0+}^{1}\psi(t) \\
&= \frac{c_1}{c_N\Gamma\left(H+\frac{1}{2}\right)\Gamma(3/2-H)}\int_0^t (t-s)^{\frac{3}{2}-H-1}s^{\frac{1}{2}-H}d\psi(s) \\
&= \frac{m}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(3/2-H)}\int_0^t (t-s)^{\frac{3}{2}-H-1}s^{\frac{1}{2}-H}d\psi(s).
\end{aligned}
$$

Putting all the terms together we obtain

$$
\frac{d\mathcal{P}_a}{d\mathcal{P}}(\psi) = \exp\left[\frac{mI_{0+}^{\frac{3}{2}-H}t^{\frac{1}{2}-H}D_{0+}^{1}\psi(t)}{2H\Gamma\left(H+\frac{1}{2}\right)\Gamma(3/2-H)} - \frac{1}{2}\frac{m^2c_1^2}{(2-2H)}T^{2-2H}\right]
$$

where the first term in the power of the exponents, matches the corresponding first term of (1.15). It should be observed that M_t is defined as a fractional integral in (1.13) and the fractional derivative part of the first term, namely $D_{0+}^{1}\psi(t)$ corresponds to the $\mathrm{d}{B_s}^H$ in (1.13). Similarly a little algebra shows the second terms in the power of the exponents exactly match as well.

3.2.6 Measurable extensions

In this subsection we present the idea of lifting maps with respect to $\mathcal{P}$ measure. The class of integrands and thus the Radon Nikodym derivative can be enlarged using this approach. The idea is to use a canonical injection from the Hilbert space H to the Banach space Ω and then ask for what class of functions do the limits make sense. These ideas have a similar motivation as Segal (Segal 1956) and Gross' (Gross 1962, 1960) work. The notion of lifting with respect to finitely additive measure has also been utilized in (Kallianpur and Karandikar 1985).

Lemma 3.2.3. *If f is a simple function on $(H,\mathcal{C}_H)$ the defining sum can be extended to define a simple function on $(\Omega,\mathcal{C})$, and hence also on $(\Omega,\mathcal{B})$.*

Proof. Suppose $f(\psi)=\sum_k a_k\chi_{A_k}(\psi)$ where $A_k=\Pi_{S_k}^{-1}(B_k)$, for subspaces $S_k \leqslant H$ and

Borel sets $B_k \subseteq S_k$. Define a new function $\widetilde{f} = \sum_k a_k \chi_{\widetilde{A}_k}(\psi)$, where $\widetilde{A}_k = \widetilde{\Pi}_k^{-1}(B_k)$ in which $\widetilde{\Pi}_k$ is the extension of Π_k to Ω (See Remark 3.1.1). □

Lemma 3.2.4. *If f is a $\mathcal{C}_H$-simple function for which the $\mathcal{C}_H$-sets are defined in terms of subspaces $Z_1, \ldots, Z_n \leqslant H$, setting Z to be the smallest subspace containing $\bigcup_k Z_k$, and supposing Z is of dimension N, and $\Psi : \mathbf{R}^N \to Z$ is the map from orthonormal coordinates to elements in H,*

$$\int_H f(\psi)\mathcal{P}(d\psi) = \int_U f(\Psi(\theta_1,\ldots,\theta_N))\,(2\pi)^{-N/2}\,e^{-\frac{1}{2}|\Psi(\theta_1,\ldots,\theta_N)|_H^2}d\theta_1\ldots d\theta_N. \quad (3.43)$$

Proof. This follows from the fact that P is the measure generated by the construction based on (3.1), and the identity $|\Psi(\theta_1,\ldots,\theta_N)|_H^2 = \sum_{k=1}^n \theta_k^2$, where the orthonormal coordinates are denoted by θ_k here and x_k in (3.1). □

The following lemma shows that it is correct to interpret P-measurable on $(\Omega, \mathcal{C})$ as *stronger* than the requirement of measurability, and hence we can regard functions which are P-measurable on $(\Omega, \mathcal{C})$ as also measurable on Ω in the conventional sense.

Lemma 3.2.5. *If f is P-measurable on $(\Omega, \mathcal{C})$ in the sense of Definition 3.2.2 (so f is an equivalence class of functions on Ω), then one of the elements of f, $\widetilde{f}$ say, is a measurable function on Ω.*

Proof. Suppose $\{\phi_k\}$, a sequence of simple functions, converges in P-measure to f. Set $\overline{f}(\psi) = \liminf_k \phi_k(\psi)$. Measurability of this limit is shown in (Malliavin 1995, Corollary 2.6.2). By Lemma 3.2.1, $P(\{\psi : \overline{f}(\psi) = \infty\}) = 0$. Now define

$$\widetilde{f}(\psi) = \begin{cases} \overline{f}(\psi), & \overline{f}(\psi) < \infty, \\ 0, & \text{otherwise}. \end{cases}$$

Since $\{\phi_k\}$ converges in P-measure, for any $\varepsilon > 0$, we can find $n > 0$ such that for all $k > n$, $|\phi_k - \phi_n|_P < \varepsilon$ and hence $|\widetilde{f} - \phi_n|_P < \varepsilon$, which shows that $\widetilde{f} \in f$. □

The following theorem provides a very general way to construct P-measurable functions on Ω with desired characteristics – the function is defined on an increasing sequence of finite-dimensional subspaces of H. It is therefore an excellent tool for constructing the nonlinear mappings on path spaces needed in applications.

Theorem 3.2.4. *Every bounded $\mathcal{P}$-measurable function, f, on H has a unique (up to $|\cdot|_P$-norm) extension to a $(\mathcal{C},P)$-measurable function, $\widetilde{f}$, on Ω with the property that if $\{\phi_k\}$ is a sequence of simple functions converging to f on H then the sequence formed by extending these simple functions to Ω (as in Lemma 3.2.3) converges in P-measure (in the sense of Definition 3.2.6) to $\widetilde{f}$. If $\overline{f} \in \widetilde{f}$,*

$$\int_\Omega \overline{f}(\psi)P(d\psi) = \int_H f(\psi)\mathcal{P}(d\psi). \tag{3.44}$$

Proof. Let f be a $\mathcal{P}$-measurable function on H and $\{\phi_k\}$ be a sequence of $\mathcal{C}_H$-simple functions such that $|f-\phi_k|_{\mathcal{P}} \to 0$. Extend ϕ_k to $\widetilde{\phi}_k$ as in Lemma 3.2.3. The series $\{\widetilde{\phi}_k\}$ is Cauchy in $|\cdot|_{(\mathcal{C},P)}$-norm, since the differences have the same $|\cdot|_{(\mathcal{C},P)}$-norms as the sequence $\{\phi_k\}$ of functions on H, so it converges to a $(\mathcal{C},P)$-measurable function on Ω.

Now suppose there was a different $(\mathcal{C},P)$-measurable function, f_1, coinciding with f on H. Since it is different, there must be some cylinder set, $S \in \mathcal{C}$, where it is different from f. But this would imply that f_1 and f differ on the non-empty set $S \cap H$, which is not possible, since f_1 and f were assumed to coincide on H.

Because both integration in the sense of Lebesgue, and in the sense in which it is defined for finitely additive set-functions (Dunford and Schwartz 1957), is defined by taking limits from simple functions, (3.44) will be true for all $\mathcal{P}$-measurable functions so long as it is true for the indicator functions of sets in $\mathcal{C}_H$. The case where f is an indicator function follows immediately from the definition of the measure P. □

Corollary 3.2.1. *Suppose (H,Ω,ι) is an abstract Wiener space, $\mathcal{P}$ is the Gauss measure (a finitely additive set function) on H, and P the measure on Ω induced by $\mathcal{P}$. Suppose $T : H \to (H,|\cdot|_\Omega)$ is $\mathcal{P}$-measurable, $\mathcal{P}\circ T^{-1} \leq \mathcal{P}$ and that*

$$\frac{d\mathcal{P}\circ T^{-1}}{d\mathcal{P}} = \rho. \tag{3.45}$$

Let $\widetilde{T}$ denote the measurable extension of T to Ω. Then the unique measurable extension $\widetilde{\rho}$ of ρ to Ω is the Radon-Nikodym derivative $\frac{dP\circ\widetilde{T}^{-1}}{dP}$.

Proof. To show that $\widetilde{\rho}$ is the Radon-Nikodym derivative it will be sufficient to show that

$$\int_\Omega \widetilde{\rho}(\psi)\overline{h}(\psi)P(d\psi) = \int_\Omega \overline{h}(\widetilde{T}(\psi))P(d\psi), \tag{3.46}$$

for all measurable bounded $\overline{h}$ defined on Ω. Because of the linearity and continuity properties of the integrals in (3.46), it will be sufficient if (3.46) holds for $\overline{h} \in \widetilde{h}$ where $\widetilde{h}$ is formed as in Theorem 3.2.4 from $h \in \mathcal{B}_n$, as in Definition 3.2.8. By Theorem 3.2.4, this follows from (3.45). □

3.3 Example Applications

A complete example in which the Radon-Nikodym derivative induced by a transformation which scales the last exit of Brownian motion is presented in Section 4.3.2.

Let (H, Ω, ι) denote the abstract Wiener space of Brownian motion on the interval $[0, \infty)$, equipped with norm

$$|\psi|_\Omega = \sup\{|\psi(t)/(|t|+1)| : t \geq 0\}.$$

Throughout this section, $\psi(t)$ is Brownian motion.

3.3.1 A First Example

Definition 3.3.1.

$$\mu_{\varepsilon,t}[\psi](s) \quad = \quad \psi(s) + \left(\frac{(\varepsilon-1)\psi(t)}{t}\right)\phi_t(s), \quad s > 0 \tag{3.47}$$

where $\phi_t(s)$ is defined as in (3.1.10).

