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# WEIGHTED REDUCED MAJOR AXIS METHOD FOR REGRESSION MODEL

**Anwar Saqr** and **Shahjahan Khan**
Deptt. of Mathematics and Computing Australian, Centre for Sustainable Catchments,
University of Southern Queensland Toowoomba, Queensland, Australia
Email: saqr.anwer@yahoo.com and khans@usq.edu.au

## ABSTRACT

The reduced major axis (RMA) method is widely used in many disciplines as a solution to errors in variables regression model, although it lacks efficiency. This paper provides an alternative view on the RMA estimator. Moreover, it introduces a new estimator to fit regression line when both variables are subject to measurement errors. The proposed weighted RMA (WR) estimator is derived based on the mathematical relationship between the vertical and orthogonal distances of the observed points and the regression line. Compared to the RMA and OLS-bisector estimators the proposed WR estimator is less sensitive to the variation of the ratio of error variances ($\lambda$). The simulation results show that the WR estimator is more consistent and efficient than the RMA and OLS-bisector estimators.

**Key Words:** Linear regression models; Measurement error models; Reflection of points; Ratio of error variances; OLS-bisector.
**2010 Mathematics Subject Classification:** Primary 62J05, Secondary 62F10.

## 1 Introduction

The reduced major axis (RMA) method is widely used in many disciplines, and it has received much attention from the experts and some have suggested that it is more useful than other methods to deal with the measurement error model (cf Sprent and Dolby, 1980; Smith, 2009; Ludbrook, 2010).

The RMA estimator was suggested as a solution to the likelihood equations in the case of the normal functional model when there is no additional information (cf Cheng and Van Ness, 1999, p. 43). This estimator is constructed based on the geometric mean of the ordinary least squares (OLS) estimator for the regression of $y$ on $x$ and the reciprocal of that of $x$ on $y$. Halfon (1985) and Draper and Yang (1997) pointed out that the RMA estimator minimizes the vertical and horizontal distances between the observed points and the regression line. Isobe et al. (1990) examined five different estimators, and pointed out that the OLS bisector (OLS-b) estimator is the best method to use, when there is no basis to distinguish between the explanatory and response variables.

Let $x$ be the observable or *manifest* explanatory variable, and let $\xi$ be the true or *latent* explanatory variable. Similarly let $\eta$ be the true value of the response variable and $y$ be the manifest response variable.

[1]On leave from Department of Statistics, Faculty of Sciences, AlJabal AlGarby University, Libya.

If the *latent* variables $\xi_j$ and $\eta_j$ are measured without error, then the simple linear regression model without ME is expressed as

$$\eta_j = \beta_0 + \beta_1 \xi_j + \epsilon_j, \quad j = 1, 2, \ldots, n, \tag{1.1}$$

where $\epsilon$ is the equation error. If there is ME in both explanatory and response variables, we can define the manifest variables as

$$x_j = \xi_j + u_j, \text{ and } y_j = \eta_j + \tau_j \qquad j = 1, 2, \ldots, n, \tag{1.2}$$

where $\eta_j$ is the $j$th realisation of the *latent* response variable, $\xi_j$ is the $j$th value of the *latent* explanatory variable, $\tau_j$ is the ME in the response variable and $u_j$ is the ME in the explanatory variable. It is assumed that,

$$\tau_j \sim N(0, \sigma_\tau^2), \; u_j \sim N(0, \sigma_u^2), \; cov(u, \tau) = 0, \text{ and } cov(u, \epsilon) = 0. \tag{1.3}$$

Note the ME in the response variable $\tau_j$ can be absorbed in the equation error $\epsilon$ which may be expressed as $e_j = \tau_j + \epsilon_j$. The simple regression model with ME in both variables and equation error $\epsilon$ is expressed as

$$y_j = \beta_0 + \beta_1 x_j + v_j, \quad j = 1, 2, \ldots, n, \tag{1.4}$$

where $v_j = e_j - \beta_1 u_j$, and $cov(u, e) = 0$, then

$$\sigma_v^2 = \sigma_e^2 + \beta_1^2 \sigma_u^2. \tag{1.5}$$

Note that the OLS method is not valid here, because the variables $x_j$ and $v_j$ in equation (1.4) are not independent.

