19th Australasian Fluid Mechanics Conference
Melbourne, Australia
8-11 December 2014

Transformation of Internal Waves at the Bottom Ledge

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Abstract

Transformation of internal gravity waves on the oceanic shelf is studied theoretically and numerically within the framework of the linear approximation. It is assumed that internal waves pass over the continental shelf experiencing partial transmission and reflection. The problem is studied for the simplified model of the shelf represented by the sharp bottom ledge. The fluid stratification is assumed to be a two-layer with the density of the upper layer being ρ_0, and the density of the lower layer being ρ_1. The theoretical approximate formulae are proposed for the transmission and reflection coefficients as the functions of an incoming wave number, density ratio, a depth of the interface between the layers, and depth ratio before and after the ledge edge. Results of direct numerical modelling of linear internal waves transformation are presented as functions of all aforementioned parameters. The modelling was undertaken by means of the numerical code MITgcm. The results obtained are analysed in details and compared against the proposed formulae.

Introduction

Internal waves, as well as surface waves, play an important role in the near-shore processes, including mixing, turbulence generation, dissipation of wave energy, transport of sediments, etc. They can affect on the engineering offshore constructions (e.g., gas and oil pipelines, platforms) and cause negative effects on the navigation of ships (due to the "dead water" effect, for instance) and submarines. Intense internal waves are usually generated by the barotropic tide when it interacts with the continental shelf. There are also some other mechanisms which generate moderate and small amplitude internal wavetrains in the open ocean. Internal waves propagating onshore experience an interaction with the non-uniform bottom relief which causes wave transformation, breaking, dissipation and leads to mixing processes. There are many papers devoted to transformation of internal waves in the coastal zones (see, e.g., [5, 16, 8, 9] and references therein).

One of the processes occurring in the coastal zone is wave transformation on the underwater obstacles, in particular, on bottom ledges. Transformation of surface waves on the step-wise bottom obstacles have been studied in many papers both in the linear approximation and in the nonlinear cases (see, e.g., [13, 4] and references therein). Both the approximate and rigorous approaches have been developed and tested against the results of numerical modelling and laboratory experiments. In the meantime, the transformation of internal waves were studied much less. The coefficients of transformation (the transmission and reflection coefficients) were obtained only for infinitely long waves in the linear approximation for two-layer model of fluid [5, 16, 8, 9]. Using these coefficients the transformation of positive polarity internal solitons of small and moderate amplitudes on a bottom ledge has been studied in those papers; then the subsequent disintegration of a transmitted wave onto secondary solitons has been calculated both theoretically and numerically within the framework of the Korteweg–de Vries and Gardner equations. The transformation of large amplitude solitary waves of negative polarity on the bottom ledge has been studied numerically within the fully nonlinear set of hydrodynamic equations [8, 9].

The problem of wavetrain transformation for waves of arbitrary length remained unresolved thus far even in the linear approximation. In this paper we present results of direct numerical modelling of transformation of small-amplitude internal wavetrains on the underwater step-wise barrier in two-layer fluid. We show that the transformation coefficients can be approximated by relatively simple formulae.

Numerical Modelling of Internal Wave Transformation

For the numerical modelling of internal wave transformation we utilized the numerical code MITgcm [10, 1], which is based on the solution of Navier–Stocks equation. We assume that the water is incompressible and inviscid fluid, and the motion is non-vortical. The latter assumption allows us to introduce the velocity potential in each layer $\mathbf{v}_{0,1} = \nabla\varphi_{0,1}$, where index 0 pertains to the upper layer, and index 1 – to the lower. The sketch of the computational domain is shown in figure 1. The impermeable and free-slip boundary conditions were prescribed at all solid boundaries including the left and right bounding walls.

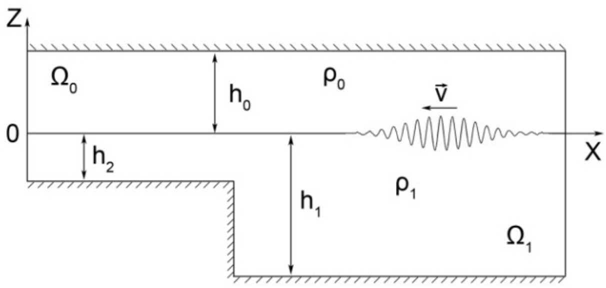

Figure 1. Sketch of the domain.