This transformation increases the magnitude of the the path ψ by the factor ε at time t by adding drift up to time t. After time t, the path is shifted vertically by the quantity $(\varepsilon-1)\psi(t)$. To simplify notation in the remainder of this section we set $M = \mu_{\varepsilon,t}$.

Lemma 3.3.1.

$$\mathcal{D}_\psi M h = \begin{cases} \varepsilon h & h = \phi_t, \\ h, & h \perp \phi_t, \end{cases} \tag{3.48}$$

Proof. This follows from the fact that M is linear, and hence has constant derivative equal to M. □

Proposition 3.3.1.

$$\frac{dP\circ M}{dP}(\psi) = \varepsilon e^{-\frac{1}{2}\left((\varepsilon-1)\psi(t)t^{-H}\right)^2-((\varepsilon-1)\psi(t))t^{-2H}\psi(t)}. \tag{3.49}$$

on H.

Proof. The $\mathcal{P}$-measurable derivative of $\mu_{\delta,t}$ is given by (3.48), and hence $\mathrm{Det}(M_\psi) = \varepsilon$. By applying Theorem 3.2.2 and using (3.18) and (3.19),

$$\begin{aligned}\frac{d\mathcal{P}\circ M}{d\mathcal{P}}(\psi) &= \varepsilon\exp\left(-\frac{1}{2}|M\psi-\psi|_H^2 - \langle M\psi-\psi,\psi\rangle_H\right)\\ &= \varepsilon e^{-\frac{1}{2}\left((\varepsilon-1)\psi(t)t^{-H}\right)^2-\left((\varepsilon-1)\psi(t)t^{-1}\right)t^{1-2H}\psi(t)}.\end{aligned}$$

□

3.3.2 A second example

Definition 3.3.2.

$$\lambda_{\delta,t}[\psi](s) = \begin{cases}\delta^H\psi(s/\delta) + \frac{(\delta-\delta^H)\psi(t)}{t}\phi_t(s/\delta), & s\le\delta t\\ (\delta-1)\psi(t)+\psi(s-(\delta-1)t), & \textit{otherwise.}\end{cases} \tag{3.50}$$

Figure 4.2 illustrates the preceding definition. The value of the transformed path *at* δt is the value of the original path at t scaled by the factor δ. The path *after* t, is shifted to the right by $(\delta-1)t$ and upwards, by $(\delta-1)\psi(t)$. The original path before t is both scaled and shifted to produce the transformed path before δt.

Definition 3.3.3. *The* modified fBm similarity *on* H *(or* Ω*) is, for all* $\delta > 0$*,*

$$\begin{aligned}\kappa_{\delta,t} : H &\to H \qquad (\Omega\to\Omega)\\ \psi &\mapsto \psi'\end{aligned}$$

where

$$\psi'(s) = \begin{cases}\delta^H\psi(s/\delta), & s\in[0,\delta t)\\ (\delta^H-1)\psi(t)+\psi(s-(\delta-1)t), & s\ge\delta t.\end{cases}$$

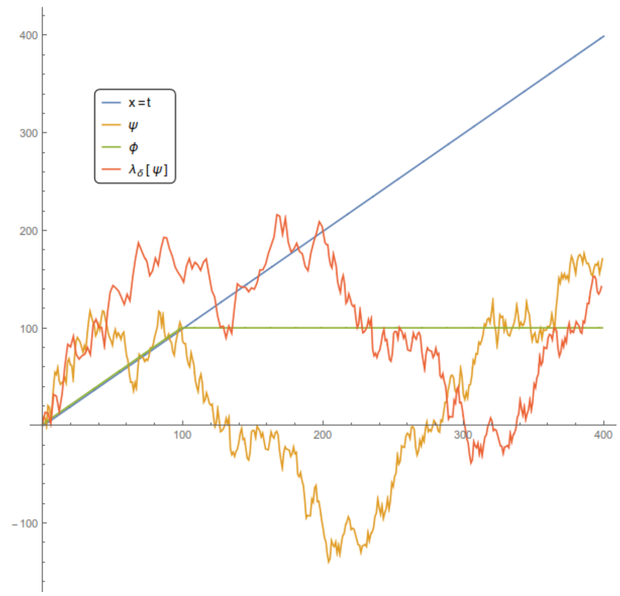

Figure 3.2: A path and its transformation by the mapping (3.50)

Note that $\kappa_{\delta,t}$ is actually a similarity on H only when $H = 0.5$. When $H = 0.5$, because it *is* a self-similarity then, we obtain:

$$\frac{d\mathcal{P}\circ\kappa_{\delta,t}}{d\mathcal{P}} = 1, \tag{3.51}$$

for all $t > 0$.

Proposition 3.3.2.

$$\lambda_{\delta,t}[\psi](s) = \kappa_{\delta,t}\circ\mu_{\delta^{1-H},t}[\psi](s) = \mu_{\delta^{1-H},\delta t}\circ\kappa_{\delta,t}[\psi](s), \quad s \geq 0. \tag{3.52}$$

Proof. In order to see the first equality, notice from (3.47), we get

$$\mu_{\delta^{1-H},t}[\psi](s) = \psi(s) + \left(\frac{(\delta^{1-H}-1)\psi(t)}{t}\right)\phi_t(s), \quad s > 0. \tag{3.53}$$

Upon applying $\kappa_{\delta,t}$ from (3.51) for $s \in [0, \delta t)$ we obtain

$$\kappa_{\delta,t}\circ\mu_{\delta^{1-H},t}[\psi](s) = \delta^H\left[\psi(\frac{s}{\delta}) + \left(\frac{(\delta^{1-H}-1)\,\psi(t)}{t}\right)\phi_t\left(\frac{s}{\delta}\right)\right],$$

as required. When $s \geq \delta t$, the effect of both $\lambda_{\delta,t}$ and of $\kappa_{\delta,t}\circ\mu_{\delta^{1-H},t}$ is to shift the path horizontally and vertically. Both transformations produce a horizontal shift of the path by $(\delta-1)t$. The vertical shift is composed of two parts: (i) the vertical shift introduced by $\kappa_{\delta,t}$ due to the $\psi(s)$ component of (3.53), which is $\left(\delta^H-1\right)\psi(t)$; and (ii) the vertical shift introduced by $\mu_{\delta^{1-H},t}$, which from (3.53) is $\left(\frac{(\delta^{1-H}-1)\psi(t)}{t}\right)\phi_t(s)$, which is then increased further by the factor δ^H by $\kappa_{\delta,t}$, giving $(\delta-\delta^H)\psi(t)$. So the net vertical shift is $(\delta-1)\psi(t)$, as required.

For the second equality when $s\in[0,\delta t)$

$$\begin{aligned}\mu_{\delta^{1-H},\delta t}\circ\kappa_{\delta,t}[\psi](s) &= \delta^H\psi\left(\frac{s}{\delta}\right)+\left[\frac{\left(\delta-\delta^H\right)\psi(t)}{\delta t}\right]\phi_{\delta t}(s)\\ &= \delta^H\psi\left(\frac{s}{\delta}\right)+\left[\frac{\left(\delta-\delta^H\right)\psi(t)}{t}\right]\phi_t(s).\end{aligned}$$

In the case when $s\geq\delta t$, both $\kappa_{\delta,t}$ and $\mu_{\delta^{1-H},\delta t}$ shift ψ vertically and horizontally. In fact $\mu_{\delta^{1-H},\delta t}$ shifts ψ only vertically. We therefore need to verify that the net horizontal and vertical shift introduced by these operations are as required.

As for the horizontal shift, this is only due to $\kappa_{\delta,t}$, which introduces a shift of $(\delta-1)t$, which is as required.

As for the vertical shift, $\kappa_{\delta,t}$ introduces a shift of $(\delta^H-1)\psi(t)$, so that the path after this operation now takes the value $\delta^H\psi(t)$ at δt. The vertical shift introduced by $\mu_{\delta^{1-H},\delta t}$, on the other hand, is $\left(\frac{(\delta^{1-H}-1)\delta^H\psi(t)}{\delta t}\right)\phi_{\delta t}(\delta t)=\left(\delta-\delta^H\right)\psi(t)$. Adding the two shifts, the net shift is $(\delta-1)\psi(t)$, as required.

□

Proposition 3.3.3.

$$\frac{dP\circ\lambda_{\delta,t}}{dP}(\psi)=\delta^{1-H}e^{-\frac{1}{2}\left((\delta^{1-H}-1)\psi(t)t^{-H}\right)^2-\left((\delta^{1-H}-1)\psi(t)t^{-1}\right)t^{1-2H}\psi(t)}.$$

on Ω.

Proof. Using (3.52), $\psi\in H$, $t>0$. Since $\kappa_{\delta,t}$ is a self-similarity, $\frac{d\mathcal{P}\circ\kappa_{\delta,t}}{d\mathcal{P}}=1$ on H. Hence, using the chain rule for Radon-Nikodym derivatives, by (3.49),

$$\frac{d\mathcal{P}\circ\lambda_{\delta,t}}{d\mathcal{P}}(\psi) \;=\; \delta^{1-H}e^{-\frac{1}{2}\left((\delta^{1-H}-1)\psi(t)t^{-H}\right)^2-\left((\delta^{1-H}-1)\psi(t)t^{-1}\right)t^{1-2H}\psi(t)}.$$

□

3.3.3 A Counter-example

In this section we give an example of a transformation of Ω which is $\mathcal{P}$-measurable but does not have a provably $\mathcal{P}$-measurable inverse, and hence Theorem 3.2.2 is *not* applicable. This highlights the importance of the $\mathcal{P}$-measurable invertibility as the condition on which successful application of this theorem relies.