There is a common recommendation to use the RMA estimator for the ME model, but without enough justifications (Smith, 2009). Jolicoeur (1975) stated that it is difficult to interpret the meaning of the slope of the RMA regression. However, the common believe is that the RMA estimator minimizes the vertical and horizontal distances between the observed points and the fitted line (cf Halfon, 1985; Draper and Yang, 1997). But it is not quite true, because it could be demonstrated that the RMA estimator minimizes the orthogonal error of the observed points with the unfitted regression line instead of the fitted regression line.

Sections 2 and 3 provide the RMA estimator and alternative way of deriving this estimator. The proposed estimator *weighted reduced major axis estimator* (WR) is introduced in Section 3. The simulation studies are conducted to compare the performances of the proposed estimator with the RMA, OLS-b, and OLS estimators in Section 4. Some concluding remarks are included in Sections 5.

## 2 Reduced major axis estimator

The RMA estimator of the slope parameter is the geometric mean of the slope of $y$ on $x$ regression line, and the reciprocal of the slope of $x$ on $y$ regression line, where $x$ and $y$ both are random (see Leng et al. 2007). It is given by

$$\hat{\beta}_{1R} = sgn(SP_{xy}) \sqrt{SS_y SS_x^{-1}} = sgn(Sp_{xy}) \, S_y S_x^{-1} \, ,$$

where $SS_x = \sum_{j=1}^{n}(x_j - \bar{x})^2$, $SS_y = \sum_{j=1}^{n}(y_j - \bar{y})^2$, $SP_{xy} = \sum_{j=1}^{n}(x_j - \bar{x})(y_j - \bar{y})$, and $S_y$ and $S_x$ are the standard deviations of $y$ and $x$ respectively.

In the literature, the RMA regression is also known as the standardized major axis (see Warton et al. 2006), and geometric mean estimator, or the line of organic correlation (cf Tessier, 1948, Kermack and Haldane, 1950, Ricker, 1973). In physics it is known as a type of standard weighting model (see Machonald and Thompson, 1992), while the astronomers call it as Strömberg's impartial line (see Feigelson and Babu, 1992).

A host of recent publications indicate that using the RMA is necessary and sufficient to fit the straight line when both the response and explanatory variables are subject to errors (see for example Levinton and Allen, 2005, Zimmerman et al, 2005, Sladek et al, 2006, and Vincent and Lailvaux, 2006). Jolicoeur (1975) and Spernt and Dolby (1980) pointed out that the RMA estimator is unbiased if and only if

$$\lambda = \beta_1^2 ,$$

where $\lambda = (\sigma_\tau^2 \sigma_u^{-2})$ is the ratio of error variances. But several other studies indicate that this assumption is unrealistic (cf Sprent and Dolby, 1980). Another competing estimator preferred by many authors is OLS-bisector (OLS-b) estimator (see e.g Isobe et al. 1990) which is given by

$$\hat{\beta}_{1OLS-b} = (\hat{\beta}_1 + \hat{\beta}_2)^{-1}\left[\hat{\beta}_1\hat{\beta}_2 - 1 + \sqrt{(1+\hat{\beta}_1^2)(1+\hat{\beta}_2^2)}\right],$$

where $\hat{\beta}_1 = S_{yx}S_x^{-2}$, and $\hat{\beta}_2 = S_y^2 S_{xy}^{-1}$.

