Nonlinear instability in generalized nonlinear phase diffusion equation

Dmitry Strunin

Department of Mathematics and Computing, Toowoomba, Queensland 4350, Australia

High-order versions of truncated generalized nonlinear phase diffusion equation of Kuramoto are analyzed. The main common feature of the equations is *nonlinear* internal energy supply. Numerical solutions are presented for some specific values of equation coefficients. Extended physical interpretations of the solutions are suggested.

§1. Introduction

In his monograph[1)] Kuramoto presents the analysis of a system of reaction-diffusion equations describing oscillators weakly coupled through a diffusion mechanism. The position of the oscillators,

$$\mathbf{U}(t) = \mathbf{U}_0[\phi(t)]$$

depends on the phase, $\phi(t)$, which, in absence of the coupling, equals time t. The diffusion coupling leads to the phase deviating from t: $\phi(\mathbf{x}, t) = t + \psi(\mathbf{x}, t)$. It was shown[1)] that slow variations of the phase deviation, ψ, satisfy the (non-dimensional) generalized nonlinear phase diffusion (GNPD) equation,

$$\begin{aligned}\partial_t\psi = a_1\nabla^2\psi + a_2(\nabla\psi)^2 + \\ b_1\nabla^4\psi + b_2\nabla^3\psi\nabla\psi + b_3(\nabla^2\psi)^2 + b_4\nabla^2\psi(\nabla\psi)^2 + b_5(\nabla\psi)^4 + \\ c_1\nabla^6\psi + c_2\nabla^5\psi\nabla\psi + c_3\nabla^4\psi\nabla^2\psi + c_4(\nabla^3\psi)^2 + c_5\nabla^4\psi(\nabla\psi)^2 + \\ c_6(\nabla^2\psi)^3 + c_7(\nabla\psi)^6 + \cdots ,\end{aligned} \tag{1·1}$$

where a_n and b_n are constant coefficients. The right-hand side of equation (1·1) is, in fact, the power series in small parameter $\nabla^2 \sim (1/L)^2$, where L is the presumably large characteristic distance over which the variations of ψ occur. The function ψ is allowed to vary considerably so that its characteristic amplitude, Ψ, is not necessarily small.

When the coefficient a_1 is positive, (1·1) reduces to a linear diffusion equation, $\partial_t\psi = a_1\nabla^2\psi$. Indeed, provided the initial variation of ψ was sufficiently small, the function evolves under the dominating influence of the diffusion term while the higher-order terms in ψ are negligible because of the smallness of Ψ and the higher-order terms in ∇ are negligible because of the smallness of $1/L$. It is the trivial fact that the diffusion equation is well-balanced in the sense that an arbitrary initial distribution does not blow up, but evolves towards a finite-amplitude (for this particular equation, zero-amplitude) settled regime. Second-order (in ∇) nonlinear truncation of (1·1) is the nonlinear phase diffusion (NPD) equation, $\partial_t\psi = a_1\nabla^2\psi + a_2(\nabla\psi)^2$. Analogously to the linear diffusion equation the NPD equation is well-balanced.[2)]

The situation changes dramatically when control parameters of the original reaction-diffusion equations are altered so that the coefficient a_1 became slightly negative: $a_1 = -\varepsilon$, $0 < \varepsilon \ll 1$. In this case the negative diffusion term becomes the source of energy resulting in an unlimited growth of the amplitude of the solution. As the amplitude grows, the nonlinear term, $a_2(\nabla\psi)^2 \sim a_2\Psi^2/L^2$, generates segments of relatively rapid variation of ψ, where L is relatively small (albeit still large enough to preserve validity of the expansion (1·1)). On those segments, the 4th-order term $b_1\nabla^4\psi \sim b_1\Psi/L^4$ becomes comparable to the 2nd-order terms. Given $b_1 < 0$, this term produces dissipative effect and smoothes out the field, ψ. With the first three terms in its right-hand side taken into account equation (1·1) reduces to the Kuramoto-Sivashinsky equation, $\partial_t\psi = a_1\nabla^2\psi + a_2(\nabla\psi)^2 + b_1\nabla^4\psi$. To estimate typical dimensions of the dissipative structures formed within the KS equation in large spatial domains, equate by the order of magnitude the terms in the right-hand side (assuming $|a_2|$, $|b_1| \sim 1$):