Definition 3.3.4.

$$\widetilde{\lambda}_\delta[\psi] \quad = \quad \widetilde{\Lambda}^{\{\tau[\psi],\delta\}}(\psi) \tag{3.54}$$

where

$$\widetilde{\Lambda}^{\{\tau,\delta\}}[\psi](t) \quad = \quad \delta^H\psi(t/\delta) + \frac{(\delta-\delta^H)\psi(\tau)}{\tau}\phi_\tau(t/\delta),$$

$t \geq 0$.

This transformation is symbolically the same as the part of Definition 3.3.2 which applies for $t \leq \tau[\psi]$, on the whole path. It is therefore much simpler conceptually than the previous transformation. However, it does not have a provably $\mathcal{P}$-measurable inverse. The obvious *candidate* for the inverse of $\widetilde{\lambda}_\delta$ is, assuming $\delta > 1$, $\widetilde{\lambda}_{\delta^{-1}}$. The reason this *fails* as an inverse is that a peak of the path occurring after the last exit may be transformed into a new *later* last exit by the transformation $\widetilde{\lambda}_\delta$, when $\delta < 1$, as illustrated in Figure 3.3.

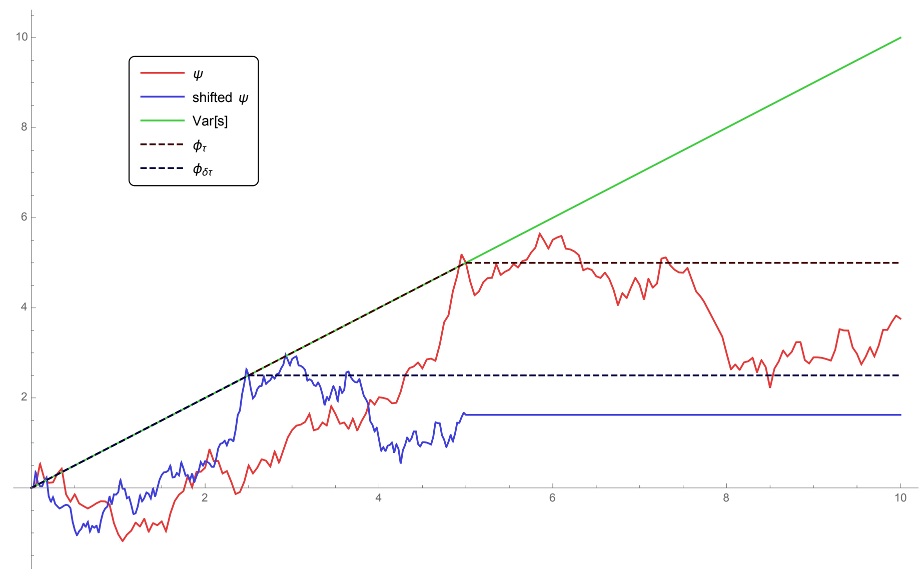

Figure 3.3: The transformation $\widetilde{\lambda}_\delta$ applied to a path causing a new last exit to appear ($\delta = 0.5$)

Chapter 4

Consistent quasi-invariant flows on Abstract Wiener Space

In this chapter we explore a certain class of quasi-invariant stochastic flows, which can be used to determine the density of a functional on a Gaussian process. These flows map the functional value, of the path, to a value δ times as large (for $\delta > 0$), and also have a consistent Radon-Nikodym derivative. When such a flow exists, the main theorem shows that there is a simple formula for the distribution of the functional, expressed in terms of the Radon-Nikodym derivative associated with the mapping.

Two examples which demonstrate the consistent quasi-invariant flows in action are presented, including one which computes the probability density of the last exit for Brownian motion.

The Girsanov theorem has been used to derive the distribution of the first passage time of Brownian motion with drift in (Karatzas and Shreve 2012) and Norros' fBm theorem was used to derive an approximation for the sup of fBm with drift in (Norros 1997). These arguments have some similarity to the approach used in this chapter.

4.1 Consistent quasi-invariant flows

Assuming a specific functional, $\xi : \Omega \to \mathbf{R}$, defined on a probability space (Ω, P) of random paths has been chosen, a *consistent* quasi-invariant stochastic flow, $\{\lambda_\delta\}_{\delta>0}$, is one with *two* consistency properties: (a) the value of the functional is transformed consistently by the flow, i.e. there is a family of mappings $\{\theta_\delta\}_{\delta>0}$, with $\theta_\delta : \mathbf{R} \to \mathbf{R}$ such that for all paths, ψ, $\xi(\lambda_\delta(\psi))) = \theta_\delta(\xi(\psi))$, and (b) the Radon-Nikodym derivative $\frac{dP\circ\lambda_\delta}{dP}$ induced by the transformation depends on the path via ξ. A formal definition is given in Section 4.1.3.

The main result of this chapter, also given in Section 4.1.3, is that when a consistent flow exists, and is known, the distribution of the functional is given by a relatively simple expression in terms of the induced Radon-Nikodym derivative.

It is not known whether consistent quasi-invariant flows always exist, or what conditions might be required for their existence. Finding an explicit form for a consistent quasi-invariant flow is, of course, a distinct, and in general more difficult problem. However, some useful special cases have been discovered and will be explored in Section 4.3.

The main advantage of the consistent quasi-invariant flow method is that it does not rely on the use of a Markov model for the underlying process. Hence it is desirable to apply this method to non-Markovian processes, for example fractional Brownian motion. This is the process used in the examples, however in the interests of simplicity, the process considered is restricted to Brownian motion in the latter part of the Section 4.3. Full treatment of a functional on *fractional* Brownian motion will be deferred to a subsequent paper.

4.1.1 Non-linear transformation of Gaussian measures

The Cameron-Martin theorem(Cameron and Martin 1944) provides a partial substitute for the concept of a density in that it provides a “likelihood ratio” between two different Gaussian measures which differ by a *shift*. This *shift* is another real-valued function of time which must come from the path space. Not all shifts lead to equivalent measures.

For that to be the case, the shift must come from the *Cameron-Martin space*, which is the space of vectors giving rise to a well-defined nonzero Radon-Nikodym derivative.

This result was extended by Girsanov(Girsanov 1960) to enable comparison of arbitrary Ito measures, i.e. measures constructed by a stochastic differential equation driven by Brownian motion.

A further extension when the transformation between measures was affine was considered in (Segal 1958) and (Feldman 1958), and to the general nonlinear case in (Ramer 1974). The resulting formula in these cases takes the form of a product of two terms, one basically the same as the Cameron-Martin formula, and the other term can be described as a Jacobian of the nonlinear transformation.

Since a Jacobian is defined in terms of the Fréchet derivative of the transformation, Ramer observed (Ramer 1974) that the result can still hold when the traditional definition of Jacobian fails. He replaced the Jacobian by an expression partially induced, by continuity, from the mapping on the Cameron-Martin space of the measure.

An alternative approach for expressing the Radon-Nikodym derivative between a Gaussian measure and its non-linear transformation was developed in chapter 3. In this chapter we make use of the result from chapter 3 for which we provided an argument in Section 3.2.4.

4.1.2 The path spaces

In this chapter we continue to work in the context of the spaces Ω and H as defined in Chapter 3 and adopt all the notation from there.

4.1.3 Definition

Let $\xi : \psi \mapsto \xi[\psi]$ denote an arbitrary functional on paths, for example, ξ could be the sup of $\psi(t) - t$, $t \geq 0$, or the last exit time (see Definition 1.3.2) of the path ψ. These two examples of a functional are shown in Figure 4.1. These examples are defined in

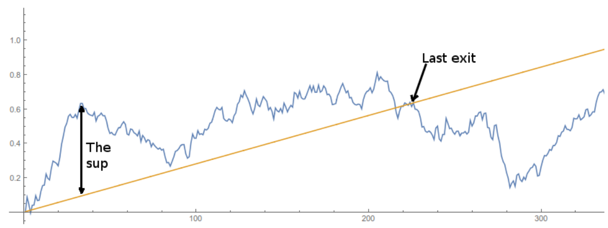

Figure 4.1: The last-exit and sup functionals, for an fBm process

terms of the *future* of the process, but there are also many examples of *causal* (non-anticipative) non-linear functionals.

Definition 4.1.1. *Suppose* $\Lambda : [0,\infty) \times \Omega \to \Omega$ *is a measurable mapping, and in terms of this mapping we define the family,* $\{\lambda_\delta\}$ *of mappings:* $\Omega \to \Omega$*, indexed by* $\delta \in [0,\infty)$*, by*

$$\lambda_\delta : \psi \mapsto \Lambda(\delta, \psi). \tag{4.1}$$

More over it is assumed that for all $delta > 0$*,* λ_δ *is P-measurable. Then* Λ *is said to be:*

(i) *a flow, if there exists a set* E *with* $P(E) = 1$ *such that* λ_1 *is the identity map on* E *and* $\lambda_{\delta_1} \circ \lambda_{\delta_2} = \lambda_{\delta_1 \delta_2}$ *for all* δ_1, δ_2 *on* E*;*

(ii) ξ-consistent*, if for* $\delta \neq 0$ *for almost all paths* $\psi \in \Omega$*,* $\xi[\lambda_\delta[\psi]] = \delta\xi[\psi]$*;*

(iii) *consistently* quasi-invariant *with respect to the functional* ξ *if there exists* $E \subseteq \Omega$ *with P-measure* 1 *such that for some function* $\eta_\delta(x)$*, for each* $\psi \in E$*, for all* $\delta > 0$*,* $\frac{dP\circ\lambda_\delta}{dP} = \eta_\delta[\xi[\psi]]$*;*

(iv) *a consistent quasi-invariant flow with respect to the functional* ξ *if all these conditions hold.*

Suppose Λ is a consistent invariant flow with respect to a certain functional, ξ, on paths. We seek to determine the density, h, of the distribution of $\xi[\psi]$ where ψ is a fractional Brownian motion process. If Λ is a consistent flow, for any $\delta > 0$, the mapping λ_δ provides a mapping of $\{\psi \in \Omega : \xi[\psi] \in (x, x+dx)$ onto $\{\psi \in \Omega : \xi[\psi] \in (\delta x, \delta(x+dx))$. The former event has probability $h_\xi(x)dx$, where h_ξ denotes the density of ξ, and the latter has probability $\delta h_\xi(\delta x)dx$.