## 3 Theoretical analysis

In the regression analysis with ME in both variables it is crucial to note the difference between the distance of the observed point from the fitted line, unfitted line, and unobserved point. Although, many authors use distance between the observed point and regression line without being specific about the fitted or unfitted lines. This issue is crucial when there is ME in both variables. The mathematical relationship between the vertical and orthogonal distances of the observed points and the fitted regression line is explained below. The fitted line of the true model (without equation error and ME) is given by

$$\eta_j = \beta_0 + \beta_1 \xi_j, \quad j = 1, 2, \ldots, n. \tag{3.1}$$

Let $(x_j, y_j)$ be the observed point and $(z_j, c_j)$ be its reflection about the fitted line (3.1), then

$$z_j = x_j \cos 2\psi + (y_j - \beta_0)\sin 2\psi, \quad \text{and } c_j = x_j \sin 2\psi - (y_j - \beta_0)\cos 2\psi + \beta_0, \tag{3.2}$$

where $\psi = tan^{-1}\beta_1$, and $\beta_0$, and $\beta_1$ are the regression parameters. For details on reflection of points please see Vaisman (1997, p. 164-169). Let the relationships between the orthogonal and vertical distance of the observed point $(x_j, y_j)$ be explained in the context of (a) fitted line, $(\eta_j = \beta_0 + \beta_1\xi_j)$, based on the *latent* variables and (b) unfitted line $(y_j = \beta_{0x} + \beta_{1x}x_j)$, based on the *manifest* variables.

There are potentially two orthogonal distances of any observed point, one from the *fitted line* (here represented by $\Upsilon$) and the other from the *unfitted line* ($Ox$). In principle, the RMA method should minimise $\Upsilon$, but in practice it minimises $Ox$. Figure 1 shows the reflection of $A = (x_j, y_j)$ about the *fitted line* $C = (z_j, c_j)$ with the orthogonal distance $\Upsilon = \overline{AB}$, and the reflection of $A = (x_j, y_j)$ about the *unfitted line* $F = (x_j^*, y_j^*)$ with the orthogonal distance $Ox = \overline{AD}$.

**(a) Fitted line.**

It is well known from the properties of the reflection process that the reflection line (which is the fitted line) is a bisector and orthogonal on the distance between the observed point $(x_j, y_j)$ and its reflection point $(z_j, c_j)$. Then the half of the square distance between the observed point $(x_j, y_j)$ and its reflection point $(z_j, c_j)$ will equal the orthogonal square distance $(\Upsilon_j^2)$ between the observed point $(x_j, y_j)$ and the fitted line. It is given by

$$4\Upsilon_j^2 \quad = \quad (z_j - x_j)^2 + (c_j - y_j)^2. \tag{3.3}$$

Then from (3.1), (3.2), and (3.3) the square orthogonal distance $(\Upsilon_j^2)$ is given by

$$\Upsilon_j^2 \quad = \quad \frac{1}{4}\left[(2x_j sin^2\psi + y_j sin2\psi - \beta_0 sin2\psi)^2 + (x_j sin2\psi - 2y_j cos^2\psi + 2\beta_0 cos^2\psi)^2\right]$$

Since $x_j = \xi_j + u_j$ , $y_j = \eta_j + e_j$ and $\beta_1 = \dfrac{sin\psi}{cos\psi}$ so

$$\begin{aligned}\Upsilon_j^2 \quad &= \quad \frac{1}{4}[(-2x_j sin^2\psi + \beta_1\xi_j sin2\psi + e_j sin2\psi)^2 + (x_j sin2\psi - 2\beta_1\xi_j cos^2\psi - 2e_j cos^2\psi)^2] \\ &= \quad u_j^2 sin^2\psi - u_j e_j sin2\psi + e_j^2 cos^2\psi.\end{aligned}$$

Then $\quad E(\Upsilon_j^2) = E(u_j^2)sin^2\psi + E(e_j^2)cos^2\psi.$

The variance of $\Upsilon_j$ is given by $\sigma_\Upsilon^2 = \sigma_u^2 sin^2\psi + \sigma_e^2 cos^2\psi$, where $E(z_j - x_j) = E(c_j - y) = 0$, and $\beta_1^2 = \sin^2\psi\, cos^{-2}\psi$, the variance of $\Upsilon_j$ becomes

$$\sigma_\Upsilon^2 \quad = \quad \left(\sigma_e^2 + \sigma_u^2 \frac{sin^2\psi}{cos^2\psi}\right) cos^2\psi = (\sigma_e^2 + \beta_1^2\sigma_u^2)cos^2\psi.$$