The fluid densities $\rho_{0,1}$ in the layers were chosen close to the real oceanic conditions, so that the traditional Boussinesq approximation (see, e.g., [3]) was applicable, i.e. $\rho_1 - \rho_0 \ll \rho_{0,1}$ and the parameter $a = \rho_0/\rho_1 = 0.9961$.

The initial perturbation on the interface between two layers was set in the form of the wavetrain with the given carrier wavelength. Its position was chosen far away from the edge of the bottom step in the domain with the depth h_1. The initial velocity potentials in both layers were chosen from the solution of the linearised problem for a monochromatic wave with the given wavelength such that the wavetrain started to move towards the bottom obstacle as shown in figure 1. The shape of the envelope was chosen in the form of Gaussian pulse with the characteristic width D which was four times greater than the wavelength of the carrier wave. Thus, the initial perturbations of basic variables were prescribed by the following equations:

$$\begin{cases} \eta(0,x) = \tilde{A}_i \exp\left[-\dfrac{(\tilde{x}-\tilde{x}_c)^2}{D^2}\right]\exp(-i\kappa\tilde{x}), \\ \varphi_0(0,x,z) = -i\dfrac{\Omega}{\kappa}\dfrac{\cosh\kappa(\tilde{z}-h_0/h_1)}{\sinh(\kappa h_0/h_1)}\eta(0,x), \\ \varphi_1(0,x,z) = i\dfrac{\Omega}{\kappa}\dfrac{\cosh\kappa(\tilde{z}+1)}{\sinh(\kappa)}\eta(0,x), \end{cases} \quad (1)$$

where $\tilde{x} = x/h_1$ and $\tilde{z} = z/h_1$ are dimensionless coordinates, $\tilde{A}_i = A_i/h_1$ is the dimensionless amplitude of the wavetrain, $\tilde{x}_c$ is the initial position of the wavetrain center. The amplitude of the perturbation was chosen so small that the linear theory was applicable, in particular, we put $\tilde{A}_i = \min(h_2/h_1, h_0/h_1, 1)/500$.

The calculation domain was covered by a mesh with different resolutions in the horizontal and vertical directions. In the horizontal direction there were at least 20 mesh nodes per a minimal wavelength, whereas in the vertical direction there were 120 nodes in the calculation domain.

Derivation of the Approximative Formulae

To obtain the formulae for the coefficients of internal wave transformation on the bottom step (the coefficients of transmission T and reflection R) we use the approach that was suggested in our recent papers for surface waves [4, 6]. The idea of derivation of the approximate formulae is based on the Lamb formula suggested in his famous book [7] for infinitely long linear waves in channels of variable cross-section. In particular, when the width of the channel is constant, but the depth changes abruptly from h_1 to h_2, Lamb's formulae read:

$$T = \frac{2}{1+c_2/c_1} = \frac{2}{1+\sqrt{h_2/h_1}}, \quad (2)$$

$$R = \frac{1-c_2/c_1}{1+c_2/c_1} = \frac{1-\sqrt{h_2/h_1}}{1+\sqrt{h_2/h_1}}, \quad (3)$$

where the transmission coefficient $T = A_t/A_i$ is the ratio of transmitted wave amplitude A_t with respect to the amplitude of incident wave A_i, similarly the reflection coefficient $R = A_r/A_i$ is the ratio of reflected wave amplitude A_r with respect to the amplitude of incident wave A_i. Then c_1 and c_2 are wave speeds in the region with the depths h_1 and h_2, respectively (see figure 1. In the case of surface waves the upper layer of infinitely large thickness h_0 has negligibly small density ρ_0, and the interface $\eta(t,x)$ plays a role of the free surface). In the long waves approximation the phase and group speeds of linear waves are equal, therefore $c_{1,2} = \sqrt{gh_{1,2}}$ are just the long wave speeds in the corresponding domains. Lamb's formulae where rigorously substantiated by Bartholomeusz [2] who derived integral equations for the determining the transformation coefficients for surface waves of arbitrary length. However, solutions to the integral equations were not obtained in his paper, but only the asymptotic analysis was performed for infinitely long waves. The transformation coefficients were derived much later by different authors using the approach suggested by Takano [14, 15] (see also [12, 11]).