4.2 The main theorem

Before the main theorem is stated and proved, we note the following.

Proposition 4.2.1. *If* $\eta_\delta(x)$ *is the Radon-Nikodym derivative function of a consistent flow, then*

$$\eta_\delta(x) = \delta^{\alpha(x)} \tag{4.2}$$

for some $\alpha : \mathbf{R} \to \mathbf{R}$*, for all* x*,* $\delta \geq 0$*.*

Proof. Let

$$\gamma_x : t \mapsto \eta_{e^t}(x). \tag{4.3}$$

By Definition 4.1.1, γ_x, satisfies

(i) $\gamma_x(0) = 1$;

(ii) $\gamma_x(u+v) = \gamma_x(u)\gamma_x(v)$.

These conditions (Dunford and Schwartz 1957, Chapter 8) determine that $\gamma_x(u) = e^{u\alpha(x)}$, for some $\alpha : \mathbf{R} \to \mathbf{R}$. Substituting $\log\delta$ for t in (4.3) now shows (4.2). □

Remark 4.2.1. *Notice that (4.3)* $\Longrightarrow \alpha = \left.\frac{\partial\eta_\delta(x)}{\partial\delta}\right|_{\delta=1}$.

Theorem 4.2.1. *If* Λ *is a consistent quasi-invariant flow for the functional* ξ*, for which the Radon-Nikodym derivative function,* $x \mapsto \eta_\delta(x)$ *satisfies a Lipschitz condition at* x *for all* $x > 0$*, then the density of* $\xi[\psi]$ *is*

$$h_\xi(x) \quad = \quad Cx^{-1}\exp\left(\int_1^x \left.\frac{\partial\eta_\delta(y)}{\partial\delta}\right|_{\delta=1} y^{-1}dy\right), \tag{4.4}$$

where C *is the appropriate normalising constant.*

Proof. By definition, if $\rho_\delta(\psi)$ is the Radon-Nikodym derivative of the measure $P \circ \lambda_\delta$ with respect to P, then

$$P \circ \lambda_\delta(A) \quad = \quad \int_A \rho_\delta(\psi)P(d\psi) \tag{4.5}$$

for all measurable $A \subseteq \Omega$ (Dunford and Schwartz 1957).

Set $A_{(x,y)} = \{\psi \in \Omega : \xi(\psi) \in (x,y)\}$, so, using Property (ii) of Definition 4.1.1, $\lambda_\delta(A_{(x,x+dx)}) = A_{(\delta x,\delta x+\delta dx)}$. Now using Property (iii) of Definition 4.1.1, and the assumption that $\eta_\delta(x)$ satisfies a Lipschitz condition at x, $\rho_\delta(\psi) = \eta_\delta(x) + O(dx)$ for all $\psi \in A_{(x,x+dx)}$, so (4.5) implies

$$P(A_{(\delta x,\delta x+\delta dx)}) = (\eta_\delta(x) + O(dx))\, P(A_{(x,x+dx)}).$$

Dividing by dx, taking into account that the length of the interval $\xi\left(A_{(\delta x,\delta x+\delta dx)}\right)$ is δdx, and letting $dx \to 0$, we obtain:

$$\delta h(\delta x) = \eta_\delta(x) h(x), \qquad \delta \geq 0, \quad x \geq 0. \tag{4.6}$$

Subtract $\delta h(x)$ from both sides and divide by δ, giving

$$h(\delta x) - h(x) = (\eta_\delta(x)/\delta - 1)\, h(x),$$

now divide by $\delta - 1$, and let $\delta \searrow 1$, giving

$$\lim_{\delta \searrow 1} \frac{h(\delta x) - h(x)}{\delta - 1} = \lim_{\delta \searrow 1} \left\{ \frac{\eta_\delta(x) - 1}{\delta - 1} h(x) + \frac{1/\delta - 1}{\delta - 1} \eta_\delta(x) h(x) \right\},$$

which, using $\lim_{\delta \to 1} \frac{1/\delta - 1}{\delta - 1} = -1$,

$$\begin{aligned}
&\implies xh'(x) = \left(\left.\frac{d\eta_\delta(x)}{d\delta}\right|_{\delta=1} - \eta_1(x) \right) h(x) \\
&\implies \frac{h'(x)}{h(x)} = \left(\left.\frac{d\eta_\delta(x)}{d\delta}\right|_{\delta=1} - 1 \right) x^{-1} \\
&\implies \frac{d}{dx} \log h(x) = \left(\left.\frac{d\eta_\delta(x)}{d\delta}\right|_{\delta=1} - 1 \right) x^{-1} \\
&\implies \log h(x) = \int_1^x \left.\frac{d\eta_\delta(y)}{d\delta}\right|_{\delta=1} y^{-1} dy - \log x + C_1 \\
&\implies h(x) = Cx^{-1} \exp\left(\int_1^x \left.\frac{d\eta_\delta(y)}{d\delta}\right|_{\delta=1} y^{-1} dy \right),
\end{aligned} \tag{4.7}$$

for some constant C. □

4.2.1 A Simple Example of a Flow

Suppose X_t is a Gaussian process which is self-similar, with family of similarities $\{\kappa_\delta\}_{\delta>0}$.

Consider the functional $\xi : \psi \mapsto \psi(1)$ and the family of mappings

$$\lambda_\delta : \psi \mapsto \psi + (\delta - 1)\xi(\psi)\phi_1, \qquad \delta > 0. \tag{4.8}$$

The flow property, and ξ-consistency are obvious. By Theorem 3.2.2 of chapter 3, using the fact (from (3.14)) that $\langle \psi, \phi_1 \rangle_H = \xi(\psi)$, and observing that in this context, in the notation of Theorem 3.2.2 from chapter 3, $\mathrm{Det}(I + D_H K(\psi)) = \delta$,

$$\frac{dP \circ \lambda_\delta}{dP}(\psi) = \delta e^{-(\delta-1)\xi(\psi)^2}, \tag{4.9}$$

which shows invariance, with $\eta_\delta(x) = \delta e^{-(\delta-1)x^2}$. Since $\left.\frac{d\eta_\delta(x)}{d\delta}\right|_{\delta=1} = 1 - x^2$, by Theorem 4.2.1, $h_\xi(x) = Ce^{-x^2/2}$, $x > 0$. The same argument applied to the functional $\psi \mapsto -\psi(1)$ shows that the density of $\psi(1)$ has the same shape on the negative real line and by symmetry each half of the density must have exactly the same weight, so according to the consistent flow theorem, the complete density is normal with mean 0 and variance 1, which is, indeed, the density of an fBm process evaluated at time 1.

4.3 The last exit of Brownian motion

Let (H, Ω, ι) denote the abstract Wiener space of Brownian motion on the interval $[0, \infty)$, equipped with norm

$$|\psi|_\Omega = \sup\{|\psi(t)/(|t|+1)| : t \geq 0\}.$$

The following functional is called the *last exit* of the process ψ.

Definition 4.3.1. *Let*

$$\tau_\alpha[\psi] \doteq \sup\{t : \psi(t) \geq \alpha t\},$$

and in particular, $\tau = \tau_1$.

4.3.1 A consistent quasi-invariant flow

The flow we define in this subsection is illustrated, in the case where $H = 0.5$, in Figure 4.2. Here is its mathematical definition :

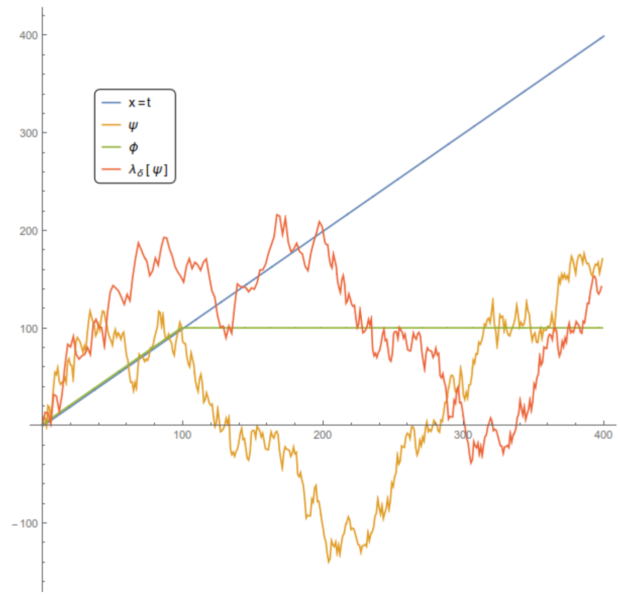

Figure 4.2: A path and its transformation by the mapping (4.10)

Definition 4.3.2.

$$\lambda_\delta[\psi](t) \quad = \quad \begin{cases} \delta^H \psi(t/\delta) + (1-\delta^{H-1})\phi_{\delta\tau}(t), & t \leq \delta\tau, \\ (\delta-1)\tau + \psi(t-(\delta-1)\tau), & \textit{otherwise.} \end{cases} \tag{4.10}$$

Figure 4.2 illustrates the preceding definition. Observe that the mapping defined in (4.10) *scales* the path before τ in the same way as the fBm similarity, while the transformed path *after* $\delta\tau$ is an exact, congruential, shift, of the original path after τ.

