

# Phase equation with nonlinear excitation for nonlocally coupled oscillators

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## Abstract

Some reaction-diffusion systems feature nonlocal interaction and, near the point of Hopf bifurcation, can be represented as a system of nonlocally coupled oscillators. Phase of oscillations satisfies an evolution PDE which takes different forms depending on the values of parameters. In the simplest case the equation is effectively a diffusion equation which is excitation-free. However, more complex forms are possible such as the Nikolaevskii equation and the Kuramoto-Sivashinsky equation incorporating linear excitation. We analyse a situation when the phase equation is based on nonlinear excitation. We derive conditions on the values of the parameters leading to the situation and show that the values satisfying the conditions exist.

## 1 Introduction

Reaction-diffusion systems exhibiting oscillatory dynamics near the Hopf bifurcation can be reduced to an evolution equation for the phase of oscillations [1, 2]

$$\begin{aligned} \partial_t \psi = & a_1 \nabla^2 \psi + a_2 (\nabla \psi)^2 + \\ & b_1 \nabla^4 \psi + b_2 \nabla^3 \psi \nabla \psi + b_3 (\nabla^2 \psi)^2 + b_4 \nabla^2 \psi (\nabla \psi)^2 + b_5 (\nabla \psi)^4 + \\ & g_1 \nabla^6 \psi + g_2 \nabla^5 \psi \nabla \psi + g_3 \nabla^4 \psi \nabla^2 \psi + g_4 (\nabla^3 \psi)^2 + g_5 \nabla^4 \psi (\nabla \psi)^2 + \\ & g_6 (\nabla^2 \psi)^3 + g_7 \nabla^3 \psi \nabla^2 \psi \nabla \psi + g_8 \nabla^3 \psi (\nabla \psi)^3 + g_9 (\nabla^2 \psi)^2 (\nabla \psi)^2 + \\ & g_{10} \nabla^2 \psi (\nabla \psi)^4 + g_{11} (\nabla \psi)^6 + \\ & e_1 \nabla^8 \psi + \dots, \end{aligned} \tag{1}$$

where  $a_n, b_n, g_n, e_n, \dots$  are constant coefficients. The right-hand side of (1) can be viewed as a power series in small parameter  $\nabla^2 \sim (1/L)^2$ , where  $L$  is the large characteristic spatial scale of variations of  $\psi$ .

Under certain conditions equation (1) can be truncated to finite forms. The most simple forms are the classical 2nd order (in  $\nabla$ ) diffusion equation and the nonlinear diffusion equation transformable into the Burgers' equation. If  $a_1 < 0$  or  $b_1 > 0$  the respective terms of (1) become anti-dissipative so that perturbations of a flat field  $\psi = \text{const}$  grow. One can say that such terms bring about excitation. In such cases a truncation must be of higher-order in  $\nabla$  in order to maintain balance, examples being the 4th order Kuramoto-Sivashinsky (KS) equation [1, 3] and the 6th order Nikolaevskii equation [4]. Note that the excitation terms in these equations are linear. As a consequence, the state  $\psi \equiv \text{const}$  turns out unstable to infinitesimal perturbations and is globally destroyed.

It is interesting to investigate whether a truncation with *nonlinear* excitation is possible. Unlike linear one, a nonlinear excitation can support localized dissipative structures sometimes referred to as auto-solitons and also complex regimes resulting from their interactions. Previously we showed that such a model is mathematically feasible [5, 6]. In the present work we execute a systematic procedure by which we derive a nonlinearly excited truncation of the phase equation from the complex Ginzburg-Landau (GL) equation with nonlocal coupling [7]. Tanaka and Kuramoto [7] mention a variety of systems linked to the nonlocal GL model: cellular slime molds, oscillating yeast cells under glycolysis and the Belousov-Zhabotinsky reaction dispersed in water-in-oil aerosol OT microemulsion.

Consider (1) and suppose that the coefficients  $b_4 = -\varepsilon$  is slightly negative. This makes the term  $-\varepsilon \nabla^2 \psi (\nabla \psi)^2$  an excitation; it can be treated as an anti-diffusion,  $-\nabla^2 \psi$ , with the nonlinear coefficient  $(\nabla \psi)^2$ . Suppose also that the coefficients  $a_1, a_2, b_1, b_2$  and  $b_3$  are small enough so that the respective terms can be neglected. The rest of the coefficients appearing in (1) with the exception of  $b_4$  are generally of order 1. When a perturbation of the flat steady state  $\psi \equiv \text{const}$  grows, the 4th order nonlinear term  $b_5 (\nabla \psi)^4$  comes into play. Its role is similar to  $(\nabla \psi)^2$  in the KS equation, which is to create regions of sharp variations of  $\psi$ . In such regions, the dissipation  $g_1 \nabla^6 \psi$  prevails (assuming  $g_1 > 0$ ) as it is of higher order in  $\nabla$ ; this prevents formation of singularities. Denoting characteristic scale of the phase variations by  $\Psi > 0$  we evaluate in absolute value:  $\nabla^2 \psi (\nabla \psi)^2 \sim \Psi^3/L^4$ ,  $(\nabla \psi)^4 \sim \Psi^4/L^4$

and  $\nabla^6\psi \sim \Psi/L^6$ . The balance between the three terms,

$$\varepsilon\Psi^3/L^4 \sim \Psi^4/L^4 \sim \Psi/L^6, \quad (2)$$

determines the scales of the dissipative structures,

$$\Psi \sim \varepsilon, \quad L \sim (1/\varepsilon)^{3/2}. \quad (3)$$

Suppose that

$$\begin{aligned} a_1 &= o(\varepsilon^6), \quad a_2 = o(\varepsilon^5), \quad b_1 = o(\varepsilon^3), \\ b_2 &= o(\varepsilon^2), \quad b_3 = o(\varepsilon^2), \quad b_4 = -\varepsilon. \end{aligned} \quad (4)$$

Let us show that these conditions make the respective terms of (1) negligible compared to the balance terms (2). For the latter, taking into account (3),

$$b_4 \nabla^2\psi(\nabla\psi)^2 \sim \varepsilon\Psi^3/L^4 \sim \varepsilon^{10}.$$

By comparison, using (3) and (4),

$$\begin{aligned} a_1 \nabla^2\psi &\sim a_1\Psi/L^2 \sim o(\varepsilon^{10}), \\ a_2 (\nabla\psi)^2 &\sim a_2\Psi^2/L^2 \sim o(\varepsilon^{10}), \\ b_1 \nabla^4\psi &\sim b_1\Psi/L^4 \sim o(\varepsilon^{10}), \\ b_2 \nabla^3\psi\nabla\psi &\sim b_2\Psi^2/L^4 \sim o(\varepsilon^{10}), \\ b_3 (\nabla^2\psi)^2 &\sim b_3\Psi^2/L^4 \sim o(\varepsilon^{10}). \end{aligned} \quad (5)$$

Now look at the rest of equation (1) starting from the term  $g_2\nabla^5\psi\nabla\psi$ . Among those all nonlinear terms are negligible relative to the balance terms because of higher order in  $\Psi \sim \varepsilon$ , and all linear terms are negligible because of higher order in  $\nabla^2 \sim 1/L^2 \sim \varepsilon^3$ . Thus, the phase equation reduces to

$$\partial_t\psi = b_4 \nabla^2\psi(\nabla\psi)^2 + b_5 (\nabla\psi)^4 + g_1 \nabla^6\psi. \quad (6)$$

Earlier [5] we solved (6) numerically as a semi-artificial model in one dimension under periodic boundary conditions and obtained smooth kink-shaped waves propagating at a constant speed.