$$\varepsilon\Psi/L^2 \sim \Psi^2/L^2 \sim \Psi/L^4\,,$$

which gives

$$\Psi \sim \varepsilon\,, \quad L \sim 1/\sqrt{\varepsilon}\,. \tag{1·2}$$

However, the ignored higher-order terms, for instance, the nonlinear source $b_4\nabla^2\psi(\nabla\psi)^2$, $b_4 < 0$ must affect the dynamics substantially if the initial field variations were not small enough or not sufficiently extended in space. To estimate the scales of such "dangerous" distortions, equate by the order of magnitude the source term and, for example, the diffusion term:

$$|b_4\nabla^2\psi(\nabla\psi)^2| \sim |\varepsilon\nabla^2\psi|\,. \tag{1·3}$$

Equation (1·3) gives the connection between the initial amplitude, Ψ_0, and width, L_0, of the dangerous distortions:

$$|b_4|\Psi_0^3/L_0^4 \sim \varepsilon\Psi_0/L_0^2\,. \tag{1·4}$$

With $|b_4| \sim 1$ (1·4) yields $\Psi_0 \sim \sqrt{\varepsilon}L_0$. Consider for instance the distortions of the scale (1·2), $L_0 \sim 1/\sqrt{\varepsilon}$; then $\Psi_0 \sim \sqrt{\varepsilon}L_0 \sim 1$. For shorter distortions, $L_0 \sim 1/\varepsilon^\alpha$, $0 < \alpha < 1/2$, we get $\Psi_0 \sim \varepsilon^{1/2-\alpha} \ll 1$. The inequality demonstrates that even small-amplitude variations can be dangerous. With respect to them the trivial state, $\psi \equiv$ const, is nonlinearly unstable.

§2. Dynamics driven by nonlinear sources

In this section we consider 6th-order (in ∇) truncations of the GNPD equation. The necessity for the equations to have such order is dictated by the presence of nonlinear energy sources. Our particular choices of the coefficients will be motivated by the possibilities to obtain interesting solutions applicable in other areas of physics as compared to the original reaction-diffusion problem considered in.[1)]

2.1. *Linearly stabilized dynamics*

Consider (1·1) in one dimension with the energy supply provided by the nonlinear source $b_4\partial_x^2\psi(\partial_x\psi)^2$, $b_4 < 0$. The order of nonlinearity of this term is 3, therefore at least 4th-order nonlinear term is needed to prevail over the source at large amplitudes. We choose the 4th-order term $b_5(\partial_x\psi)^4$. However, this term alone has no dissipative effect similarly to the term $(\partial_x\psi)^2$ in the KS equation. Its function is to create steep sections on the topography of ψ with relatively small widths, L. The term is thus responsible for the transfer of the energy towards smaller spatial scales. To prevent the steep sections from becoming singularities, a higher-order linear dissipative term is required to smooth out the field. By the order of magnitude, $(\partial_x\psi)^4 \sim \Psi^4/L^4$, therefore the dissipation must be expressed by at least 6th-order derivative in order to dominate at relatively small L: $\partial_x^6\psi \sim \Psi/L^6$. Such balancing mechanism is similar to that of the KS equation. Essential element of the balance is the linear dissipation, which, in the final analysis, is responsible for the stabilization. For the 6th-order truncation in question, a multitude of different settled regimes can be obtained using different values of the coefficients of the equation and different wave numbers. We present one numerical solution for the coefficients $a_1 = 6.0$ (of the diffusion term), $a_2 = 0$, $b_1 = 0$, $b_2 = 0$, $b_3 = 18.0$ (of the 2nd-order-of-nonlinearity source), $b_4 = -3.3$ (of the "main" 3rd-order nonlinear source), $b_5 = 0.4$ (of the 4th-order nonlinear energy transfer term), $c_1 = 15.0$ (of the 6th-order derivative expressing the dissipation), $c_2 = c_3 = c_4 = c_5 = c_6 = c_7 = 0$. Periodic boundary conditions were used and the spectral Galerkin method with harmonic basis functions was employed in the numerical scheme. The reason behind the particular choice of the coefficients is to get a rotating auto-wave solution of saw-like shape similar to that found in solid-phase combustion.[3)] The settled regime is displayed in Fig. 1, left, where two periods are shown and x is given in conditional units. When put onto a cylindrical surface such a wave performs a helical motion. Kinematically and dynamically the wave well corresponds to a spinning combustion front,[3)] interpreted as the line dividing hot combustion products (located below the line in Fig. 1) from cold unreacted mixture (located above the line). For this phenomenon the phase ψ, may be associated with the distance passed by the front along a burning cylindrical sample.