4.3.2 The Radon-Nikodym derivative of λ_δ

Definition 4.3.3. *For any sequence of subsets* $\{Z_k\}_{k=0}^{\infty}$ *of* **R**, *we say they* become dense *in* $S \subseteq \mathbf{R}$, *if for any* $t \in S$ *and* $\varepsilon > 0$, $\exists K > 0$, *such that for* $k > K$, $\exists s \in Z_k$ *with* $|s-t| < \varepsilon$.

Definition 4.3.4. *With* $a > 1$ *(typically only slightly bigger than 1), and* $m \geq 2$, *let* $Q_{a,m} = \{a^k : k = 1-m, 2-m, \ldots, m\}$, $Q_m = Q_{2^{1/m}, m2^{m-1}}$. *Note that* $N_m \doteq |Q_m| = m2^m$.

The sequence of sets $Q_2 \subseteq Q_3 \subseteq \ldots$ becomes dense in $[0, \infty)$.

Definition 4.3.5. *Let*

$$\tau_\alpha^{\{m\}}[\psi] \doteq \sup\{x : x \in Q_m \text{ and } \psi(x) \geq \alpha x\}.$$

and, in particular, $\tau^{\{m\}} \doteq \tau_1^{\{m\}}$.

Lemma 4.3.1. $\tau^{\{m\}}$ *converges to* τ *almost surely and in probability.*

Proof. Because the additional sample-points in Q_m relative to Q_{m-1} can't change the fact that ψ crosses the line $x = t$ in the j'th interval of Q_m, $\{\tau^{\{m\}}\}_{m>0}$ is an *increasing* sequence. By definition of the space Ω , necessarily $\frac{\psi(t)}{|t|+1} \to 0$, so for each $\varepsilon > 0$, $\exists B_\varepsilon > 0$ such that if $\mathcal{B}_\varepsilon = \left\{\psi : \forall k > 0, \tau^{\{m\}}(\psi) < B_\varepsilon\right\}$ then $P(\mathcal{B}_\varepsilon) > 1 - \varepsilon$. Thus, on $\mathcal{B}_\varepsilon$, the sequence $\{\tau^{\{m\}}\}_{m>0}$ is increasing and bounded. Any such sequence has at least one accumulation point, and this accumulation point must be unique, i.e. all the sequences in $\mathcal{B}_\varepsilon$ converge. Since $\varepsilon > 0$ was arbitrarily chosen, $\{\tau^{\{m\}}\}_{m>0}$ converges almost surely on Ω.

Since almost sure convergence implies convergence in probability, convergence in probability also holds. □

Lemma 4.3.2. *If* $\{\lambda_\delta\}_{\delta>0}$ *is a flow on* H *and* $\{g_\delta\}_{\delta>0}$ *is a family of real-valued functions on* H *such that*

$$g_{\delta_1}(\psi) g_{\delta_2} \circ \lambda_{\delta_1}(\psi) \quad = \quad g_{\delta_1\delta_2}(\psi), \qquad \delta_1, \delta_2 > 0, \tag{4.11}$$

and

$$g_\delta(\psi) \quad = \quad \delta^{\alpha(\psi)} + O\left((\delta-1)^2\right), \tag{4.12}$$

uniformly in ψ*, for* $\delta \in (0,\infty)$ *near* 1*, for some function* $\alpha : H \to (-\infty,\infty)$*, then*

$$g_\delta(\psi) = \delta^{\alpha(\psi)},$$

$\delta > 0$.

Proof. Choose $n > 0$ and set $\delta_1 = 1$ and $\delta_0 = \delta_2^{1/n} \approx 1 + \frac{\delta_2 - 1}{n}$. Then, applying (4.11) n times, with δ_0 in place of δ_2 and δ_1 increasing from 1 to δ_2 by multiples of δ_0, and using (4.12) to evaluate each term, we find

$$g_{\delta_2}(\psi) = \delta_2^{\alpha(\psi)} + O\left(\frac{(\delta_2-1)^2}{n}\right) \tag{4.13}$$

for δ_2 near 1. Since n is arbitrary, the conclusion follows.

Second proof From (4.11), using the "well-known fact" (Dunford and Schwartz 1957, Chapter 8) that the only solution of the functional equation $f(a+b) = f(a)f(b)$, for all $a, b \in (-\infty,\infty)$ is $f(a) = e^{\alpha a}$, for some $\alpha \in (-\infty,\infty)$, any family of functions $\{g_\delta(\psi)\}$ satisfying (4.11) must take the form $g_\delta(\psi) = \delta^{\beta(\psi)}$, for some function β. However, $b \doteq \beta(\psi) \neq a \doteq \alpha(\psi)$ for any ψ contradicts (4.12), so $\beta = \alpha$. □

Lemma 4.3.3. τ *is* $\mathcal{P}$*-measurable on* H.

Proof. Because $\tau^{\{m\}}$, restricted to H, is $\mathcal{C}_H$-simple, by the previous lemma, $\{\tau^{\{m\}}\}$ is a sequence of $\mathcal{C}_H$-simple functions converging in $\mathcal{P}$-measure to τ. □

Now let us consider a family of transformations $\{\lambda_\delta\}$ defined by

$$\lambda_\delta[\psi] \quad = \quad \psi'$$

where, for ψ such that $\tau[\psi] = \widetilde{\tau}$,

$$\psi' = \lambda_{\delta,\widetilde{\tau}}[\psi]. \tag{4.14}$$

Proposition 4.3.1. $\lambda_{\delta,t}^{-1} = \lambda_{\delta^{-1},\delta t}$ *and* $\lambda_\delta^{-1} = \lambda_{\delta^{-1}}$.

Proof. Using the first equality of Proposition 3.3.2 for $\lambda_{\delta,t}$ and the second for $\lambda_{\delta^{-1},\delta t}$ we find

$$\begin{aligned}
\lambda_{\delta^{-1},\delta t} \circ \lambda_{\delta,t} &= \mu_{\delta^{H-1},\delta^{-1}\delta t} \circ \kappa_{\delta^{-1},\delta t} \circ \kappa_{\delta,t} \circ \mu_{\delta^{1-H},t} \\
&= \mu_{\delta^{H-1},t} \circ \mu_{\delta^{1-H},t} \\
&= I.
\end{aligned}$$

Now for the second equation, suppose $\tau[\psi] = \widetilde{\tau}$. Then $\lambda_\delta[\psi] = \lambda_{\delta,\widetilde{\tau}}[\psi]$ which is a path with last exit at $\delta\widetilde{\tau}$. Hence, when applied to this path, ψ' say, $\lambda_{\delta^{-1}}[\psi'] = \lambda_{\delta^{-1},\delta\widetilde{\tau}}[\psi']$, and so

$$\begin{aligned}
\lambda_{\delta^{-1}} \circ \lambda_\delta[\psi] &= \lambda_{\delta^{-1},\delta\widetilde{\tau}} \circ \lambda_{\delta,\widetilde{\tau}}[\psi] \\
&= \psi.
\end{aligned}$$

Since ψ was arbitrary, $\lambda_{\delta^{-1}} \circ \lambda_\delta = I$ also, as required. □

Proposition 4.3.2.

$$\frac{dP\circ\lambda_\delta}{dP}(\psi)=\delta^{(1-H)\left(1-t^{2-2H}\right)} \tag{4.15}$$

on H.

Proof. Proposition 4.3.1 is also true for the version of λ_δ which uses the approximation $\tau^{\{m\}}$ in place of τ. Further details of this point are given below.

We can therefore apply Proposition 3.2.10 with τ in place of π, using the sequence $\tau^{\{m\}}\to\tau$, to conclude

$$\frac{d\mathcal{P}\circ\lambda_\delta}{d\mathcal{P}}(\psi) \quad = \quad \delta^{1-H}e^{-\frac{1}{2}\left((\delta^{1-H}-1)\tau^{-H}\right)^2-(\delta^{1-H}-1)\tau^{1-2H}\tau}$$

which, using $e^{O((1-\delta)^2)}=1+O(\delta-1)^2$, and $e^{\delta-1}=\delta+O(\delta-1)^2$,

$$\begin{aligned}
&= \delta^{1-H}e^{-\left(\delta^{1-H}-1\right)\tau^{2-2H}}+O\left((\delta-1)^2\right)\\
&= \delta^{1-H}\delta^{-(1-H)\tau^{2-2H}}+O\left((\delta-1)^2\right)\\
&= \delta^{(1-H)\left(1-\tau^{2-2H}\right)}+O\left((\delta-1)^2\right)\\
&= \delta^{(1-H)\left(1-\tau[\psi]^{2-2H}\right)}
\end{aligned}$$

by Lemma 4.3.2. The result now follows from Theorem 3.2.2 and Corollary 3.2.1.