Then the relationship between the variance of the orthogonal distance and the variance of the vertical distance is given by formula

$$\sigma_\Upsilon^2 = \sigma_v^2 cos^2\psi = \sigma_v^2(1 + \beta_1^2)^{-1}. \tag{3.4}$$

Note that both the vertical and orthogonal distances measure the distance between the observed point $(x_j, y_j)$ and the fitted line, but it does not measure the distance between the observed point $(x_j, y_j)$ and the unobserved point $(\xi_j, \eta_j)$. Under certain assumptions such as $\lambda = 1$ or $\beta_1 = 1$ the distance between the observed point and the unobserved point is equal to the double of the orthogonal distance, where the distance between the observed point and the unobserved point $(\Delta)$ is given by

$$\Delta^2 = (x_j - \xi_j)^2 + (y_j - \eta_j)^2 = (u_j^2 + _j^2),$$

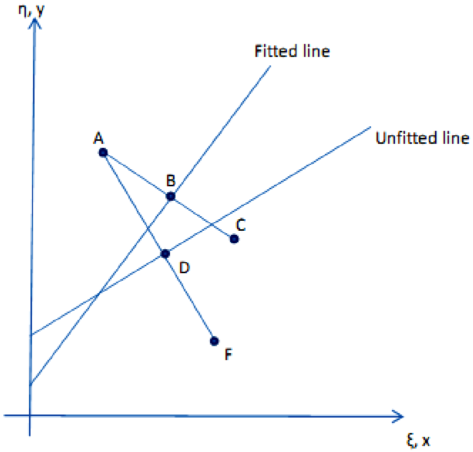


Figure 1: Graph of two orthogonal distances ($\overline{AB} = \Upsilon$, and $\overline{AD} = Ox$) between the observed point and the fitted and unfitted lines.

where $u_j$, and $e_j$ are the ME in the explanatory and response variables respectively. From (1.3) the variance of the distance ($\Delta$) is $\sigma^2_\Delta = \sigma^2_e + \sigma^2_u$. From (1.5) and if $\lambda = 1$ then $\sigma^2_\Delta = 2\sigma^2_e$.

**(b) Unfitted line.**

In order to find the relationship between the observed point $(x_j, y_j)$ and the unfitted line we follow the similar procedure as in case (a) except replacing the parameters of the fitted line, $\psi = tan^{-1}\beta_1$, $\beta_0$, and $\beta_1$ by the coefficients of the unfitted line (the ME model) $\hat{\theta} = tan^{-1}\hat{\beta}_{1x}$, $\hat{\beta}_{0x}$, and $\hat{\beta}_{1x}$ respectively. Then we get

$$x^*_j \;=\; x_j cos2\hat{\theta} - (y_j - \hat{\beta}_{0x}) sin2\hat{\theta}, \text{ and } y^*_j = x_j sin2\hat{\theta} - (y_j - \hat{\beta}_{0x}) cos2\hat{\theta} + \hat{\beta}_{0x},$$

where $(x^*_j, y^*_j)$ is the reflection point of the observed point $(x_j, y_j)$ about the unfitted line.

The relationship between the sample variance of the orthogonal distance $(Ox)$ and vertical distance $(v)$ is given by

$$S^2_{Ox} \;=\; S^2_v cos^2\hat{\theta} = S^2_v (1 + \hat{\beta}^2_{1x})^{-1}. \tag{3.5}$$

Also the relationship between the observed point and unfitted population line becomes

$$\sigma^2_{Ox} \;=\; \sigma^2_v cos^2\theta = \sigma^2_v (1 + \beta^2_{1x})^{-1}. \tag{3.6}$$