Our plan is to derive (6) from the complex Ginzburg-Landau equation relevant to nonlocal reaction-diffusion systems. Tanaka and Kuramoto [7] analysed the system of this type

$$\partial_t \mathbf{X} = \mathbf{f}(\mathbf{X}) + \hat{\delta} \nabla^2 \mathbf{X} + k \mathbf{g}(S), \quad (7)$$

$$\tau \partial_t S = -S + D \nabla^2 S + h(\mathbf{X}). \quad (8)$$

Here  $\mathbf{X}$  is the vector representing concentrations of reactants,  $\hat{\delta}$  is a diagonal matrix responsible for diffusion,  $\mathbf{f}$  and  $\mathbf{g}$  are the vector functions,  $k$ ,  $\tau$  and  $D$  are constants. Implicitly system (7)–(8) involves a bifurcation parameter  $\mu$  which, when changing from  $\mu < 0$  to  $\mu > 0$ , brings about oscillatory instability. Note that equation (8) is linear; this allows to solve it for  $S$  in terms of  $\mathbf{X}$  and substitute in (7). Provided that the parameter  $k$  is small,  $k \sim O(|\mu|)$ , this leads to the complex nonlocal Ginzburg-Landau equation

$$\partial_t A = \mu \sigma A - \beta |A|^2 A + \delta \nabla^2 A + k \eta' \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') A(\mathbf{r}', t), \quad (9)$$

where  $A$  is loosely proportional to  $\mathbf{X}$ ;  $\sigma$ ,  $\beta$ ,  $\delta$  and  $\eta'$  are parameters. It is assumed that

$$\text{Re } \sigma > 0. \quad (10)$$

The coupling function  $G$  satisfies the normalization condition

$$\int G(\mathbf{r}) d\mathbf{r} = 1. \quad (11)$$

The complex parameter  $\delta$  is responsible for diffusion and, therefore, should have positive real part,

$$\text{Re } \delta > 0. \quad (12)$$

Using (11), it is convenient to write (9) in the form

$$\partial_t A = \mu \sigma' A - \beta |A|^2 A + \delta \nabla^2 A + k \eta' \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}', t) - A(\mathbf{r}, t)], \quad (13)$$

where

$$\sigma' = \sigma + k \eta' / \mu. \quad (14)$$

The system is assumed supercritical,

$$\text{Re } \sigma' > 0. \quad (15)$$

Some parameters in (13) can be eliminated by changing variables. The imaginary part of the coefficient  $\mu\sigma'$  vanishes after transforming  $A \rightarrow A \exp[i\mu \text{Im} \sigma' t]$  and the diffusion coefficient  $D$  as well as  $\text{Re} \beta$  and  $\mu \text{Re} \sigma'$  become unity by rescaling  $A$ ,  $t$  and  $\mathbf{r}$ . Eventually (13) is transformed to

$$\begin{aligned} \partial_t A = & A - (1 + ic_2)|A|^2 A + (\delta_1 + i\delta_2)\nabla^2 A \\ & + K(1 + ic_1) \int d\mathbf{r}' G(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}', t) - A(\mathbf{r}, t)], \end{aligned} \quad (16)$$

where  $c_1$ ,  $c_2$ ,  $\delta_1$ ,  $\delta_2$  and  $K$  are real constants. Condition (12) is equivalent to

$$\delta_1 > 0. \quad (17)$$

The complex amplitude  $A$  is connected to the real phase of oscillations,  $\varphi$ , via

$$A = ae^{-i\varphi}, \quad (18)$$

where  $a$  is the real amplitude. We consider a one-dimensional case keeping the symbol  $\nabla$  for convenience. In one dimension

$$G(x) = \frac{1}{2}(\zeta + i\eta)e^{-(\zeta+i\eta)|x|}, \quad (19)$$

where

$$\zeta = \left( \frac{1 + \sqrt{1 + \theta^2}}{2D} \right)^{1/2}, \quad \eta = \left( \frac{-1 + \sqrt{1 + \theta^2}}{2D} \right)^{1/2}. \quad (20)$$

Here the parameter  $\theta = \omega_0\tau$  is proportional to the basic frequency of oscillations,  $\omega_0$ , and characteristic time  $\tau$ . The diffusion coefficient  $D$  becomes unity upon the mentioned rescaling of  $x$  leading to (16). The form (19)–(20) expresses the physical fact that effective coupling radius,  $1/\zeta$ , is proportional to the square root of the diffusion coefficient,  $D^{1/2}$ . We refer to [7] for more details.

As for the phase, it is convenient to analyse its departure from  $c_2 t$ , defined by

$$\varphi = c_2 t + \psi. \quad (21)$$

For  $\psi$  we will derive the phase equation (1) using (16) and (18). During this procedure the integral term is decomposed in a power series in  $\nabla$ . Each coefficient in equation (1) turns out to be a combination of the parameters  $\delta_1$ ,  $\delta_2$ ,  $c_1$ ,  $c_2$ ,  $K$  and  $\theta$ .

It is important to note that the value of the parameter  $K$  is limited. After transferring from (13) to (16),  $K(1 + ic_1)$  appears as the combination

$$\frac{k\eta'/\mu}{\operatorname{Re} \sigma'} = K(1 + ic_1).$$

Taking real part and using (10), (14) and (15), we have

$$K = \frac{\operatorname{Re}(k\eta'/\mu)}{\operatorname{Re} \sigma'} = \frac{\operatorname{Re} \sigma' - \operatorname{Re} \sigma}{\operatorname{Re} \sigma'} = 1 - \frac{\operatorname{Re} \sigma}{\operatorname{Re} \sigma'} < 1. \quad (22)$$

The results above were obtained by Tanaka and Kuramoto [7]. Tanaka subsequently showed [8, 9] that, when the values of the parameters are appropriately chosen, equation (16) reduces to the Kuramoto-Sivashinsky equation or the Nikolaevskii equation. We present and briefly discuss these equations in Section 5.

## 2 Complex GL equation with two nonlocal terms

Tanaka and Kuramoto commented that the Ginzburg-Landau equation (16) “involves six independent parameters  $c_1$ ,  $c_2$ ,  $K$ ,  $\theta$ ,  $\delta_1$  and  $\delta_2$ ” [7] (p.026219-5). Our aim in the paper is to satisfy the six conditions (4) by choosing the values of the parameters. However, our computations indicate that to achieve this we seem to need more than six independent parameters.

Therefore, we add one more reactant into (7)–(8); this will allow to raise the number of independent parameters to nine. By selecting their values we will be able to satisfy in Section 4 the conditions (4) subject to all necessary restrictions. So we consider an extended reaction-diffusion system

$$\partial_t \mathbf{X} = \mathbf{f}(\mathbf{X}) + \hat{\delta} \nabla^2 \mathbf{X} + k_1 \mathbf{g}_1(S_1) + k_2 \mathbf{g}_2(S_2), \quad (23)$$

$$\tau_1 \partial_t S_1 = -S_1 + D \nabla^2 S_1 + h_1(\mathbf{X}), \quad (24)$$

$$\tau_2 \partial_t S_2 = -S_2 + D \nabla^2 S_2 + h_2(\mathbf{X}), \quad (25)$$

where  $k_1 \sim k_2 \sim O(|\mu|)$ . The extra term  $k_2 \mathbf{g}_2(S_2)$  in (23) leads to the extra integral term in the respective nonlocal Ginzburg-Landau equation,

$$\partial_t A = \mu \sigma A - \beta |A|^2 A + \delta \nabla^2 A$$

$$+k_1\eta_1' \int d\mathbf{r}' G_1(\mathbf{r} - \mathbf{r}') A(\mathbf{r}', t) + k_2\eta_2' \int d\mathbf{r}' G_2(\mathbf{r} - \mathbf{r}') A(\mathbf{r}', t), \quad (26)$$

where each coupling function  $G_n$  carries its own  $\theta_n$ ,  $n = 1, 2$ . The coupling functions satisfy the normalization conditions

$$\int G_1(\mathbf{r}) d\mathbf{r} = \int G_2(\mathbf{r}) d\mathbf{r} = 1.$$

Using these conditions, equation (26) is modified to

$$\begin{aligned} \partial_t A &= \mu\sigma' A - \beta|A|^2 A + \delta\nabla^2 A \\ &+ k_1\eta_1' \int d\mathbf{r}' G_1(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}', t) - A(\mathbf{r}, t)] \\ &+ k_2\eta_2' \int d\mathbf{r}' G_2(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}', t) - A(\mathbf{r}, t)], \end{aligned} \quad (27)$$

where

$$\sigma' = \sigma + k_1\eta_1'/\mu + k_2\eta_2'/\mu. \quad (28)$$

Rescaling (27) in the similar way to (13), we obtain

$$\begin{aligned} \partial_t A &= A - (1 + ic_2)|A|^2 A + (\delta_1 + i\delta_2)\nabla^2 A \\ &+ K_1(1 + ic_{11}) \int d\mathbf{r}' G_1(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}') - A(\mathbf{r})] \\ &+ K_2(1 + ic_{12}) \int d\mathbf{r}' G_2(\mathbf{r} - \mathbf{r}') [A(\mathbf{r}') - A(\mathbf{r})], \end{aligned} \quad (29)$$

where

$$K_1(1 + ic_{11}) = \frac{k_1\eta_1'/\mu}{\text{Re } \sigma'}, \quad K_2(1 + ic_{12}) = \frac{k_2\eta_2'/\mu}{\text{Re } \sigma'}. \quad (30)$$