To explain the invertibility of the approximations to λ_δ in more detail, define

$$\lambda_{\delta,\tau_1^{\{k\}}}[\psi] \quad = \quad \psi'$$

where

$$\psi'(t) \quad = \quad \lambda_{\delta,\tilde{\tau}}[\psi](t), \quad t\geq 0$$

in which

$$\tilde{\tau}=\tau_1^{\{k\}}(\psi). \tag{4.16}$$

Since $\tau_1^{\{k\}}(\psi)=x\Leftrightarrow\tau_\delta^k(\lambda_{\delta,\tau_1^{\{k\}}}(\psi))=x$,

$$\lambda^{-1}_{\delta,\tau_1^{\{k\}}}=\lambda_{\delta^{-1},\tau_\delta^k},$$

and so $\lambda^{-1}_{\delta,\tau_1^{\{k\}}}$ is invertible as required in order to apply Proposition 3.2.10. □

4.3.3 The last-exit density

Example 4.3.1.. The last exit from Brownian Motion

Theorem 4.2.1 now applies, and gives $H_\tau(x) = \int_0^x h_\tau(u)du$ as the distribution of the last exit, where

$$h_\tau(x) = Cx^{-1}\exp\left(\int_1^x \frac{\partial\eta_\delta(y)}{\partial\delta}\bigg|_{\delta=1} y^{-1}dy\right)$$

$$= Cx^{-1}\exp\left(\int_1^x (1-H)(1-\tau^{2-2H})\tau^{-1}d\tau\right) \quad (4.17)$$

by Proposition 4.3.2

$$= Cx^{-1}x^{1-H}\exp\left(-\frac{1}{2}x^{2-2H}\right)$$

$$= Cx^{-H}\exp\left(-\frac{1}{2}x^{2-2H}\right), \quad (4.18)$$

for $x \geq 0$. So (4.18) is the density of the last exit from the region $x > t$, by Brownian motion. A formula which implies (4.18) was derived in (Salminen 1988). The last exit from a linear boundary with $x_0 = 0$ has also been interpreted as the length of a busy period (weighted by its length) and under this interpretation its density was determined to be (4.18) by a completely different method in (Salminen and Norros 2001, Corollary 3.8). Integrating h_τ we find $C = \frac{2-2H}{\sqrt{2\pi}}$, and

$$H_\tau(x) = \text{Erfc}\left(\frac{x^{1-H}}{\sqrt{2}}\right), \quad x \geq 0.$$

□

Chapter 5

Probability densities of functionals of fBm with drift

In this chapter, we first present a connection between an approximation of the probability distribution of sup of fBm with drift and the Generalized Gamma distribution. Some simplified expressions are also provided for when the Hurst parameter takes certain special values.

We then proceed to derive transport equations for the family of probability densities of the sup and the first passage for fBm with linear drift. It is also shown that a previously proposed approximation for the density of the sup satisfies the corresponding PDE, as well as other necessary constraints for the density of the sup.

5.1 Supremum of fBm with drift

5.1.1 The model and related work

By Theorem 11.11 in (Lifshits 1995), the supremum of fBm with negative drift has a density, for all $0 < H < 1$, except for one possible atom, which must occur at the minimal value for which the buffer level has non-zero probability density, i.e. 0. It is also known that there is no atom at the origin, so the sup has a density, which we denote henceforth by $\phi(x,\mu)$.

Let $X(t)$ denote an arithmetic fractional Brownian process with drift, corresponding to the SDE of the form

$$\mathrm{d}X(t) = \mu\mathrm{d}t + \sigma\mathrm{d}B^H(t), \qquad X(0) = x_0 \tag{5.1}$$

where B^H is an fBm with $H \in [0,1]$. When $\mu < 0$, $Q = \sup_{t\geq 0} X_t$ is well-defined almost surely and an approximation for its probability density function according to (Chen et al. 2013) is:

$$\frac{P(Q \in (x, x+dx))}{dx} \approx \widetilde{\phi}(x,\mu) \doteq \frac{\nu\alpha^{\frac{\beta}{\nu}}}{\Gamma\left(\frac{\beta}{\nu}\right)} x^{\beta-1} e^{-\alpha x^{\nu}}, \tag{5.2}$$

where

$$\alpha = \frac{(1-H)^{2H-2}|\mu|^{2H}}{2H^{2H}\sigma_1^2}, \tag{5.3}$$

$\beta = \frac{1}{H} - 1$, and $\nu = 2(1-H)$.

This approximation was obtained by inferring the asymptotic form of the density of Q from that of the complimentary distribution given in (Hüsler and Piterbarg 1999). The condition that the density is proper then determines the constant. Simulation results were not able to find any discrepancy of this formula over the range of parameter values of the density. The mean, standard deviation, and higher order moments of Q, as predicted by (5.2) were also checked by simulation in (Chen et al. 2015) and no discrepancy between the approximation (5.2) could be detected for any choice of the parameters.

Observe that this approximation of the supremum of fBm's probability distribution is known as the Generalised Gamma distribution. The Generalised Gamma distribution is a special case of the Amoroso distribution (Amoroso 1925, Crooks 2010, Omori 1995) used originally to model income rates. The Amoroso distribution has the following density

$$f_X(x; a, d, g, p) = \frac{p\,(x-g)^{d-1} \exp\left(-\left(\frac{x-g}{a}\right)^{p}\right)}{a^d \Gamma\left(\frac{d}{p}\right)}, \tag{5.4}$$

where $a > 0$, $d > 0$, $p > 0$, $g \in \mathrm{R}$ and $x \geq g$ and $\Gamma(\cdot)$ denotes the gamma function. Substituting $g = 0$, $d = \beta$, $p = \nu$ and $a = \alpha^{-1/\nu}$ in (5.4) we obtain (5.2), so it is a Generalised Gamma density. Stacy (Stacy 1962) defined the Generalised Gamma distribution as the special case of the Amoroso distribution where $g = 0$ and identified the cumulative distribution function (CDF) and moment-generating function (MGF), in this case, as

$$F(x;a,d,p) = \frac{\gamma\left(\frac{d}{p}, \left(\frac{x}{a}\right)^p\right)}{\Gamma\left(\frac{d}{p}\right)}, \tag{5.5}$$

and

$$M(t;a,d,p) = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!} \frac{\Gamma\left(\frac{d+k}{p}\right)}{\Gamma\left(\frac{d}{p}\right)}, \tag{5.6}$$

respectively, where $\gamma(\cdot)$ in (5.5) denotes the lower incomplete gamma function. The mean and variance when $g = 0$ are

$$\mathrm{E}[Q] = a\frac{\Gamma((d+1)/p)}{\Gamma(d/p)}, \tag{5.7}$$

and

$$\mathrm{Var}[Q] = a^2\left[\frac{\Gamma((d+2)/p)}{\Gamma(d/p)} - \left(\frac{\Gamma((d+1)/p)}{\Gamma(d/p)}\right)^2\right], \tag{5.8}$$

respectively (Crooks 2010). From the MGF, the third central moment is

$$\begin{aligned}\mathrm{E}[(Q-\mathrm{E}[Q])^3] &= \frac{a^3\Gamma(\frac{d+3}{p})}{\Gamma(d/p)} + 2\left(a\frac{\Gamma((d+1)/p)}{\Gamma(d/p)}\right)^3 \\ &\quad - \frac{3a^3\Gamma\left(\frac{d+1}{p}\right)\Gamma\left(\frac{d+2}{p}\right)}{\Gamma^2(d/p)}.\end{aligned} \tag{5.9}$$

5.1.2 The mean, variance, third central moment and Skewness

Replacing d, p and a with their expressions in terms of α and H, we find:

$$\mathrm{E}[Q] \approx \alpha^{\frac{-1}{2-2H}} \frac{\Gamma\left(\frac{(1-H)/H+1}{2-2H}\right)}{\Gamma\left(\frac{1-H}{H(2-2H)}\right)} = \alpha^{\frac{1}{2H-2}} \frac{\Gamma\left(\frac{1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)}, \tag{5.10}$$

$$\mathrm{Var}[Q] \approx \frac{\alpha^{\frac{1}{H-1}}}{\Gamma\left(\frac{1}{2H}\right)}\left[\Gamma\left(\frac{1+H}{2H(1-H)}\right) - \frac{\Gamma^2\left(\frac{1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)}\right], \tag{5.11}$$

and

$$\mathrm{E}[(Q-\mathrm{E}[Q])^3] \approx \alpha^{\frac{3}{2H-2}} \left[\frac{\Gamma\left(\frac{2H+1}{2H(1-H)}\right)}{\Gamma\left(\frac{1}{2H}\right)} + \frac{2\Gamma^3\left(\frac{1}{2H(1-H)}\right)}{\Gamma^3\left(\frac{1}{2H}\right)} - \frac{3\Gamma\left(\frac{H+1}{2H(1-H)}\right)\Gamma\left(\frac{1}{2H(1-H)}\right)}{\Gamma^2\left(\frac{1}{2H}\right)} \right]. \tag{5.12}$$

The skewness of Q, can be expressed as

$$\mathrm{Skewness}[Q] = \frac{\mathrm{E}[(Q-\mathrm{E}[Q])^3]}{(\mathrm{Var}[Q])^{3/2}}. \tag{5.13}$$

By (5.11) and (5.12), α is canceled out in (5.13), so skewness$[Q]$ is a function of only one parameter, H.

5.1.3 Simplifications of $\mathrm{E}[Q]$ and $\mathrm{Var}[Q]$ for certain H values

Consider a single server queue with constant service rate, C [B/s], fed by an fBm input process with the Hurst parameter H, with mean input rate m [B/s], and variance per unit time, σ_1^2. Introducing the mean net input, $\mu = m - C$, we can characterize the fBm queueing model by three parameters: H, μ and σ_1^2. In the Brownian case of $H = 0.5$, where $\sigma^2 = m$ in a Poisson process, (5.3) reduces to

$$\alpha = \frac{2|\mu|}{\sigma^2}.$$

Thus, (5.10) and (5.11) can be further simplified as

$$\mathrm{E}[Q] = \left(\frac{2|\mu|}{\sigma^2}\right)^{-1} = \frac{\sigma^2}{2|\mu|}, \tag{5.14}$$

and

$$\mathrm{Var}[Q] = \left(\frac{2|\mu|}{\sigma^2}\right)^{-2}\left(2 - \frac{1^2}{1}\right) = \left(\frac{\sigma^2}{2|\mu|}\right)^2 = (\mathrm{E}[Q])^2. \tag{5.15}$$

An alternative way to obtain $\mathrm{E}[Q]$ and $\mathrm{Var}[Q]$ for the case $H = 0.5$ is to use the exact solution for $P(Q > x)$ from (Harrison 1985), which is

$$P(Q > x) = e^{\frac{2\mu}{\sigma^2}x}.$$

Then the mean and variance of X are obtained by

$$\mathrm{E}[Q] = \int_0^\infty P(Q > x)dx,$$

and

$$\mathrm{Var}[Q] = 2\int_0^\infty xP(Q > x)dx - \left(\int_0^\infty P(Q > x)dx\right)^2,$$

which are consistent with (5.14) and (5.15). Since $|\mu| = C - m$, we can express $\mathrm{E}[Q]$ by (5.14)

$$\mathrm{E}[Q] = \frac{\sigma^2}{2(C-m)},$$

which is half of that for the equivalent M/M/1 queueing system, where $\sigma^2 = m$ and $\mathrm{E}[X] = m/(C-m)$. It is also equal to the mean of an equivalent M/D/1 queue under heavy traffic (where the utilization approaches 1) based on the Pollaczek Khintchine formula.