From (3.4) and (3.6) the relationship between the variances of the orthogonal distance in cases (a) and (b) is given by

$$\sigma^2_\Upsilon \;=\; \sigma^2_{Ox}\, cos^2\psi cos^{-2}\theta = \sigma^2_{Ox}(1 + \beta^2_{1x})(1 + \beta^2_1)^{-1}. \tag{3.7}$$

Note that in general, $\sigma_\Upsilon^2 < \sigma_{Ox}^2$, and they are equal if and only if there is no measurement error. Therefore, any method that minimizes $\sigma_{Ox}^2$, will not work well, and that is what is happening with the RMA method. The next section shows that the RMA method is minimizing $\sigma_{Ox}^2$, rather than $\sigma_\Upsilon^2$.

From (3.5) the sum of squares of orthogonal distance ($SS_{Ox}$) between the observed point $(x_j, y_j)$ and the unfitted line ($\hat{y}_j = \hat{\beta}_{0x} + \hat{\beta}_{1x} x_j$), it can derived the RMA estimator can be derived by different procedures in order to understand its working mechanism as follows:

$$\begin{aligned} SS_{Ox} &= SS_v cos^2\hat{\theta} = \sum_{j=1}^{n}(y_j - \hat{\beta}_{0x} - \hat{\beta}_{1x}x_j)^2 cos^2\hat{\theta} \\ &= \sum_{j=1}^{n}((y_j - \bar{y}) - \hat{\beta}_{1x}(x_j - \bar{x}))^2 cos^2\hat{\theta} = \sum_{j=1}^{n}((y_j - \bar{y})cos\hat{\theta} - (x_j - \bar{x})sin\hat{\theta})^2 \quad (3.8) \end{aligned}$$

Let $Q_1 = sin\hat{\theta}$, and $Q_2 = cos\hat{\theta}$. Then

$$SS_{Ox} = \sum_{j=1}^{n}((y_j - \bar{y})Q_2 - (x_j - \bar{x})Q_1)^2.$$

Differentiating $SS_{Ox}$ w.r.t. $Q_1$ and $Q_2$, and setting the derivatives to zero, we get

$$\begin{aligned} \frac{\partial SS_{Ox}}{\partial Q_1} &= 2\sum_{j=1}^{n}((y_j - \bar{y})Q_2 - (x_j - \bar{x})Q_1)(-(x_j - \bar{x})) = 0, \\ \frac{\partial SS_{Ox}}{\partial Q_2} &= 2\sum_{j=1}^{n}((y_j - \bar{y})Q_2 - (x_j - \bar{x})Q_1)(y_j - \bar{y}) = 0, \end{aligned}$$

or equivalently

$$Q_1 S_x^2 = Q_2 S_{yx}, \quad (3.9)$$

$$Q_2 S_y^2 = Q_1 S_{yx}. \quad (3.10)$$

From (3.9), (3.10) and $\hat{\beta}_{1x} = Q_1 Q_2^{-1}$ we get two estimators of the slope

$$\hat{\beta}_{11} = S_{yx} S_x^{-2} \text{ and } \hat{\beta}_{12} = S_y^2 S_{yx}^{-1}. \quad (3.11)$$

Then the RMA is the geometric mean of the estimators in (3.11), that is,

$$\hat{\beta}_{1RMA} = sgn\{S_{yx}\} \sqrt{S_y^2 S_x^{-2}}.$$

It is clear that the RMA estimator is derived by minimizing the orthogonal distance between the observed point $(x_j, y_j)$ and unfitted line. Hence it does not minimizes the vertical and horizontal distances between observed points and the fitted line.

**Remark:** The RMA estimator is based on the minimization of the orthogonal distance between the observed points and the regression line. Since there are two regression lines, namely the fitted (latent) and unfitted (manifest) lines, it is essential to clarify the orthogonal distance between an observed point and either the fitted line or the unfitted line. Obviously the orthogonal distance of a point from the fitted line is most likely to be different from the unfitted line.