Now we have the 9 independent parameters at our disposal:  $\delta_1$ ,  $\delta_2$ ,  $c_{11}$ ,  $c_{12}$ ,  $c_2$ ,  $K_1$ ,  $K_2$ ,  $\theta_1$  and  $\theta_2$  (the latter two are part of  $G_1$  and  $G_2$ ).

Let us derive restrictive conditions on  $K_1$  and  $K_2$  in place of (22). Taking real part of (30) and summing up, we get

$$K_1 + K_2 = \frac{\text{Re}(k_1\eta_1'/\mu) + \text{Re}(k_2\eta_2'/\mu)}{\text{Re } \sigma'}.$$

Then, using (28), (10) and (15),

$$K_1 + K_2 = \frac{\text{Re } \sigma' - \text{Re } \sigma}{\text{Re } \sigma'} = 1 - \frac{\text{Re } \sigma}{\text{Re } \sigma'} < 1. \quad (31)$$

Another restriction to be met is positiveness of the coefficient at  $\nabla^6\psi$  in (6),

$$g_1 > 0. \quad (32)$$

It is necessary to ensure that the term is dissipative.

### 3 Derivation of the phase equation

We start with the derivation of the phase equation for the Ginzburg-Landau equation (16) and then will generalize it for the extended variant (29). Denote

$$I_0 = K(1 + ic_1) \int dx' G(x - x')[A(x') - A(x)]. \quad (33)$$

Substituting (41) into (16) and separating real and imaginary parts, we obtain

$$\partial_t a = a - a^3 + \delta_1 \nabla^2 a - \delta_1 a (\nabla \varphi)^2 + 2\delta_2 \nabla a \nabla \varphi + \delta_2 a \nabla^2 \varphi + \text{Re } I. \quad (34)$$

$$\partial_t \varphi = c_2 a^2 + 2\delta_1 \frac{\nabla a \nabla \varphi}{a} + \delta_1 \nabla^2 \varphi - \delta_2 \frac{\nabla^2 a}{a} + \delta_2 (\nabla \varphi)^2 - \frac{1}{a} \text{Im } I, \quad (35)$$

where

$$I = \frac{I_0}{e^{-i\varphi}}. \quad (36)$$

Spatial variations of both the real amplitude  $a$  and phase  $\varphi$  are assumed slow, therefore  $\nabla \sim 1/L \sim \varepsilon_1$  is a small parameter. For consistency with (3) we state

$$\varepsilon_1 = \varepsilon^{3/2}.$$

As we said in the previous section, the integral  $I$  in (34)–(35) can be represented as a series in  $\nabla$ . Since  $A(x') - A(x) = (x' - x)\nabla A + (1/2)(x' - x)^2 \nabla^2 A + \dots$  and  $G(x) = G(-x)$  the leading term in such series is of order  $\nabla^2$ .

We face a typical centre manifold situation involving the fast variable  $a$  and slow variable  $\varphi$ . The amplitude  $a$  is quickly attracted to the vicinity of the equilibrium state,  $a_0 = 1$ , by means of  $\partial_t a = a - a^3$  and then is weakly affected by the perturbations. The phase is subject to the almost constant force  $c_2 a^2 \approx c_2 a_0^2 = c_2$  plus the weak influence of the perturbations. Thus, the phase and amplitude end up changing slowly in space and time about



$\varphi = c_2 t$  and  $a_0 = 1$ . The centre manifold theory says that there exists a manifold to which the dynamics are attracted exponentially quickly,

$$a = a[\nabla\varphi]. \quad (37)$$

Equation (37) manifests a stiff connection between the amplitude and phase on the attractor. This link allows to eliminate the amplitude from (35) and thus obtain a closed equation for the phase.

Rescale the variables via  $t_1 = \varepsilon_1^2 t$  and  $x_1 = \varepsilon_1 x$ . Then

$$\nabla = \varepsilon_1 \nabla_1, \quad \partial_t = \varepsilon_1^2 \partial_{t_1}. \quad (38)$$

Expand the amplitude into the series

$$a(x) = 1 + \varepsilon_1^2 a_2(x) + \varepsilon_1^4 a_4(x) + \varepsilon_1^6 a_6(x) + \dots \quad (39)$$

Using (38), (21) and (39) in the phase equation (35) and the amplitude equation (34), we obtain, for the phase departure introduced in (21),

$$\begin{aligned} \varepsilon_1^2 \partial_{t_1} \psi &= 2c_2 \varepsilon_1^2 a_2 + 2c_2 \varepsilon_1^4 a_4 + c_2 \varepsilon_1^4 a_2^2 + 2c_2 \varepsilon_1^6 a_6 \\ &+ 2\delta_1 \varepsilon_1^4 \nabla_1 a_2 \nabla_1 \psi + \delta_1 \varepsilon_1^2 \nabla_1^2 \psi - \delta_2 \varepsilon_1^4 \nabla_1^2 a_2 - \delta_2 \varepsilon_1^6 \nabla_1^2 a_4 \\ &+ \delta_2 \varepsilon_1^2 (\nabla_1 \psi)^2 - \frac{1}{a} \text{Im } I + \dots \end{aligned} \quad (40)$$

and, for the amplitude,

$$\begin{aligned} \varepsilon_1^4 \partial_{t_1} a_2 + \varepsilon_1^6 \partial_{t_1} a_4 + \dots &= -2\varepsilon_1^2 a_2 - \varepsilon_1^4 (2a_4 + 3a_2^2) - \varepsilon_1^6 (2a_6 + 6a_2 a_4 + a_2^3) \\ &+ \delta_1 \varepsilon_1^2 (\varepsilon_1^2 \nabla_1^2 a_2 + \varepsilon_1^4 \nabla_1^2 a_4 + \dots) - \delta_1 (1 + \varepsilon_1^2 a_2 + \varepsilon_1^4 a_4 + \dots) \varepsilon_1^2 (\nabla_1 \psi)^2 \\ &+ 2\delta_2 \varepsilon_1 (\varepsilon_1^2 \nabla_1 a_2 + \varepsilon_1^4 \nabla_1 a_4 + \dots) \varepsilon_1 \nabla_1 \psi \\ &+ \delta_2 (1 + \varepsilon_1^2 a_2 + \varepsilon_1^4 a_4 + \dots) \varepsilon_1^2 \nabla_1^2 \psi + \text{Re } I. \end{aligned} \quad (41)$$