When $H = 1 - \frac{1}{2n}$, for $n = 1, 2, \ldots$, we have $\frac{1}{2H(1-H)} = \frac{1}{2H} + n$, and

$$\begin{aligned}\Gamma\left(\frac{1}{2H(1-H)}\right) &= \Gamma\left(\frac{1}{2H} + n\right)\\ &= \left(n - 1 + \frac{1}{2H}\right)\left(n - 2 + \frac{1}{2H}\right)\cdots\left(1 + \frac{1}{2H}\right)\frac{1}{2H}\Gamma\left(\frac{1}{2H}\right).\end{aligned}$$

Consequently, (5.10) can be expressed without the Gamma function as

$$\mathrm{E}[Q] = \alpha^{\frac{1}{2H-2}}\left[\frac{1}{2(1-H)} - 1 + \frac{1}{2H}\right]\cdots\left(1 + \frac{1}{2H}\right)\frac{1}{2H}. \tag{5.16}$$

5.1.4 Consistency of the density under self similar transformation

We now demonstrate that the probability density in (5.2) matches the supremum density for a fbm process, even after applying the self-similarity transform to it.

It's noteworthy, although fractional Brownian motion is a self similar process, arithmetic fractional Brownian motion is a self-affine process, as defined below.

Definition 5.1.1. *A stochastic process is said to be self-similar if there exists $H > 0$ such that for any scaling factor $a > 0$, the processes $X(at)$ and $|a|^H X(t)$ have the same law. H is termed the self-similarity exponent of the process X.*

Definition 5.1.2. *X is said to be self-affine if there exists $H > 0$ such that for any $a > 0$ the processes $X(at)$ and $|a|^H X(t)$ have the same law up to centering:*

$$\exists b_a : [0,\infty[\mapsto \mathrm{R}, \quad X(at) \simeq \left(b_a(t) + |a|^H X(t)\right) \quad \forall t \geq 0$$

Upon applying the transformation

$$\{X_t\}_{t\geq 0} \mapsto \left\{a^{-H}X(at)\right\}_{t\geq 0}, \tag{5.17}$$

the mean of a process X with drift μt becomes $a^{1-H}\mu t$ and the volatility remains unchanged. This may be regarded as a self-similarity of the *family* of fBm processes with drift.

For the transformed process upon substituting μa^{1-H} in place of μ in α and xa^{-H} in place of x and by multiplying the right-hand side of Eq. (5.2) with a Jacobian term a^{-H} can be presented as follows

$$\begin{aligned}
&= \frac{a^{-H}\nu a^{\frac{(1-H)2H\beta}{\nu}}\alpha^{\frac{\beta}{\nu}}}{\Gamma\left(\frac{\beta}{\nu}\right)} a^{-H(\beta-1)}x^{\beta-1}e^{-a^{(1-H)2H}\alpha a^{-H\nu}x^{\nu}} \\
&= a^{-H+\frac{(1-H)2H\beta}{\nu}-H(\beta-1)}\frac{\nu\alpha^{\frac{\beta}{\nu}}}{\Gamma\left(\frac{\beta}{\nu}\right)}x^{\beta-1}e^{-a^{(1-H)2H-H\nu}\alpha x^{\nu}}
\end{aligned}$$

Substituting for β and ν in the power of a in the initial coefficient, we obtain

$$-H + \frac{(1-H)2H\beta}{\nu} - H(\beta - 1) = -\left(\frac{1}{H} - 2\right)H + \left(\frac{1}{H} - 1\right)H - H = 0$$

while the power of a in the exponential term is $(1-H)2H - H\nu = 0$ which reduces the transformed probability density function to match the density in (5.2).

5.2 The Self Similarity PDEs

5.2.1 Self Similarity PDE for supremum of fBm with drift

A self-similarity of the probability law for the sup was derived in (Norros 1997, Theorem 2.1). Consider, on the one hand, an fBm process $X(t)$ with drift μ, and on the other hand an fBm process originally with drift $a^{H-1}\mu$ transformed by the fBm similarity (5.17). In the second case, using the original density for the sup of this process and transforming it appropriately, taking into account the Jacobian factor required in front of the density, it's sup has density $x \mapsto a^H \phi(a^H x, a^{H-1}\mu)$. These processes have the same probability measure, and hence same distribution for the sup, so for any $a > 0$,

$$\phi(x,\mu) = a^H \phi(a^H x, a^{H-1}\mu), \tag{5.18}$$

$x \geq 0$, $H \in (0,1)$, $\mu < 0$, $\sigma > 0$. We may express this by saying that the fBm self-similarity induces a similarity relationship on *the family of densities* of the sup over the range of different possible values of the drift, μ.

In (Norros 1997), it is argued that (5.18) is useful in relation to applications of fBm as a model of traffic even without having an explicit form for ϕ_Q. In addition, the law (5.18) is an important tool in the important task of finding explicit solutions, or successively more accurate approximations for the distribution of Q. The self-similarity of fBm can be used to carry out simulations much more efficiently (Chen et al. 2013), and (5.18) makes it possible for validation of the approximation to be comprehensive rather than exploratory.

In the case of fBm, research into the distribution of Q has taken place over several decades already, and builds on the special case of Brownian motion, where an explicit result is known and can be derived in a variety of ways. Although (5.18) does not uniquely identify the solution, ϕ_Q, it plays a role in several of the methods which work in the case of BM.

For example, a direct approach to solving for ϕ_Q can obviously be based on finding the ratio between $\phi_Q(x,\mu)$ and $\phi_Q(x+dx,\mu)$ for all x, with μ fixed. However, because of (5.18), it is equally effective to find the ratio between $\phi_Q(x,\mu)$ and $\phi_Q(x,\mu+d\mu)$ for all μ, with x fixed. In other words if we can explicitly compare the distribution of the sup for two cases with different drift, we will be able to solve the key problem. Such a comparison is provided, in fact, by the Girsanov theorem (Klebaner 2005, Øksendal 2003). This approach works explicitly in the case of BM, and in the case of fBm it has also been applied, but because the form of the Girsanov theorem (Norros et al. 1999) in this case is more complex it provides an approximation rather than an exact solution for ϕ_Q (Norros 1997, Proposition 5.3).

Let us now check that this self-similarity applies to $\widetilde{\phi}$. For the transformed process upon substituting μa^{1-H} in place of μ in α and xa^{-H} in place of x and by multiplying the right-hand side of Eq. (5.2) with a Jacobian term a^{-H} reduces the transformed probability density function to match the density in (5.2). Thus, (5.2) satisfies (5.18). Adding and subtracting identical terms to (5.18) we get

$$
\begin{aligned}
& a^{-H}\phi(a^{-H}x, a^{1-H}\mu) - \phi(a^{-H}x, a^{1-H}\mu) + \phi(a^{-H}x, a^{1-H}\mu) - \phi(x, a^{1-H}\mu) \\
& = \phi(x,\mu) - \phi(x, a^{1-H}\mu).
\end{aligned}
$$

Dividing by $1-a$ and upon letting $a \to 1$ gives

$$
\begin{aligned}
& \frac{a^{-H}\phi(a^{-H}x, a^{1-H}\mu) - \phi(a^{-H}x, a^{1-H}\mu)}{1-a} + \frac{\phi(a^{-H}x, a^{1-H}\mu) - \phi(x, a^{1-H}\mu)}{1-a} \\
& = \frac{\phi(x,\mu) - \phi(x, a^{1-H}\mu)}{1-a}.
\end{aligned} \tag{5.19}
$$

The first term in (5.19) can be rewritten as

$$
\frac{a^{-H}\phi(a^{-H}x, a^{1-H}\mu) - \phi(a^{-H}x, a^{1-H}\mu)}{a^{-H}-1} \times \frac{a^{-H}-1}{1-a} \to H\phi(x,\mu)
$$

as $a \to 1$. The second term in (5.19) can be rewritten as

$$
\frac{\phi(a^{-H}x, a^{1-H}\mu) - \phi(x, a^{1-H}\mu)}{a^{-H}x - x} \times \frac{a^{-H}x - x}{1-a} \to Hx\frac{\partial\phi(x,\mu)}{\partial x}
$$

as $a \to 1$. A similar limit is clear for the RHS. So, letting $a \to 1$ in (5.19) gives

$$
H\phi(x,\mu) + Hx\frac{\partial\phi(x,\mu)}{\partial x} = (1-H)\mu\frac{\partial\phi(x,\mu)}{\partial\mu}
$$

or, rearranged,

$$
Hx\frac{\partial\phi(x,\mu)}{\partial x} + (H-1)\mu\frac{\partial\phi(x,\mu)}{\partial\mu} = -H\phi(x,\mu). \tag{5.20}
$$

This type of PDE is known as a *transport equation* (Zauderer 2011) and can be solved by the *method of characteristics*. A *characteristic* is a path through the parameter space of the PDE on which the values of a solution evolve according to an ordinary differential equation. The equations of the characteristic curve may be expressed invariantly by the Lagrange-Charpit equations (Gyunter 1934)

$$
\frac{dx}{Hx} = \frac{d\mu}{(H-1)\mu} = \frac{d\phi}{-H}. \tag{5.21}
$$

We now try find two functions $\Pi(x,\mu,\phi)$, $\Psi(x,\mu,\phi)$ such that $d\Pi = d\Psi = 0$. The general solution is then given by $F(\Pi,\Psi) = 0$ with F an arbitrary function. Using

$$\frac{dx}{d\mu} = \frac{Hx}{(H-1)\mu} \implies (H-1)\mu dx - Hx d\mu = 0 \implies \Pi = x - \mu^{\frac{H}{H-1}}$$

and

$$\frac{dx}{-x} = d\phi \implies d(\phi + \log(x)) = 0 \implies \Psi = \phi + \log(x).$$

Putting these together we obtain the general solution as

$$F\left[x - \mu^{\frac{H}{H-1}}, \phi + \log(x)\right] = 0 \implies \phi = -\log(x) + f\left[x - \mu^{\frac{H}{H-1}}\right]$$

where f is any differentiable function.

