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ORIGINAL ARTICLE

Statistical properties of multi-year droughts in climate models and observations with corresponding theoretical estimates

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Abstract
Multi-year droughts (MYDs) can have major impacts on ecosystems, agriculture, water resources, economies, people and societies. Here, we examine statistical properties of MYDs consisting of uninterrupted sequences of years in which annual precipitation falls below a given threshold. We examine statistics including the proportion of years that are part of droughts of duration $\geq n$ years, the proportion of droughts that have duration $\geq n$ years, and both the duration and the number of droughts with duration $\geq n$ years, in thirty-eight 200-year-long simulations of CMIP6 coupled global climate models under pre-industrial control conditions. We also derive formulae that approximate the average value of these and other statistics using simple stochastic models. The theoretical values obtained agree reasonably well with their Multi-Model Mean (MMM) climate model counterparts over the globe. Regional contrasts in the value of the statistics for MYDs consisting of uninterrupted sequences of years with below-average precipitation can be largely explained by spatial variations in the percentile of the mean. Corresponding formulae that account for non-zero temporal autocorrelation tend to agree somewhat more closely with the MMM values. MYD statistics are estimated using observational data and compared with theoretical and climate model estimates. The formulae incorporating non-zero autocorrelation provide a better estimate of the observational values than do MMM values or formulae derived assuming zero autocorrelation.

1 Introduction

Multi-year droughts (MYDs) can have a profound impact on ecosystems, agriculture, water and coastal resources, economies, human health, and livelihoods (e.g., Archer et al. 2022; Arthi 2018; Boisier et al. 2016; Chong et al. 2021; Devanand et al. 2024; Endfield et al. 2006; Garreaud et al. 2015; Gonzalez et al. 2018; Goulden and Bales 2019; Lin et al. 2022; McMichael 2012; Power et al. 2005; van Dijk et al. 2013). MYDs have been studied using historical information, observations—both instrumental and paleo reconstructed, and/or climate model simulations (see, e.g., Lin et al. 2022 for a review). Studies using climate model simulations have examined the extent to which anthropogenic and natural processes contributed to impactful MYDs that have already occurred, the characteristics and causes of MYDs in the past—including historical and preindustrial periods, and past and projected future changes in MYD properties (e.g., Ault et al. 2016; Bjarke et al. 2024; Cook et al. 2020, 2022; Devanand et al. 2024; Falster et al 2024; Garreaud et al. 2015; Gessner et al. 2022; Lin et al. 2022; Rakovec et al. 2022; Rauniyar and Power 2020; Rigden et al. 2024; Stevenson et al. 2022; Taschetto et al. 2016; Williams et al. 2020; Wu et al. 2022; Chen et al. 2025). Other studies have described processes responsible for drought and drought duration over land, highlighting the importance of changes in the frequency of weather systems that influence rainfall, sea surface temperature anomalies including those linked to coherent sources of variability like the El Nino-Southern Oscillation, local land–atmosphere feedbacks, natural and anthropogenic external forcing, internal atmospheric variability (see e.g. Lin et al. 2022; Holgate et al. 2024).

In this investigation, we are interested in documenting and understanding some of the statistical properties of

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MYDs in climate models under preindustrial control conditions. This issue has been investigated previously in a few recent studies. For example, statistical properties of MYDs over Europe and North America (Gessner et al. 2022), Australia (Taschetto et al. 2016; Falster et al. 2024), and Chile (Garreau et al. 2015) have been examined in preindustrial simulations, while Wu et al. (2022) examined statistics of MYDs averaged over the globe and over all of the continents except Antarctica.

We will examine many properties of droughts, including MYDs, across the entire globe. Our focus is on droughts consisting of uninterrupted sequences (UISs) of years in which annual precipitation falls below a given threshold, immediately preceded and followed by a year in which precipitation meets or exceeds the threshold. We further assume that the probability of precipitation falling below this threshold in a randomly selected year is p. We will refer to these droughts as UIS(p) droughts.

We recognise that this is a simple approach and that numerous other definitions of drought and MYD are available. Nevertheless, it is instructive to determine the extent to which the statistical properties of MYDs and their spatial variation in climate models using one specific definition can be understood using (simple) mathematical formulae. The formulae are based on the statistics for MYDs that arise from random changes in precipitation from one year to the next. If the formulae provide good estimates of the climate model results, then they lend support to the dominance of stochastic variability in driving MYD statistics.

Formulae for some statistical properties of UISs have been obtained previously as part of the so-called "*Theory of runs*" (e.g., Mood 1940; Yevjevich 1967; Makri and Psillakis 2010). By exploiting the fact that the output from each climate model we use spans 200 years, the theories and associated mathematics outlined in this paper are much simpler. This simplification enables us to provide formulae for the average value of a much larger number of quantities than provided in these previous studies, both with and without temporal correlation.