2.2. *Non-linearly stabilized dynamics*

Different type of the balance occurs when a nonlinear source is directly counterbalanced by another *nonlinear* term. Let us choose $b_3(\partial_x^2\psi)^2$, $b_3 > 0$, to be the only source; it is obviously positive. The local growth of ψ driven by the source can be stopped by a higher power of $\partial_x^2\psi$, for instance $c_6(\partial_x^2\psi)^3$ with $c_6 > 0$. Indeed, as ψ grows its second derivative becomes more and more negative. The term $c_6(\partial_x^2\psi)^3$ is also negative; hence it stunts the growth at large amplitudes. Again, many possible settled regimes can be obtained using different values of the coefficients and wave number. We give one example using the following coefficients: $a_1 = 0$, $a_2 = -1.0$ (of the 2nd-order nonlinear energy transfer term), $b_1 = -10.0$ (of the 4th-order derivative representing the dissipative term), $b_2 = 0$, $b_3 = 6.4$,

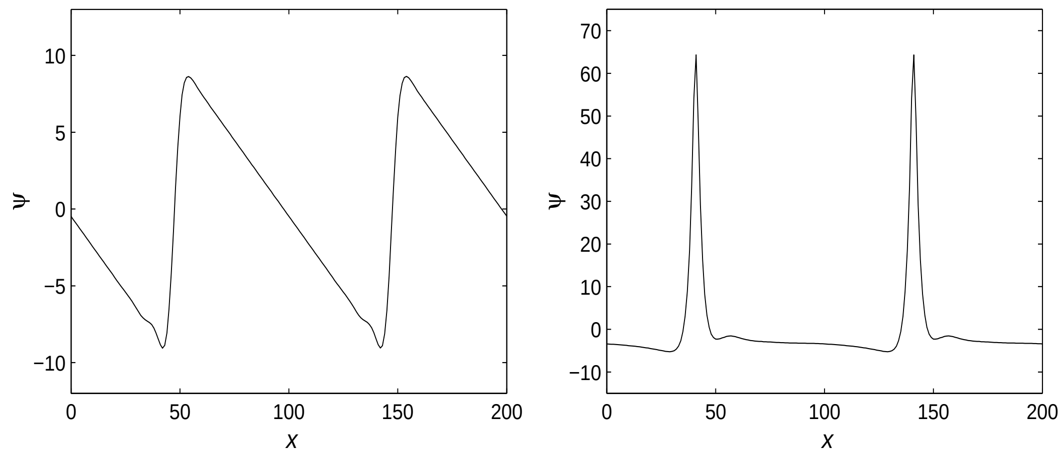

Fig. 1. Left: saw-like wave travelling to the left (the wave number $k = 0.14$). Right: train of solitons travelling to the right ($k = 0.05$).

$b_4 = b_5 = c_1 = c_2 = c_3 = c_4 = c_5 = 0$, $c_6 = 0.3$ (of the stabilizing 3rd-order nonlinear term), $c_7 = 0$. The resulting equation looks like an extension of the Hamilton-Jacobi (HJ) equation in quantum mechanics, where ψ represents the physical quantity called action,[5),6)] With $b_1 = b_3 = c_6 = 0$ and $a_2 \neq 0$ the classical form of the HJ equation follows. Within this equation, any localized perturbation of spatially uniform field shrinks to a point, that is the particle appears to be a point. In contrast, within the extended HJ equation it is possible to adjust the coefficient values in such a way that to obtain self-sustained extended structures—auto-solitons (Fig. 1, right). The auto-soliton can be a useful mathematical object to model an extended elementary particle. Strunin[6)] managed to obtain completely stationary solitons by additionally adopting some non-zero value for the coefficient a_1. It is the *stationary* soliton that is needed to model the particle at rest state. The regime presented in Fig. 1 is not stationary, however it presents certain interest because the model with close values of the coefficients may happen to be useful for modelling the resting particle in 2 or 3 dimensions. Recently Sivashinsky[4)] put forward the idea that in 2 or 3 dimensions the travelling soliton may assume the form of a solitary *rotating* wave. Such an eventuality would be desirable from quantum mechanical point of view since oscillating nature of such a wave (breather) would reflect wavy character of the particle. The perspective to find the breather solution gives an exciting incentive for special investigation of the extended HJ equation.

§3. Conclusions

Investigated are new truncated versions of the generalized nonlinear phase diffusion equation of Kuramoto, where energy is internally supplied by nonlinear sources. Depending on a type of an energy balance, two kinds of a truncation are presented: the first where a balance takes place between a nonlinear source and a higher-order "energy transfer" term combined with a high-order linear dissipation; and the second where a balance occurs between a nonlinear source and a higher-order nonlinear stabilizing term. Numerical solutions are presented for selected values of coefficients leading to physically interesting solutions.

References

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