Note that since $\widetilde{\phi}$ also satisfies the fBm similarity (5.18), it also satisfies (5.20).

From (Hüsler and Piterbarg 1999),

$$P(Q > x)/x^{\frac{2H^2-3H+1}{H}} e^{\left(-\frac{x^{2-2H}(1-H)^{2H-2}|\mu|^{2H}}{2H^{2H}\sigma_1^2}\right)} \to C \tag{5.22}$$

as $x \to \infty$ for some constant $C > 0$.

Let $\Phi(x) = \int_x^\infty \phi(x)dx$, and similarly for $\widetilde{\Phi}$. By (Chen et al. 2013), in addition to (5.20), ϕ satisfies the constraint

$$\Phi(x,\mu)/\widetilde{\Phi}(x,\mu) \to c$$

for some $c > 0$, as $x \to \infty$. Also, $\int_0^\infty \phi(x,\mu)dx = 1$, $\mu < 0$ and as $\mu \to -\infty$, $\phi(x,\mu)$ approaches an impulse with mass 1 concentrated at 0. The following theorem shows that these conditions are all satisfied by $\widetilde{\phi}$.

Theorem 5.2.1. $\widetilde{\phi}$ *satisfies the PDE (5.20) and all the constraints on* ϕ *as identified above.*

Proof. All of this has already been shown except that $\widetilde{\phi}$ converges to an impulse as $\mu \to -\infty$. This follows from (5.18). □

If the PDE and the constraints given above imposed a unique solution, we could assert $\phi = \widetilde{\phi}$. However, we can disprove uniqueness of these conditions as follows. Choose

$\mu_1 < 0$ and $x_1 > 0$, and set $\phi_1(x,\mu_1) = \widetilde{\phi}(x,\mu_1)$ for all $x > x_1$. Now set $\phi_1(x,\mu_1)$ for $x \leq x_1$ in any way which ensures that it is a density. Finally, set $\phi_1(x,\mu)$ for $\mu \neq \mu_1$ by the self-similarity. Then ϕ_1 also satisfies the PDE and all the given constraints.

A transport PDE for many quantities related to ϕ (e.g. $\frac{d\phi}{dx}$, $\frac{d\phi}{d\mu}$), can be derived from (5.18) by transforming the PDE in the usual way. Solving for any such quantity will therefore produce a solution for any other, so, in a sense, all such problems are equivalent. More significantly, an approximate solution of a quantity resulting from such a transformation will lead to an approximate solution for the others and if a condition which determines a unique solution of one such PDE is known approximately, it will lead to an approximate solution for all such PDEs.

5.2.2 Self Similarity PDE for First passage time of fBm with drift

Let $X(t)$ denote fractional Brownian process with drift μt, i.e.

$$X(t) = B^H(t) + \mu t, \quad t > 0, \tag{5.23}$$

where B^H is an fBm with Hurst parameter $H \in [0,1]$. Fractional Brownian motion is a self similar process. The process $X(t)$ is self-affine, that is to say, on applying the transformation $\{X_t\}_{t\geq 0} \mapsto \left\{a^{-H}X(at)\right\}_{t\geq 0}$, the mean of the process becomes $a^{1-H}\mu t$ and the volatility remains unchanged.

Let $\phi(t,\mu,x)dt = P(\tau_x \in (t,t+dt))$. The self-similarity of X implies

$$\phi(t,\mu,x) = a\phi(at, a^{1-H}\mu, a^{-H}x), \tag{5.24}$$

for any $a > 0$.

Adding and subtracting identical terms to (5.24) we get

$$\begin{aligned}&\phi(t,\mu,x) - a\phi(t,\mu,x) + a\phi(t,\mu,x) - a\phi(at,\mu,x) + a\phi(at,\mu,x) - a\phi(at,a^{1-H}\mu,x)\\&= -a\phi(at,a^{1-H}\mu,x) + a\phi(at,a^{1-H}\mu,a^{-H}x)\end{aligned}$$

Dividing by $1-a$ and taking limits as $a \to 1$ gives

$$Hx\frac{\partial\phi(t,\mu,x)}{\partial x} + t\frac{\partial\phi(t,\mu,x)}{\partial t} + (1-H)\mu\frac{\partial\phi(t,\mu,x)}{\partial\mu} = -\phi(t,\mu,x). \tag{5.25}$$

This partial differential equation in 3 variables is also a *transport equation* and can be solved by the *method of characteristics*. In this case characteristic equations (ODEs) are

$$\frac{dx}{ds} = Hx \quad \frac{dt}{ds} = t \quad \frac{d\mu}{ds} = (1-H)\mu \quad \text{and} \quad \frac{d\phi}{ds} = -\phi.$$

Solving which we get

$$x(s) = e^{Hs+c_1} \quad t(s) = e^{s+c_2} \quad \mu(s) = e^{(1-H)s+c_3} \quad \text{and} \quad \phi(s) = e^{-s+c_4}.$$

Eliminating s between x and t, we get $x - t^H$ = a constant, $\mu - t^{1-H}$ = a constant and $\phi - t^{-1}$ is a constant, and the general solution is

$$F\left[x - t^H, \mu - t^{1-H}, \phi - t^{-1}\right] = 0.$$

Explicitly

$$\phi = f\left(x - t^H, \mu - t^{1-H}\right) + \frac{1}{t} \tag{5.26}$$

for some arbitrary smooth functions f.

Chapter 6

Conclusions and future work

In Chapter 1, a generalisation of the Girsanov theorem from (Norros et al. 1999) was provided, which can be applied to fBm processes of a wider parameter range.

In Chapter 2 we proved by showing a contradiction with the theorem on continuity of Pickands' constant, that the initial boundary value problem approach of studying first passage problem of Brownian motion does not carry over to fBm by replacing the partial differential equation.

The evaluation of the Radon-Nikodym derivative of a measure relative to the same measure composed with a nonlinear mapping is of great importance in applications and has been the subject of study for many years. The main results of this thesis extended that of (Ramer 1974) and in particular make it easier to apply this type of result to problems from applied probability. Evaluation of these Radon-Nikodym derivatives by first considering an analogous concept on the Cameron-Martin space of the Gaussian measure and then extending the formula to the whole space enables a larger class of nonlinear functions to be successfully investigated.

Being able to infer the Radon-Nikodym derivative from its form on the Cameron-Martin space relied on an assumption of $\mathcal{P}$-measurability (a special kind of measurability), and this concept of measurability was developed further in this thesis. In particular, the $\mathcal{P}$-measurable derivative of a $\mathcal{P}$-measurable function was defined and shown to coincide with the stochastic derivative, under certain assumptions. The stochastic derivative in turn coincides with the Malliavin derivative when they both exist (Di Nunno et al. 2009). The main theorem's results, expressed the Radon-Nikodym derivative as the Jacobian of a transformation $\times$ the classical Cameron-Martin theorem.

It would be fruitful, and a possible future direction of research to investigate, if the kernel function associated with reproducing kernel Hilbert space of a fractional Brownian motion process can be shown to be a Hilbert-Schmidt operator, then via Sazanov's theorem (Sazonov 1958) there is a countably additive measure. This may make the Radon-Nikodym derivative theory easier to apply as one can lift Gaussian measure and there will be two countably additive measures. More specifically this may come in handy for applications, where a transformation of Ω is $\mathcal{P}$-measurable but does not have a provably $\mathcal{P}$-measurable inverse, as shown in 3.3.3.

Based on this generalisation of the Ramer theorem, a generalised version of Girsanov theorem for fractional Brownian motion, which is not restricted to changes of linear drift only was presented. We also demonstrated an example application of this theory, in which it was used to find the Radon-Nikodym derivative of a measure relative to a domain-transformation of the measure.

This thesis also introduces the concept of a *consistent quasi-invariant stochastic flow*, and showed how it can be used to determine the density of a functional defined on paths of Brownian motion. This method was then applied to the last exit of a Brownian motion process from a linear boundary and was used to determine the density of this functional.

The flows studied in this thesis were only used to determine the density of a functional defined on paths of Brownian motion. One could try to generalise this by instead applying it to a functional defined on paths of fractional Brownian motion.

As part of this research, we have established, for the first time, the link between an approximation of the probability distribution of Supremum of Fractional Brownian motion and the Amoroso distribution (or its special case of the Generalized Gamma distribution). This adds an important application to the long list of applications of the Amoroso distribution (Nadarajah and Gupta 2007, Lienhard and Meyer 1967, Mees and Gerard 1984). This link has provided new closed-form approximations for the mean, variance, third central moment and skewness of an fBm queue. Simplified expressions for the mean and variance for a range of cases were also provided.

Furthermore a long-standing problem of considerable interest and importance has an approximate solution as previously proposed in (Chen et al. 2013). In this thesis, the evidence for this approximation has been developed further. Additionally transport partial differential equations were derived based on the similarity law, which with the right additional conditions can give the exact probability density functions.

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