Unfortunately, in the aforementioned papers the transformation coefficients were obtained numerically by means of solution of the truncated set of infinite number of algebraic equations. In such form the results obtained are not handy for practical applications and analysis of their dependence on hydrological parameters. In the paper [4] there were suggested the approximative formulae which represent the transformation coefficients in the closed forms convenient for the analysis and practical application. It was shown that there is a good agreement between the results of direct numerical modelling of surface wave transformation on a bottom step and predications on the basis of approximative formulae. In the subsequent paper [6] the accuracy of the approximative formulae was studied thoroughly by comparison with the results of rigorous theory, direct numerical calculations, and the low of of energy flux conservation. In particular, it was shown that the maximal error for the transmission coefficient does not exceed 5% of the exact value (for the reflection coefficient the error is much greater, but this coefficient is not so important in practice).

In the derivation of approximate formulae for the transformation coefficients it was assumed that the structure of Lamb's formulae (2) and (3) remains the same, where however either group or phase speeds should be used. It was found that the results of direct numerical calculations of transmitted and reflected surface waves can be well approximated if one uses group speeds in the formula for the transmission coefficient and phase speeds for the reflection coefficient.

The same approach we are suggesting in this paper for the transformation coefficients of internal waves. The dispersion relation between the wave frequency and wavenumber can be easily derived for internal waves in two-layer fluid (see, e.g., [7, 3]). In the particular case of rigid-lid approximation filtering surface waves, it reads

$$\tilde{\omega}^2 = \frac{\kappa(1-a)}{a\coth(\kappa h_0/h_1) + \coth\kappa}, \quad (4)$$

$$\tilde{\omega}^2 = \frac{q(1-a)}{a\coth(qh_0/h_1) + \coth(qh_2/h_1)}, \quad (5)$$

where $\tilde{\omega}^2 = \omega^2(\rho_1 h_0 + \rho_0 h_1)/[g(\rho_1 - \rho_0)h_0 h_1]$, $a = \rho_0/\rho_1$ is the density ratio, $\kappa = k_1 h_1$ is the dimensionless wavenumber of the incident and reflected waves, $q = k_2 h_1$ is the dimensionless wavenumber of the transmitted wave.

From these dispersion relations one can derive the expressions for the group and phase speeds in front of the bottom step and behind it (see figure 1). The group speeds $\tilde{V}_{g1} \equiv d\tilde{\omega}/d\kappa$ and $\tilde{V}_{g2} \equiv d\tilde{\omega}/dq$, as well the phase speeds $\tilde{V}_{p1} \equiv \tilde{\omega}/\kappa$ and $\tilde{V}_{p2} \equiv \tilde{\omega}/q$ can be readily calculated from the dispersion relations (4) and (5).

In the long-wave approximation, when κ, $\kappa h_0/h_1$, q, $qh_0/h_1 \ll 1$, the corresponding group and phase speeds become equal, $V_{g1} = V_{p1} \equiv c_1$ and $V_{g2} = \tilde{V}_{p2} \equiv c_2$, we obtain:

$$c_1 = \sqrt{\frac{(1-a)(h_0/h_1)}{a+h_0/h_1}}, \qquad c_2 = \sqrt{\frac{(h_2/h_1)(1-a)(h_0/h_1)}{a(h_2/h_1)+h_0/h_1}}. \quad (6)$$

Substituting these expressions into Eqs. (2) and (3) instead of c_1 and c_2, we obtain the formulae for the transformation coefficients of infinitely long internal waves:

$$T = \frac{2}{1+Q_l}, \qquad R = \frac{1-Q_l}{1+Q_l}, \quad (7)$$

where

$$Q_l = \sqrt{\frac{h_2}{h_1}\frac{a+h_0/h_1}{a(h_2/h_1)+h_0/h_1}}. \quad (8)$$

These formulae reduce to Lamb's formulae (2) and (3) for surface waves expressed in terms of depth ratio, if the density of the upper layer becomes negligibly small ($a \to 0$), and the density interface becomes a free surface.