Collecting terms  $\sim \varepsilon_1^2$  in (41) we obtain

$$0 = -2a_2 - \delta_1 (\nabla_1 \psi)^2 + \delta_2 \nabla_1^2 \psi + (\text{Re } I)_2, \quad (42)$$

where  $(\text{Re } I)_2$  denotes the coefficient at  $\varepsilon_1^2$  in  $\varepsilon_1$ -series for  $\text{Re } I$  that we need yet to determine. Inserting (18) into (33) we get

$$I_0 = K(1 + ic_1) \int dx' G(x - x') [a(x') e^{-i\varphi(x')} - a(x) e^{-i\varphi(x)}] \quad (43)$$

implying that  $a$  and  $\varphi$  also depend on  $t$ . We want to decompose the integrand into a series. Using the Taylor series

$$\begin{aligned} a(x') &= a(x) + \nabla a(x)\Delta x + \frac{1}{2}\nabla^2 a(x)\Delta x^2 + \dots \\ &= a(x) + \varepsilon_1 \nabla_1 a(x)\Delta x + \frac{1}{2}\varepsilon_1^2 \nabla_1^2 a(x)\Delta x^2 + \dots, \end{aligned} \quad (44)$$

where

$$\Delta x = x' - x,$$

and substituting the  $\varepsilon_1$ -series (39) into (44) and then the latter together with the coupling function (19) into (43) we get

$$\begin{aligned} I_0 &= \frac{K}{2}(1 + ic_1)(\zeta + i\eta)e^{-i\varphi} \int dx' e^{-(\zeta+i\eta)|\Delta x|} \\ &\quad \times [(e^{i\Delta\varphi} - 1) + \varepsilon_1^2 a_2(x) (e^{i\Delta\varphi} - 1) \\ &\quad + \varepsilon_1^3 \nabla_1 a_2(x)\Delta x e^{i\Delta\varphi} \\ &\quad + \varepsilon_1^4 \left( a_4(x)e^{i\Delta\varphi} + \frac{1}{2}\nabla_1^2 a_2(x)\Delta x^2 e^{i\Delta\varphi} - a_4(x) \right) \\ &\quad + \varepsilon_1^5 \left( \nabla_1 a_4(x)\Delta x e^{i\Delta\varphi} + \frac{1}{3!}\nabla_1^3 a_2(x)\Delta x^3 e^{i\Delta\varphi} \right) \\ &\quad + \varepsilon_1^6 \left( a_6(x)e^{i\Delta\varphi} + \frac{1}{2}\nabla_1^2 a_4(x)\Delta x^2 e^{i\Delta\varphi} + \frac{1}{4!}\nabla_1^4 a_2(x)\Delta x^4 e^{i\Delta\varphi} - a_6(x) \right) + \dots], \end{aligned} \quad (45)$$

where

$$\Delta\varphi = \varphi - \varphi'$$

( $\varphi' = \varphi(x')$ ). In order to determine  $(\text{Re } I)_2$  we need to extract cumulative component  $\sim \varepsilon_1^2$  from (45). We need the Taylor series

$$e^{i\Delta\varphi} = 1 + i\Delta\varphi - \frac{1}{2}\Delta\varphi^2 - \frac{1}{3!}i\Delta\varphi^3 + \frac{1}{4!}\Delta\varphi^4 + \frac{1}{5!}i\Delta\varphi^5 + \dots \quad (46)$$

and also the Taylor series for  $\Delta\varphi$ ,

$$\begin{aligned} \Delta\varphi &= -(\varphi' - \varphi) = -(\psi' - \psi) \\ &= -\varepsilon_1 \nabla_1 \psi \Delta x - \frac{1}{2}\varepsilon_1^2 \nabla_1^2 \psi \Delta x^2 + \dots \end{aligned} \quad (47)$$

Substituting (47) into (46) we find

$$e^{i\Delta\varphi} - 1 = i \left( -\varepsilon_1 \nabla_1 \psi \Delta x - \frac{1}{2} \varepsilon_1^2 \nabla_1^2 \psi \Delta x^2 + \dots \right) - \frac{1}{2} \left( \varepsilon_1 \nabla_1 \psi \Delta x + \frac{1}{2} \varepsilon_1^2 \nabla_1^2 \psi \Delta x^2 + \dots \right)^2 + \dots \quad (48)$$

Now we substitute (48) into (45) and take into account that the integrals involving odd powers of  $\Delta x$  give zero. The leading term  $\sim \varepsilon_1$  in (45) comes from  $e^{i\Delta\varphi} - 1$ , it is  $(-i\varepsilon_1 \nabla_1 \psi \Delta x)$ ; upon integration it gives zero. In order  $\sim \varepsilon_1^2$

$$(\text{Re } I)_2 = \text{Re} \left\{ \frac{K}{2} (1 + ic_1) (\zeta + i\eta) \int dx' e^{-(\zeta+i\eta)|\Delta x|} \times \left[ -i \frac{1}{2} \nabla_1^2 \psi \Delta x^2 - \frac{1}{2} (\nabla_1 \psi)^2 \Delta x^2 \right] \right\}. \quad (49)$$

Using(49), we express  $a_2$  from (42),

$$a_2 = \frac{\delta_2}{2} \nabla_1^2 \psi - \frac{\delta_1}{2} (\nabla_1 \psi)^2 + \frac{1}{2} (\text{Re } I)_2 = \frac{\delta_2}{2} \nabla_1^2 \psi - \frac{\delta_1}{2} (\nabla_1 \psi)^2 + \frac{K}{4} \left\{ (\zeta - c_1 \eta) \int dx' e^{-\zeta|\Delta x|} \left[ -\frac{1}{2} (\nabla_1 \psi)^2 \Delta x^2 \cos(\eta|\Delta x|) - \frac{1}{2} \nabla_1^2 \psi \Delta x^2 \sin(\eta|\Delta x|) \right] - (c_1 \zeta + \eta) \int dx' e^{-\zeta|\Delta x|} \left[ -\frac{1}{2} \nabla_1^2 \psi \Delta x^2 \cos(\eta|\Delta x|) + \frac{1}{2} (\nabla_1 \psi)^2 \Delta x^2 \sin(\eta|\Delta x|) \right] \right\}. \quad (50)$$

The terms  $(\nabla_1 \psi)^2$  and  $\nabla_1^2 \psi$  do not depend on  $x'$  so (50) contains the integrals

$$C_n(\zeta, \eta) = \int_{-\infty}^{\infty} dx' e^{-\zeta|x'-x|} \cos(\eta|x'-x|) (x'-x)^n,$$

$$S_n(\zeta, \eta) = \int_{-\infty}^{\infty} dx' e^{-\zeta|x'-x|} \sin(\eta|x'-x|) (x'-x)^n$$

for  $n = 2$ . Tables of integrals give

$$C_2(\zeta, \eta) = \frac{4\zeta(\zeta^2 - 3\eta^2)}{(\zeta^2 + \eta^2)^3}, \quad S_2(\zeta, \eta) = \frac{4\eta(3\zeta^2 - \eta^2)}{(\zeta^2 + \eta^2)^3}.$$

Later we will also need cases  $n = 4$  and  $n = 6$ ,

$$\begin{aligned} C_n(\zeta, \eta) &= \frac{2\Gamma(n+1)}{(\zeta^2 + \eta^2)^{(n+1)/2}} \cos[(n+1) \arctan(\eta/\zeta)] , \\ S_n(\zeta, \eta) &= \frac{2\Gamma(n+1)}{(\zeta^2 + \eta^2)^{(n+1)/2}} \sin[(n+1) \arctan(\eta/\zeta)] . \end{aligned} \quad (51)$$

Straightforward manipulations in (50) lead to the answer

$$a_2 = \alpha_1 \nabla_1^2 \psi - \alpha_2 (\nabla_1 \psi)^2 , \quad (52)$$

where

$$\begin{aligned} \alpha_1 &= \frac{\delta_2}{2} - \frac{K}{8} S_2(\zeta - c_1 \eta) + \frac{K}{8} C_2(c_1 \zeta + \eta) , \\ \alpha_2 &= \frac{\delta_1}{2} + \frac{K}{8} C_2(\zeta - c_1 \eta) + \frac{K}{8} S_2(c_1 \zeta + \eta) . \end{aligned} \quad (53)$$