# 4 The weighted RMA estimator

In this section we introduce the weighted RMA (WR) estimator. The proposed estimator minimises the orthogonal distance between the observed point $(x_j, y_j)$ and the unfitted regression line. This estimator is based on the relationship (3.5) between the vertical and orthogonal distances of the observed points and the unfitted regression line. The proposed estimator is derived as follow
Multiply equation (3.9) by $S_y^2$, and equation (3.10) by $S_{yx}$, we get

$$Q_1 S_x^2 S_y^2 = Q_2 S_{yx} S_y^2 \tag{4.12}$$

$$Q_1 S_{yx}^2 = Q_2 S_{yx} S_y^2, \tag{4.13}$$

from equations (4.12) and (4.13) we get

$$Q_1(S_x^2 S_y^2 + S_{yx}^2) = Q_2 2 S_{yx} S_y^2 \tag{4.14}$$

$$(S_x^2 S_y^2 + S_{yx}^2) sin\hat{\theta} = 2 S_{yx} S_y^2 cos\hat{\theta}. \tag{4.15}$$

From (4.14) and (4.15) we define an estimator

$$\hat{\beta}_{1O} = \frac{sin\hat{\theta}}{cos\hat{\theta}} = \frac{2 S_{yx} S_y^2}{S_y^2 S_x^2 + S^2 yx}, \tag{4.16}$$

which is simplified as follows

$$\hat{\beta}_{1O} = \frac{2 S_y^2 S_x^{-2}}{S_y^2 S_{yx}^{-1} + S_{yx} S_x^{-2}} = \frac{2 \hat{\beta}_{1R}^2}{(\hat{\beta}_1 + \hat{\beta}_2)} = \omega \ \hat{\beta}_{1R}, \tag{4.17}$$

where $\omega = \hat{\beta}_{1R} \hat{\beta}_{OLS-m}^{-1}$, and $\hat{\beta}_{OLS-m}$ is the mean of the slope of the $OLS(y|x)$ and that of $OLS(x|y)$, that is, $\hat{\beta}_{OLS-m} = (\hat{\beta}_{1xy} + \hat{\beta}_{1yx})/2$. Note that the estimators $\hat{\beta}_{1O}$, $\hat{\beta}_{1RMA}$, and $\hat{\beta}_{1OLS-mean}$ are equal if $\omega = 1$, and $\hat{\beta}_{1O}$ is the estimator obtained to minimising the $Ox$.

It is well known that when $\sigma_{yx} > 0$ the slope of the unfitted line $\beta_{1x}$ is less than the true slope $\beta_1$ of the fitted model. Then the true slope $\beta_1$ is located between $\hat{\beta}_{1O}$ and $\hat{\beta_1}_x$ estimators. Hence the proposed WR estimator is defined as the mean of $\hat{\beta}_{1S}$ and $\beta_{1x}$, and is given by

$$\hat{\beta}_{1WR} = (\hat{\beta}_{1O} + \beta_{1x})/2.$$

Note that the WR estimator and the other estimators mentioned in this paper are used when there is no prior information on the error variances. In order to demonstrate that the performance of WR estimator is better than the RMA, OLS, and OLS-b estimators when $\lambda$ is known or misspecified, we provide the results of extensive simulation studies in the next section.

# 5 Simulation studies

We perform extensive simulations to illustrate that the proposed WR estimator is relatively unbiased and consistent compared to the RMA, OLS, and OLS-b estimators. It is more so when $\lambda$ is large. In this section we compare the WR estimator to the RMA estimator, OLS and OLS-b estimator for a wide range of values of $\lambda$ $(0.1 \leq \lambda \leq 19)$. These studies demonstrate that the WR estimator is not sensitive to the ratio of error variances $\lambda$, whereas the RMA estimator grows larger as the value of $\lambda$ increases. Also the WR estimator preforms consistently better than the OLS-b, and OLS estimators. The results based on 10,000 replications of samples size $n = 100$, $\beta_0 = 0$, and $\beta_1 = 0.6$ of normal structural model, where $x \sim N(0, 100)$, are presented in the following graphs.