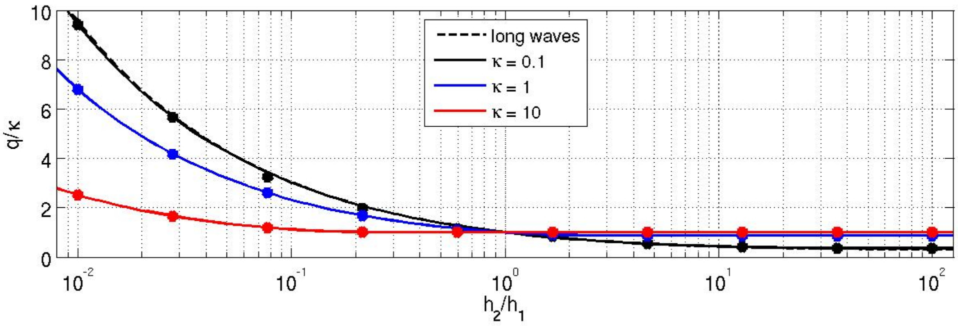

Figure 2. Transmitted to incident wavenumber ratio against the depth ratio h_1/h_2 for $h_0/h_1 = 10$. Solid lines represent solutions of equation (10), points are numerical results.

Results of our numerical modelling of the transformation process show that in the general case of arbitrary wavelength, the transmission coefficient T can be well approximated when the group speeds V_{g1} and V_{g2} are used in Eq. (2) for T instead of $c_{1,2}$. In the meantime, the reflection coefficient R can be satisfactorily approximated when the phase speeds V_{p1} and V_{p2} are used in Eq. (3) instead of $c_{1,2}$. Notice, that the same results were obtained for surface wave transformation [4, 6].

After substitution of the corresponding expressions for the group and phase velocities into (2) and (3), one obtains the following approximate formulae for the transformation coefficients:

$$T = \frac{2}{1+Q}, \qquad R = \frac{1-\kappa/q}{1+\kappa/q}, \tag{9}$$

where

$$Q = \frac{\kappa}{q}\frac{D[\kappa,\ 1,\ \kappa(h_0/h_1),\ 1]}{D[\kappa,\ E(\kappa h_0/h_1),\ \kappa(h_0/h_1),\ E(\kappa)]}\times \frac{D[q(h_2/h_1),\ E(qh_0/h_1),\ q(h_0/h_1),\ E(qh_2/h_1)]}{D[q(h_2/h_1),\ 1,\ q(h_0/h_1),\ 1]},$$

$$D(\alpha,\ \beta,\ \gamma,\ \delta) = a\gamma\tanh\alpha + \delta\tanh\beta,$$

$$E(\alpha) = 1 + \alpha\frac{\operatorname{sech}^2\alpha}{\tanh\alpha}.$$

The relationship between the wavenumbers q and κ of the transmitted and incident waves can be found from the frequency conservation low – wave frequency remains unchanged in any stationary system. Equating Eqs. (4) and (5), we obtain the transcendental equation, which can be solved numerically by any appropriate procedure:

$$\frac{\kappa}{q} = \frac{a\coth(\kappa h_0/h_1) + \coth(\kappa)}{a\coth(qh_0/h_1) + \coth(qh_2/h_1)}. \tag{10}$$

In the next section we illustrate graphically expressions for the transformation coefficients (9) and dependence of q/κ on depth ratio h_2/h_1. We also present the comparison of the suggested approximative formulae with the results of direct numerical simulations for different position of the density interface (pycnocline) in two-layer fluid.

Discussion of Results Obtained and Conclusion

With the help of properly adapted simulation code MITgcm we have performed more than 100 runs to model the transformation of small-amplitude internal waves on bottom step in two-layer fluid. For the dimensionless wavenumber of the incident wave the following three values were taken: $\kappa = \{0.1,\ 1.0,\ 10.0\}$. For each of these wavenumbers we have performed runs for three values of the depth ratio $h_0/h_1 = \{0.1,\ 1.0,\ 10.0\}$; this depth ratio characterises the relative thicknesses of fluid layer. Then, the calculations were performed for 20 differen values of the bottom layer thicknesses h_2/h_1 varying from 0.01 to 100 (the logarithmic scale was used). This range of variation of h_2/h_1 includes both the cases when $h_2/h_1 < 1$ and $h_2/h_1 > 1$; in other words this pertains to waves travelling toward the bottom jump both with the decreasing depth and with the increasing depth. We kept the depth h_1 constant in all calculations.