In order  $\sim \varepsilon_1^3$  the amplitude equation (41) says

$$0 = (\text{Re } I)_3 . \quad (54)$$

Let us show that this is true. From (45) using (48) we find

$$\begin{aligned} (\text{Re } I)_3 &= \text{Re} \left\{ \frac{K}{2} (1 + ic_1) (\zeta + i\eta) \int_{-\infty}^{\infty} dx' e^{-(\zeta + i\eta)|\Delta x|} \right. \\ &\quad \times \left[ -i \frac{1}{3!} \nabla_1^3 \psi \Delta x^3 - \frac{1}{2} \nabla_1 \psi \nabla_1^2 \psi \Delta x^3 + \frac{1}{3!} i \nabla_1^3 \psi \Delta x^3 \right. \\ &\quad \left. \left. - a_2 i \nabla_1 \psi \Delta x + \nabla_1 a_2 \Delta x \right] \right\} . \end{aligned}$$

This integral is equal to zero as it involves only odd powers of  $\Delta x$ . Grouping terms  $\sim \varepsilon_1^4$  in the amplitude equation (41) we have

$$\begin{aligned} a_4 &= -(1/2) \partial_{t_1} a_2 - (3/2) a_2^2 + (1/2) \delta_1 \nabla_1^2 a_2 - (1/2) \delta_1 a_2 (\nabla_1 \psi)^2 \\ &\quad + \delta_2 \nabla_1 a_2 \nabla_1 \psi + (1/2) \delta_2 a_2 \nabla_1^2 \psi + \frac{1}{2} (\text{Re } I)_4 . \end{aligned} \quad (55)$$

$(\text{Re } I)_4$  is the coefficient at  $\varepsilon_1^4$  in the series for  $\text{Re } I$ .

Here we make the crucial remark that the time derivative in (55) can be ignored straight away as justified by the following. Having placed  $a_2$

expressed by (52) under the time derivative in (55) and subsequently into the phase equation (40) we get the terms of higher orders than the time derivative  $\varepsilon_1^2 \partial_{t_1} \psi \sim \varepsilon_1^2$  already present in (40), for example  $2c_2 \varepsilon_1^4 \alpha_1 \partial_{t_1} \nabla_1^2 \psi \sim \varepsilon_1^4$ . Such terms can be ignored.

Now let us determine  $(\text{Re } I)_4$  by grouping terms  $\sim \varepsilon_1^4$  in (45),

$$\begin{aligned}
(\text{Re } I)_4 = & \text{Re} \left\{ \frac{K}{2} (1 + ic_1) (\zeta + i\eta) \int dx' e^{-(\zeta+i\eta)|\Delta x|} \right. \\
& \times \left[ \left( -i \frac{1}{24} \nabla_1^4 \psi \Delta x^4 - \frac{1}{8} (\nabla_1^2 \psi)^2 \Delta x^4 - \frac{1}{6} \nabla_1 \psi \nabla_1^3 \psi \Delta x^4 \right. \right. \\
& \quad \left. \left. + i \frac{1}{4} (\nabla_1 \psi)^2 \nabla_1^2 \psi \Delta x^4 + \frac{1}{24} (\nabla_1 \psi)^4 \Delta x^4 \right) \right. \\
& \left. + a_2 \left( -i \frac{1}{2} \nabla_1^2 \psi \Delta x^2 - \frac{1}{2} (\nabla_1 \psi)^2 \Delta x^2 \right) + \frac{1}{2} \nabla_1^2 a_2 \Delta x^2 \right] \left. \right\}. \tag{56}
\end{aligned}$$

Substituting (52) into (56) and the result into (55) (with the time derivative removed) and rearranging we come to

$$a_4 = \beta \nabla_1^4 \psi + \beta_1 (\nabla_1^2 \psi)^2 + \beta_2 \nabla_1 \psi \nabla_1^3 \psi + \beta_3 \nabla_1^2 \psi (\nabla_1 \psi)^2 + \beta_4 (\nabla_1 \psi)^4 \tag{57}$$

with the coefficients

$$\begin{aligned}
\beta = & \alpha_1 \frac{\delta_1}{2} + \frac{K}{8} (\zeta - c_1 \eta) C_2 \alpha_1 + \frac{K}{8} (\eta + c_1 \zeta) S_2 \alpha_1 \\
& - \frac{K}{4 \cdot 24} (\zeta - c_1 \eta) S_4 + \frac{K}{4 \cdot 24} (\eta + c_1 \zeta) C_4, \\
\beta_1 = & -\frac{3}{2} \alpha_1^2 - \delta_1 \alpha_2 + \frac{\delta_2}{2} \alpha_1 + \frac{K}{32} (\zeta - c_1 \eta) C_4 - \frac{K}{32} (\eta + c_1 \zeta) S_4 \\
& - \frac{K}{4} (\zeta - c_1 \eta) C_2 \alpha_2 - \frac{K}{4} (\eta + c_1 \zeta) S_2 \alpha_2 \\
& - \frac{K}{8} (\zeta - c_1 \eta) S_2 \alpha_1 + \frac{K}{8} (\eta + c_1 \zeta) C_2 \alpha_1, \\
\beta_2 = & -\delta_1 \alpha_2 + \delta_2 \alpha_1 - \frac{K}{24} (\zeta - c_1 \eta) C_4 - \frac{K}{24} (\eta + c_1 \zeta) S_4 \\
& - \frac{K}{4} (\zeta - c_1 \eta) C_2 \alpha_2 - \frac{K}{4} (\eta + c_1 \zeta) S_2 \alpha_2,
\end{aligned} \tag{58}$$

$$\begin{aligned}
\beta_3 &= 3\alpha_1\alpha_2 - \frac{\delta_1}{2}\alpha_1 - \frac{5\delta_2}{2}\alpha_2 \\
&\quad - \frac{K}{8}(\zeta - c_1\eta)C_2\alpha_1 - \frac{K}{8}(\eta + c_1\zeta)S_2\alpha_1 \\
&\quad + \frac{K}{16}(\zeta - c_1\eta)S_4 - \frac{K}{16}(\eta + c_1\zeta)C_4 \\
&\quad + \frac{K}{8}(\zeta - c_1\eta)S_2\alpha_2 - \frac{K}{8}(\eta + c_1\zeta)C_2\alpha_2, \\
\beta_4 &= -\frac{3}{2}\alpha_2^2 + \frac{K}{4 \cdot 24}(\zeta - c_1\eta)C_4 + \frac{K}{4 \cdot 24}(\eta + c_1\zeta)S_4 \\
&\quad + \frac{K}{8}(\zeta - c_1\eta)C_2\alpha_2 + \frac{K}{8}(\eta + c_1\zeta)S_2\alpha_2 + \frac{\delta_1}{2}\alpha_2.
\end{aligned} \tag{59}$$

In order  $\sim \varepsilon_1^5$  the amplitude equation (41) requires

$$0 = (\text{Re } I)_5.$$

Let us show that this is true. Using (45), we obtain

$$\begin{aligned}
(\text{Re } I)_5 &= \text{Re} \left\{ \frac{K}{2}(1 + ic_1)(\zeta + i\eta) \int_{-\infty}^{\infty} dx' e^{-(\zeta + i\eta)|\Delta x|} \right. \\
&\times \left[ -i \frac{1}{5!} \nabla_1^5 \psi \Delta x^5 - \frac{1}{2 \cdot 3!} \nabla_1^2 \psi \nabla_1^3 \psi \Delta x^5 - \frac{1}{4!} \nabla_1 \psi \nabla_1^4 \psi \Delta x^5 \right. \\
&\quad + i \frac{1}{8} \nabla_1 \psi (\nabla_1^2 \psi)^2 \Delta x^5 + i \frac{3}{3!^2} (\nabla_1 \psi)^2 \nabla_1^3 \psi \Delta x^5 \\
&\quad + \frac{2}{4!} (\nabla_1 \psi)^3 \nabla_1^2 \psi \Delta x^5 - i \frac{1}{5!} (\nabla_1 \psi)^5 \Delta x^5 \\
&\quad - ia_2(x) \frac{1}{3!} \nabla_1^3 \psi \Delta x^3 - a_2(x) \frac{1}{2} \nabla_1 \psi \nabla_1^2 \psi \Delta x^3 + ia_2(x) \frac{1}{3!} (\nabla_1 \psi)^3 \Delta x^3 \\
&\quad - i \frac{1}{2} \nabla_1^2 \psi \nabla_1 a_2 \Delta x^3 - \frac{1}{2} (\nabla_1 \psi)^2 \nabla_1 a_2 \Delta x^3 \\
&\quad + ia_4(x) \nabla_1 \psi \Delta x - i \frac{1}{2} \nabla_1^2 a_2 \nabla_1 \psi \Delta x^3 \\
&\quad \left. \left. + \nabla_1 a_4(x) \Delta x + \frac{1}{3!} \nabla_1^3 a_2 \Delta x^3 \right] \right\} = 0
\end{aligned}$$