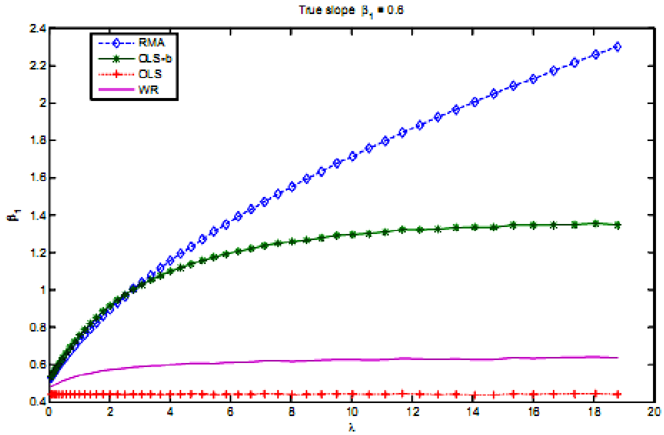


Figure 2: Plot of the mean slope of four different estimators against $0.08 \leq \lambda \leq 19$. when $\beta_0 = 0, \beta_1 = 0.6$.

It is clear from Figure 2 that the values of the OLS-bisector estimator are away from the true values of $\beta_1$. The values of the RMA estimator are far above the true values of $\beta_1$. As $\lambda$ increases, the RMA estimator appears to grow large. Clearly the proposed WR estimator is much closer to the true values of $\beta_1$ than the other three estimators.

Figure 3 gives useful indications about the statistical properties of the estimators. The measurement error makes the spread of the RMA estimator the highest. While the spread of the OLS-b estimator appears to be better than that of the RMA estimator, though they are not small. The OLS estimator is consistently an under estimate of $\beta_1$. Moreover, it is inappropriate for ME models, and hence we do not compare it with the WR estimator. Sarach and Celik (2011) discussed eight different regression techniques, and pointed out that the OLS-bisector estimator is nearer to the real value than all other estimators, and the mean squares error of OLS-bisector is smaller than other estimators. The current study reveals that the WR estimator is consistently better than the OLS-b estimator in term of the closeness of $\hat{\beta}_{1WG}$ to $\beta_1$.

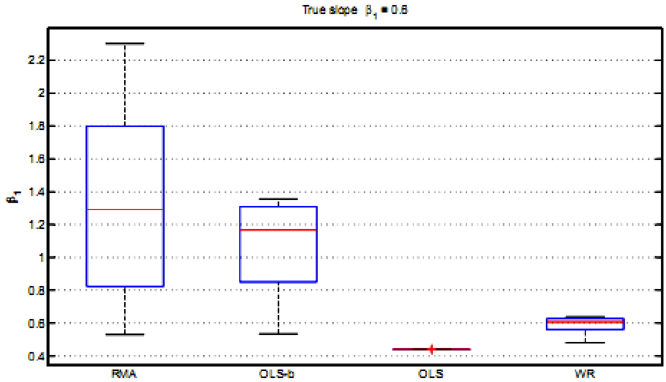


Figure 3: Graph of the distribution of the mean slope of four different estimators when $\beta_0 = 0, \beta_1 = 0.6$, and $0.08 \leq \lambda \leq 19$.

## 6 Concluding Remarks

This paper proposes a new estimator based on the mathematical relationship between the vertical and orthogonal distances of the observed points with fitted and unfitted lines. This estimator is appropriate for fitting straight lines when both variables are subject to measurement errors, especially when there is no basis for distinguishing between response and explanatory variables. Moreover, the WR estimator is appropriate to the normal structural model even when $\lambda$ is misspecified. The graphs in Figures 2 and 3 provide clear evidence that the WR is much closer to the true slope than the other competing estimators. The values of the proposed estimator are nearer to the real value than the RMA, OLS-b, and OLS estimators. Therefore, the proposed estimator possesses better statistical proprieties than the other estimators. Moreover, the new method is stable and works well for different values of $\lambda$.

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