Figure 2 shows the dependences of wavenumber ratios of the transmitted and incident waves as functions of depth ratios h_2/h_1. Similar to surface waves [4, 6], the wavenumber increases when a wave enters into the shallower layer from the deep layer and decreases when a wave enters into the deeper layer from the shallower one. As one can see from figure 2, the longer is the incident wave, the greater is the change of its wavenumber after transformation on the bottom step – cf. black and red lines in figure 2. Notice that a wave with $\kappa = 0.1$ can be treated as the infinitely long wave – the corresponding solid black line for the case $\kappa = 0.1$ is indistinguishable from the dashed line for the case $\kappa \to 0$.

Coefficients of transformation are depicted in figure 3 against the depth ratio h_2/h_1. These coefficients are non-monotonic function. In the linear approximation the amplitude of a transmitted internal wave can increase up to twice with respect to the amplitude of initial wave, if it travels towards the step with the decreasing total depth. In this case the reflection coefficient goes to one. other details are clearly seen in figure 3. It is interesting to note that in the case of long wave transformation with $\kappa = 0.1$ both the coefficient of transmission and reflection asymptotically approach the values which are close to 0.5 when the ratio h_2/h_1 goes to infinity. This is in a contrast with the transformation coefficients for long surface waves, where the transmission coefficient monotonically goes to zero, and the reflection coefficient goes to one in the same limit (see [4, 6] for details). Furthermore, the transmission coefficient of long waves may become even greater than 0.5 when h_0/h_1 decreases, that is when the thickness of the upper layer decreases. The inverse situation occurs when the thickness of the upper layer increases.

Another interesting finding of our research pertains to the reflectionless transmission of short internal waves travelling from the shallower domain to the deeper domain (see red line in figure 3 for $\kappa = 10$). The reflection coefficient vanishes in this case, and the transmission coefficient approaches to unity. The transmission coefficient for the same wavenumber also turns to unity at some depth ratio $h_2/h_1 < 1$. But in this case the reflection coefficient is not equal to zero. This means that the amplitude of

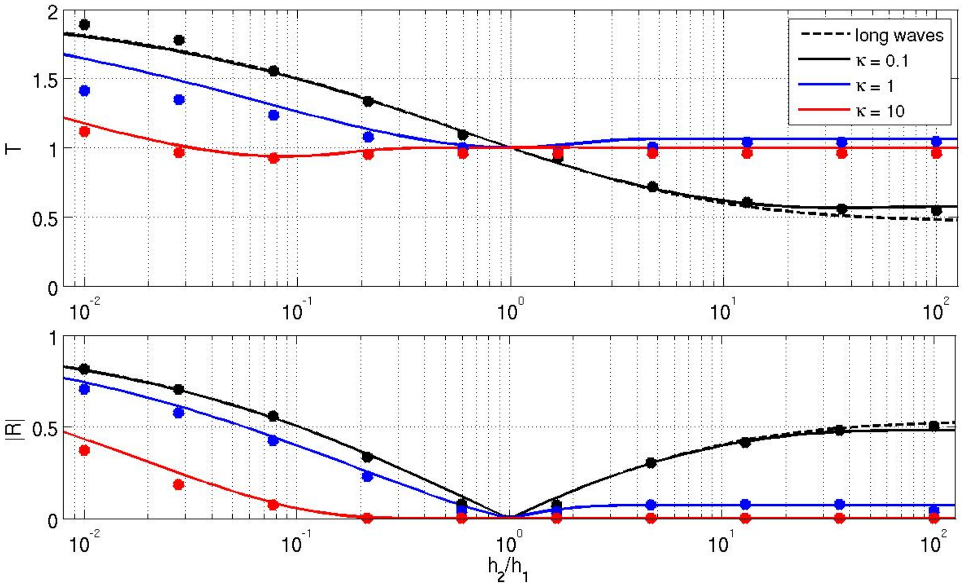

Figure 3. Coefficients of transmission (upper frame) and reflection (lower frame) as functions of the depths ratio h_2/h_1 for the particular value of the depth ratio upper to lower layers $h_0/h_1 = 10$ (solid lines – approximative formulae, points – numerical results)

the transmitted wave remains the same as the amplitude of the incident wave, but the wavelength becomes different. The same effect was discovered for surface waves [4, 6].