as only odd powers of  $\Delta x$  are involved here. Grouping terms  $\sim \varepsilon_1^6$  in the amplitude equation (41) we have

$$a_6 = -\frac{1}{2}\partial_{t_1}a_4 - 3a_2a_4 - \frac{1}{2}a_2^3 + \frac{\delta_1}{2}\nabla_1^2a_4 - \frac{\delta_1}{2}a_4(\nabla_1\psi)^2 + \delta_2\nabla_1a_4\nabla_1\psi + \frac{\delta_2}{2}a_4\nabla_1^2\psi + \frac{1}{2}(\text{Re } I)_6. \quad (60)$$

The time derivative  $\partial_{t_1}a_4$  in (60) can be immediately ignored for the same reason as  $\partial_{t_1}a_2$  in (55). From the rest of equation (60) we need to retain only linear contribution that is the terms proportional to  $\nabla_1^6\psi$  because all the nonlinear terms of order  $\varepsilon_1^6$  are negligible as we argued in Section 1. Such linear terms may come only from two terms in (60):

$$a_6 = \text{linear terms from } \left\{ \frac{1}{2}\delta_1\nabla_1^2a_4 + \frac{1}{2}(\text{Re } I)_6 \right\}, \quad (61)$$

where

$$(\text{Re } I)_6 = \text{Re} \left\{ \frac{K}{2}(1 + ic_1)(\zeta + i\eta) \int dx' e^{-(\zeta + i\eta)|\Delta x|} \times \left[ -i\frac{1}{6!}\nabla_1^6\psi\Delta x^6 + \frac{1}{2}\nabla_1^2a_4\Delta x^2 + \frac{1}{4!}\nabla_1^4a_2\Delta x^4 \right] \right\}. \quad (62)$$

Using  $a_2$  from (52) and  $a_4$  from (57) and doing some algebra we obtain

$$a_6 = \nabla_1^6\psi \left[ \frac{1}{2}\delta_1\beta + \frac{K}{8}(\zeta - c_1\eta)C_2\beta + \frac{K}{4 \cdot 4!}(\zeta - c_1\eta)C_4\alpha_1 - \frac{K}{4 \cdot 6!}(\zeta - c_1\eta)S_6 + \frac{K}{4 \cdot 6!}(c_1\zeta + \eta)C_6 + \frac{K}{8}(c_1\zeta + \eta)S_2\beta + \frac{K}{4 \cdot 4!}(c_1\zeta + \eta)S_4\alpha_1 \right] \quad (63)$$

with  $C_6$  and  $S_6$  given by (51) and  $\beta$  given by (58).

We are getting closer to achieving the goal of transforming the phase equation (40) into a closed form. To this end we determined all the components of the amplitude, namely  $a_2$ ,  $a_4$  and  $a_6$  appearing in (40), in terms of  $\psi$ . The next task is to express the input from  $(1/a) \text{Im } I$  in terms of  $\psi$ . First, we decompose  $\text{Im } I$  into the series

$$\text{Im } I = \varepsilon_1(\text{Im } I)_1 + \varepsilon_1^2(\text{Im } I)_2 + \varepsilon_1^3(\text{Im } I)_3 + \dots$$

and take into account that all the odd terms disappear because  $I_1 = I_3 = \dots = 0$  as we explained earlier. Thus, we have

$$\begin{aligned} a^{-1} \text{Im } I &= (1 + \varepsilon_1^2 a_2 + \varepsilon_1^4 a_4 + \varepsilon_1^6 a_6 + \dots)^{-1} \\ &\times [\varepsilon_1^2 (\text{Im } I)_2 + \varepsilon_1^4 (\text{Im } I)_4 + \varepsilon_1^6 (\text{Im } I)_6 + \dots] \\ &= -\varepsilon_1^4 a_2 (\text{Im } I)_2 + \varepsilon_1^2 (\text{Im } I)_2 + \varepsilon_1^4 (\text{Im } I)_4 + \varepsilon_1^6 (\text{Im } I)_6 + \dots \end{aligned} \quad (64)$$

To find  $(\text{Im } I)_2$  we use the expression under Re in (49),

$$\begin{aligned} (\text{Im } I)_2 &= \text{Im} \left\{ \frac{K}{2} (1 + ic_1) (\zeta + i\eta) \int dx' e^{-(\zeta+i\eta)|\Delta x|} \right. \\ &\quad \left. \times \left[ -i \frac{1}{2} \nabla_1^2 \psi \Delta x^2 - \frac{1}{2} (\nabla_1 \psi)^2 \Delta x^2 \right] \right\} \\ &= \frac{K}{2} \left[ -(\zeta - c_1 \eta) C_2 \frac{1}{2} \nabla_1^2 \psi - (c_1 \zeta + \eta) C_2 \frac{1}{2} (\nabla_1 \psi)^2 \right. \\ &\quad \left. + (\zeta - c_1 \eta) S_2 \frac{1}{2} (\nabla_1 \psi)^2 - (c_1 \zeta + \eta) S_2 \frac{1}{2} \nabla_1^2 \psi \right]. \end{aligned} \quad (65)$$

To determine  $(\text{Im } I)_4$  we use the expression under Re in (56),

$$\begin{aligned} (\text{Im } I)_4 &= \frac{K}{2} \left\{ (\zeta - c_1 \eta) \left[ \frac{1}{8} (\nabla_1^2 \psi)^2 S_4 + \frac{1}{6} \nabla_1 \psi \nabla_1^3 \psi S_4 - \frac{1}{24} (\nabla_1 \psi)^4 S_4 \right. \right. \\ &\quad \left. \left. + a_2 \frac{1}{2} (\nabla_1 \psi)^2 S_2 - \frac{1}{2} \nabla_1^2 a_2 S_2 - \frac{1}{24} \nabla_1^4 \psi C_4 \right. \right. \\ &\quad \left. \left. + \frac{1}{4} (\nabla_1 \psi)^2 \nabla_1^2 \psi C_4 - a_2 \frac{1}{2} \nabla_1^2 \psi C_2 \right] \right. \\ &\quad \left. + (\eta + c_1 \zeta) \left[ -\frac{1}{8} (\nabla_1^2 \psi)^2 C_4 - \frac{1}{6} \nabla_1 \psi \nabla_1^3 \psi C_4 + \frac{1}{24} (\nabla_1 \psi)^4 C_4 \right. \right. \\ &\quad \left. \left. - a_2 \frac{1}{2} (\nabla_1 \psi)^2 C_2 + \frac{1}{2} \nabla_1^2 a_2 C_2 - \frac{1}{24} \nabla_1^4 \psi S_4 \right. \right. \\ &\quad \left. \left. + \frac{1}{4} (\nabla_1 \psi)^2 \nabla_1^2 \psi S_4 - a_2 \frac{1}{2} \nabla_1^2 \psi S_2 \right] \right\}, \end{aligned} \quad (66)$$

with  $a_2$  given by (52). In order to determine  $(\text{Im } I)_6$  we need to take imaginary part of the expression under Re in (62) and retain only linear part,

$$(\text{Im } I)_6 = \text{linear terms from } \left\{ \frac{K}{2} (\zeta - c_1 \eta) \right.$$



$$\begin{aligned} & \times \left[ -\frac{1}{6!} \nabla_1^6 \psi C_6 - \frac{1}{2} \nabla_1^2 a_4 S_2 - \frac{1}{4!} \nabla_1^4 a_2 S_4 \right] \\ & + \frac{K}{2} (c_1 \zeta + \eta) \left[ \frac{1}{2} \nabla_1^2 a_4 C_2 + \frac{1}{4!} \nabla_1^4 a_2 C_4 - \frac{1}{6!} \nabla_1^6 \psi S_6 \right] \Big\} . \end{aligned}$$

Inserting  $a_2$  expressed by (52) and  $a_4$  by (57) this leads to

$$\begin{aligned} (\text{Im } I)_6 = & \nabla_1^6 \psi \frac{K}{2} \left\{ (\zeta - c_1 \eta) \left[ -\frac{1}{6!} C_6 - \frac{1}{2} S_2 \beta - \frac{1}{4!} S_4 \alpha_1 \right] \right. \\ & \left. + (c_1 \zeta + \eta) \left[ \frac{1}{2} C_2 \beta + \frac{1}{4!} C_4 \alpha_1 - \frac{1}{6!} S_6 \right] \right\} . \end{aligned} \quad (67)$$

Substituting (64) and (65)–(67) into (40) and returning to the unscaled operators  $\partial_t$  and  $\nabla$  we finally obtain the phase equation in closed form,

$$\begin{aligned} \partial_t \psi = & \nabla^2 \psi \left[ 2c_2 \alpha_1 + \delta_1 + \frac{K}{4} (\zeta - c_1 \eta) C_2 - \frac{K}{4} (c_1 \zeta + \eta) S_2 \right] \\ & + (\nabla \psi)^2 \left[ -2c_2 \alpha_2 + \delta_2 + \frac{K}{4} (c_1 \zeta + \eta) C_2 - \frac{K}{4} (\zeta - c_1 \eta) S_2 \right] \\ & + \nabla^4 \psi \left[ 2c_2 \beta - \delta_2 \alpha_1 + \frac{K}{48} (\eta + c_1 \zeta) S_4 + \frac{K}{48} (\zeta - c_1 \eta) C_4 \right. \\ & \quad \left. - \frac{K}{4} (\eta + c_1 \zeta) C_2 \alpha_1 + \frac{K}{4} (\zeta - c_1 \eta) S_2 \alpha_1 \right] \\ & + \nabla \psi \nabla^3 \psi \left[ 2c_2 \beta_2 + 2\delta_1 \alpha_1 + 2\delta_2 \alpha_2 - \frac{K}{12} (\zeta - c_1 \eta) S_4 \right. \\ & \quad \left. + \frac{K}{12} (\eta + c_1 \zeta) C_4 - \frac{K}{2} (\zeta - c_1 \eta) S_2 \alpha_2 + \frac{K}{2} (\eta + c_1 \zeta) C_2 \alpha_2 \right] \\ & + (\nabla^2 \psi)^2 \left[ 2c_2 \beta_1 + 2c_2 \alpha_1^2 + 2\delta_2 \alpha_2 - \frac{K}{16} (\zeta - c_1 \eta) S_4 \right. \\ & \quad \left. + \frac{K}{16} (\eta + c_1 \zeta) C_4 - \frac{K}{2} (\zeta - c_1 \eta) S_2 \alpha_2 + \frac{K}{2} (\eta + c_1 \zeta) C_2 \alpha_2 \right] \end{aligned} \quad (68)$$

$$\begin{aligned}
& +\nabla^2\psi(\nabla\psi)^2 \left[ 2c_2\beta_3 - 4c_2\alpha_1\alpha_2 - 4\delta_1\alpha_2 - \frac{K}{8}(\zeta - c_1\eta)C_4 - \frac{K}{8}(\eta + c_1\zeta)S_4 \right] \\
& +(\nabla\psi)^4 \left[ 2c_2\beta_4 + 2c_2\alpha_2^2 + \frac{K}{48}(\zeta - c_1\eta)S_4 - \frac{K}{48}(\eta + c_1\zeta)C_4 \right] \\
& +\nabla^6\psi \left[ 2c_2 \left( \frac{1}{2}\delta_1\beta + \frac{K}{8}(\zeta - c_1\eta)C_2\beta + \frac{K}{4 \cdot 4!}(\zeta - c_1\eta)C_4\alpha_1 \right. \right. \\
& \quad \left. \left. - \frac{K}{4 \cdot 6!}(\zeta - c_1\eta)S_6 + \frac{K}{4 \cdot 6!}(c_1\zeta + \eta)C_6 \right. \right. \\
& \quad \left. \left. + \frac{K}{8}(c_1\zeta + \eta)S_2\beta + \frac{K}{4 \cdot 4!}(c_1\zeta + \eta)S_4\alpha_1 \right) \right. \\
& \quad \left. -\beta\delta_2 - \frac{K}{2}(\zeta - c_1\eta) \left( -\frac{1}{6!}C_6 - \frac{1}{2}S_2\beta - \frac{1}{4!}S_4\alpha_1 \right) \right. \\
& \quad \left. -\frac{K}{2}(c_1\zeta + \eta) \left( \frac{1}{2}C_2\beta + \frac{1}{4!}C_4\alpha_1 - \frac{1}{6!}S_6 \right) \right] + \dots .
\end{aligned} \tag{69}$$

Equation (68)–(69) is in the form (1) with the coefficients  $a_1, a_2, b_1, b_2, b_3, b_4, b_5$  and  $g_1$  being combinations of the independent parameters  $c_1, c_2, K, \delta_1, \delta_2$  and  $\theta$  (the latter is present via  $\zeta$  and  $\eta$ , see (20)).

For the extended Ginzburg-Landau equation (29), which is our primary interest, the phase equation is straightforwardly obtained by replacing

$$\begin{aligned}
K(\zeta - c_1\eta)S_n &\rightarrow K_1(\zeta_1 - c_{11}\eta_1)S_{n1} + K_2(\zeta_2 - c_{12}\eta_2)S_{n2}, \\
K(\eta + c_1\zeta)S_n &\rightarrow K_1(\eta_1 + c_{11}\zeta_1)S_{n1} + K_2(\eta_2 + c_{12}\zeta_2)S_{n2}, \\
K(\zeta - c_1\eta)C_n &\rightarrow K_1(\zeta_1 - c_{11}\eta_1)C_{n1} + K_2(\zeta_2 - c_{12}\eta_2)C_{n2}, \\
K(\eta + c_1\zeta)C_n &\rightarrow K_1(\eta_1 + c_{11}\zeta_1)C_{n1} + K_2(\eta_2 + c_{12}\zeta_2)C_{n2} \\
&\quad (n = 2, 4, 6)
\end{aligned}$$

in (68)–(69) and also in the expressions for  $\alpha_1$  and  $\alpha_2$ , (53), and for  $\beta, \beta_1, \beta_2, \beta_3$  and  $\beta_4$ , (58)–(59).