Thus, we obtained quite satisfactory agreement between the data of approximate formulae and results of direct numerical calculations. The approximative formulae describes not qualitatively, but even quantitatively the main features of transformation coefficients for internal waves in two-layer fluid. Suggested formulae are capable to predict correctly even such specific features as the values and positions of local extrema on the curves, as well as asymptotic values of the coefficients. Some minor discrepancies between the theoretical and numerical data can be explained by the approximative character of the suggested formulae and numerical errors caused by discretisation of the domain and data processing.

Acknowledgments

This study was supported by the State Project of Russian Federation in the field of scientific activity, Task No 5.30.2014/K. The authors are grateful to the management of the Intel Inc. head-office in Nizhny Novgorod (Russia) for the provision of computational facilities. Sergey Semin is also grateful to the administration of University of Southern Queensland (Australia) for the hospitality during his 10-month occupational traineeship program in 2013–2014.

References

[1] Adcroft, J., et al., MITgcm User Manual. *MIT Department of EAPS*, Cambridge, MA, 2008.

[2] Bartholomeusz, L.F., The reflection of long waves at a step, *Proc. Camb. Philos. Soc.*, **54**, 1958, 106–118.

[3] Brekhovskikh, L.M., Goncharov, V.V., *Mechanics of Continua and Wave Dynamics*, Springer, Berlin, 1994.

[4] Giniyatullin, A.R., et al., Transformation of narrowband wavetrains of surface gravity waves passing over a bottom step, *Math. Model. of Nat. Proc.*, **9(5)**, 2014, 73–82.

[5] Grimshaw, R., Pelinovsky, E., Talipova, T., Fission of a weakly nonlinear interfacial solitary wave at a step, *Geophys. and Astrophys. Fluid Dyn.*, **102(2)**, 2008, 179–194.

[6] Kurkin, A.A., Semin S.V., Stepanyants, Y.A., Surface water waves transformation over a bottom ledge, *Izvestiya, Atmos. and Oceanic Phys.*, 2014, to be published.

[7] Lamb, H. *Hydrodynamics*, Cambridge Univ. Press, Cambridge, 1932.

[8] Maderich, V., et al., The transformation of an interfacial solitary wave of elevation at a bottom step, *Nonlin. Processes in Geophys.*, **16**, 2009, 33–42.

[9] Maderich, V., et al., Interaction of a large amplitude solitary wave of depression with bottom step, *Phys. of Fluids*, **22**, 2010, 076602.

[10] Marshal, J., et al., Hydrostatic, quasi-hydrostatic, and nonhydrostatic ocean modeling, *J. Geophys. Res.*, **102**, 1997, 5733–5752.

[11] Massel, S.R., Harmonic generation by waves propagating over a submerged step, *Coastal Eng.*, **7**, 1983, 357–380.

[12] Newman J.N., Propagation of water waves over an infinite step, *J. Fluid Mech.*, **23**, 1965, 339–415.

[13] Pelinovsky, E.N., *Hydrodynamics of Tsunami Waves*, Nizhny Novogrod, IAP RAS, 1996 (in Russian).

[14] Takano, K., Effets d'un obstacle parallelepipedique sur la propagation de la houle, *La Houille Blanche*, **15**, 1960, 247–267.

[15] Takano, K., Effet d'un changement brusque de profondeur sur une houle irrotationnelle, *La mer*, **5(2)**, 1967, 100–116.

[16] Talipova, T.G., et al., Internal solitary waves transformation over the bottom step: loss of energy, *Phys. Fluids*, **25**, 2013, 032110.