## 4 Computational results

We solved the system

$$a_1 = a_2 = b_1 = b_2 = b_3 = 0, \quad b_4 = -\varepsilon \tag{70}$$

numerically using a program written REDUCE. In the program, we assigned some value to  $\varepsilon$  and also to 3 of the independent parameters,  $\theta_1$ ,  $\theta_2$  and  $c_{12}$ . Those values are not unique but taken arbitrarily from a range of acceptable values. The following system of 14 equations was solved: the 6 main equations (70), 2 equations (53) representing  $\alpha_1$  and  $\alpha_2$ , 5 equations (58)–(59) representing  $\beta$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$  and  $\beta_4$  and 1 equation representing  $g_1$ . As solution the values of the following 14 parameters were found:  $\alpha_1$ ,  $\alpha_2$ ,  $\beta$ ,  $\beta_1$ ,  $\beta_2$ ,  $\beta_3$ ,  $\beta_4$ ,  $c_{11}$ ,  $c_2$ ,  $K_1$ ,  $K_2$ ,  $\delta_1$ ,  $\delta_2$  and  $g_1$ . Here we report the results of just two computations which demonstrate that all the conditions and restrictions are satisfied. We chose

$$\theta_1 = 5, \quad \theta_2 = 10, \quad c_{12} = 3.$$

For the first computation we took  $\varepsilon = 0$  implying

$$b_4 = 0.$$

The result, up to the fifth decimal digit without rounding, is

$$\delta_1 = 4.06892, \quad K_1 = 17.14055, \quad K_2 = -28.03970,$$

$$\delta_2 = -1.75789, \quad c_2 = 1.68100, \quad c_{11} = 0.69815.$$

It is important to know the sign of the coefficient at the sixth derivative,

$$g_1 = 1.01076.$$

We are satisfied to observe that all three necessary restrictions on the parameters are met, namely

$$g_1 > 0, \quad \delta_1 > 0, \quad K_1 + K_2 < 1. \quad (71)$$

For the second computation we took slightly negative  $b_4 = -\varepsilon$ . For computational efficiency  $\varepsilon$  was created as a product of one of the unknowns,  $K_1$ , and a small number,

$$b_4 = -K_1 \cdot 0.00001. \quad (72)$$

The result is

$$\delta_1 = 4.06912, \quad K_1 = 17.14113, \quad K_2 = -28.04100,$$

$$\delta_2 = -1.75795, \quad c_2 = 1.68106, \quad c_{11} = 0.69819$$

and

$$g_1 = 1.01078.$$

Formula (72) means that

$$b_4 = -\varepsilon = -0.00017141.$$

Clearly we can make  $b_4$  as close to zero as we like. The result of this computation only slightly differs from that for  $b_4 = 0$  so that the conditions (71) remain satisfied.

Thus, we satisfied the conditions (70) and restrictions (71). The conditions result, as we showed in Section 1, in the phase scale  $\Psi \sim \varepsilon$  and length scale  $L \sim (1/\varepsilon)^{3/2}$ . The higher-order terms in  $\psi$  and  $\nabla$  in the phase equation (1) can be safely ignored.

Previously we used equation (6) as a model for solid flames [5]. The spinning auto-waves obtained in [5] demonstrate robustness of the dynamical balance behind (6). For a two-dimensional version of (6) and rectangular shape of domain we also obtained complex, seemingly chaotic regimes [10].

## 5 Scaling

The table shows the order of magnitude of the coefficients of the phase equation (1) leading to different truncations and scalings. Where important, signs of the coefficients are shown.

$a_1$	$a_2$	$b_1$	$b_2$	$b_3$	$b_4$	$b_5$	$g_1$	truncation
+1	1	1	1	1	1	1	1	<i>diffusion eq.</i>
<i>excitation</i> - $\varepsilon$	1	-1	1	1	1	1	1	<i>Kuramoto-Sivashinsky eq.</i>
+ $\varepsilon^2$	1	<i>excitation</i> + $\varepsilon$	1	1	1	1	+1	<i>Nikolaevskii eq.</i>
$o(\varepsilon^6)$	$o(\varepsilon^5)$	$o(\varepsilon^3)$	$o(\varepsilon^2)$	$o(\varepsilon^2)$	<i>excitation</i> - $\varepsilon$	1	+1	<i>eq. (6)</i>

Table. Hierarchy of truncations of the phase equation (1).

For the Kuramoto-Sivashinsky equation

$$\partial_t \psi = -\varepsilon \nabla^2 \psi + (\nabla \psi)^2 - \nabla^4 \psi$$

the balance  $\varepsilon\Psi/L^2 \sim \Psi^2/L^2 \sim \Psi/L^4$  gives

$$\Psi \sim \varepsilon, \quad L \sim 1/\sqrt{\varepsilon}. \quad (73)$$

The time scale is determined from  $\Psi/T \sim \varepsilon\Psi/L^2$ ,

$$T \sim L^2/\varepsilon \sim 1/\varepsilon^2. \quad (74)$$

Hence, from (73) and (74) we have the scaling relations

$$\psi = \varepsilon\psi_1(r_1, t_1), \quad r_1 = \sqrt{\varepsilon}r, \quad t_1 = \varepsilon^2t.$$

Different scaling takes place for the Nikolaevskii equation

$$\partial_t\psi = \varepsilon^2\nabla^2\psi + \varepsilon\nabla^4\psi + \nabla^6\psi + (\nabla\psi)^2.$$

The balance  $\varepsilon^2\Psi/L^2 \sim \varepsilon\Psi/L^4 \sim \Psi/L^6 \sim \Psi^2/L^2$  gives

$$\Psi \sim \varepsilon^2, \quad L \sim 1/\sqrt{\varepsilon}$$

and, using  $\Psi/T \sim \varepsilon^2\Psi/L^2 \sim \varepsilon^5$ ,

$$T \sim \Psi/\varepsilon^5 \sim 1/\varepsilon^3.$$

Thus, the scaling relations are

$$\psi = \varepsilon^2\psi_1(r_1, t_1), \quad r_1 = \sqrt{\varepsilon}r, \quad t_1 = \varepsilon^3t.$$

For the nonlinearly excited equation (6),

$$\partial_t\psi = -\varepsilon\nabla^2\psi(\nabla\psi)^2 + (\nabla\psi)^4 + \nabla^6\psi,$$

as we determined in (3),

$$\Psi \sim \varepsilon, \quad L \sim (1/\varepsilon)^{3/2}.$$

Therefore, from  $\Psi/T \sim \Psi^4/L^4 \sim \varepsilon^{10}$ ,

$$T \sim 1/\varepsilon^9.$$

Thus, the scaling is

$$\psi = \varepsilon\psi_1(r_1, t_1), \quad r_1 = \varepsilon^{3/2}r, \quad t_1 = \varepsilon^9t.$$

## 6 Conclusions

We derived a nonlinearly excited truncated phase equation (6) governing the dynamics of reaction-diffusion systems with nonlocal coupling. The systems are described by the nonlocal complex Ginzburg-Landau equation with nine independent parameters. The form (6) is valid under the conditions (4) which we satisfied by selecting the values of the independent parameters.

## References

- [1] Y. Kuramoto, T. Tsuzuki, Persistent propagation of concentration waves in dissipative media far from thermal equilibrium, *Progr. Theor. Phys.* 55 (1976) 356–369.
- [2] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Berlin: Springer-Verlag, 1984.
- [3] G.I. Sivashinsky, Nonlinear analysis of hydrodynamical instability in laminar flames, *Acta Astronaut.* 4 (1977) 1177–1206.
- [4] V.N. Nikolaevskii, in *Recent Advances in Engineering Science*, edited by S.L. Koh, C.G. Speciale, *Lecture Notes in Engineering*, 39 Berlin: Springer, 1989, 210.
- [5] D.V. Strunin, Autosoliton model of the spinning fronts of reaction, *IMA J. Appl. Math.* 63 (1999) 163–177.
- [6] D.V. Strunin, Nonlinear instability in generalized nonlinear phase diffusion equation, *Progr. Theor. Phys. Suppl.* N 150 (2003) 444–448.
- [7] D. Tanaka, Y. Kuramoto, Complex Ginzburg-Landau equation with nonlocal coupling, *Phys. Rev. E* 68 (2003) 026219.
- [8] D. Tanaka, Chemical turbulence equivalent to Nikolaevskii turbulence, *Phys. Rev. E* 70 (2004) 015202(R).
- [9] D. Tanaka, Turing instability leads oscillatory systems to spatiotemporal chaos, *Progr. Theor. Phys. Suppl.* N 161 (2006) 119–126.

- [10] D.V. Strunin, S.A. Suslov, Phenomenological approach to 3D spinning combustion waves: numerical experiments with a rectangular rod, *Int. J. Self-propagating High-temperature Synthesis* 14 (2005) 33–39.