The primary aims of this investigation are to (i) derive simple mathematical formulae for several key statistics of UIS MYDs, (ii) estimate the value of these same statistics across the globe in preindustrial control simulations using the latest generation of climate models (Coupled Model Intercomparison Project phase 6, CMIP6; Eyring et al. 2016) and observations, (iii) determine the degree of agreement between the modeled, theoretical and observed estimates, and (iv) use the theoretical estimates to better understand the statistical properties of MYDs and their regional contrasts. We will also derive formulae that take temporal correlation into account, and we will also examine the statistical properties of UIS MYDs using the average precipitation (μ) as the threshold.

The methods used in this study are described in Sect. 2. Mathematical formulae for various statistical properties of UIS(p) droughts, assuming zero year-to-year temporal autocorrelation in precipitation ("*Theory I*"), are derived in Sect. 3. The formulae are analyzed and compared with results from Monte Carlo simulations in Sect. 4. The statistical properties of MYDs in climate models under preindustrial forcing are presented in Sect. 5, along with a comparison with Theory. 1 Formulae for drought statistics that account for nonzero autocorrelation ("*Theory II*") are then provided in Sect. 6. Results using *Theory II* are compared with those using *Theory I* and from the climate models in the same section. MYD statistics using observational data are examined in Sect. 7 and a comparison between the observational estimates and estimates obtained from both theory and individual climate models is provided in the same section. The impact of sampling error is discussed in Sect. 8 and the results in Sects. 3, 4, 5, 6, 7, and 8 are summarized and discussed in Sect. 9.

2 Methods and data

The derivations in Sects. 3 and 6 assume that the precipitation time series are infinitely long. They are in fact 200 years long. We will see, however, that the approximations are accurate for a wide range of statistics.

Monte Carlo experiments are conducted to assess the accuracy of the formulae derived in Sect. 3. We conducted numerous simulations, each 200 years long to match the climate model simulations, if the annual precipitation data has no autocorrelation. For each year we obtain a real number randomly selected from a uniform distribution over the interval [0, 1]. If the random number is less than p then the year is considered dry, otherwise the year is considered wet. Recall that p is the probability of precipitation falling below the threshold used to define dry years.

We used the preindustrial control (piControl) experiment from 38 CMIP6 models (Table S1). We obtained monthly total precipitation (variable *pr* and unit kg m^{-2} s^{-1}) from the last 200 years of each experiment, converted to mm per month and summed across calendar years (January–December) to obtain annual precipitation and focus on MYDs. Analyses were conducted at the models' native resolution and re-gridded to a common 1° resolution using bilinear interpolation for calculating multi-model means (MMMs) and for plotting.

We also use annual observational data for the period 1900–2014 derived from the monthly Global Precipitation Climatology Centre dataset version 2.3 (GPCC; Schneider et al. 2011).

3 Derivation of formulae for statistical properties of droughts with zero autocorrelation

In this section we derive mathematical formulae which approximate a range of important statistics of drought, including MYDs that consist of UIS of years with precipitation below a given percentile. In this section we assume that precipitation has zero autocorrelation from one year to the next. The statistics, the associated formulae, and their equation numbers are summarized in Table 1.

Here p is the probability that precipitation falls below a given threshold in any given year, n is the length (in years) of the drought. See text for definition and further details.

The derivations begin by supposing that we have an infinitely long time series of annual precipitation data in a stable climate, and that the precipitation is a white noise process for which the autocorrelation at all nonzero time lags is zero. We will revisit this assumption in Sect. 6. We further suppose that the probability that the precipitation falls below a given threshold equals p each year, and the probability that it does not is $q = 1 - p$.

3.1 P_n= proportion of years in an n-year drought

A one-year drought (hereafter 1YD) must have a sequence qpq, so that the probability that a randomly selected year is in a 1YD is given by $pq^2 = 1pq^2$.

A 2YD must have a sequence $qppq$. A randomly selected year is in a 2YD if it is either the first or second year with a p, and so the probability that a randomly selected year is in a 2YD is given by $2p^2q^2$.

A 3YD must have a sequence $qpppq$. A randomly selected year is in a 3YD if it is either the first, second or third year has a p, and so the probability that a randomly selected year is in a 3YD is given by $3p^3q^2$.

From this information, we can deduce that the probability that a randomly selected year is in an nYD, where $n = 1, 2, 3, \ldots$ is given by

$$P_n = np^n(1-p)^2. \tag{1}$$

3.2 Q_n= the probability that a randomly selected year is part of a drought of duration $k \geq n$

In this previous subsection we examined the proportion of years in a drought of duration equal to n years. It is also of interest to know the proportion of years in a drought of duration $\geq n$ years, which is equal to the probability that a randomly selected year is part of a drought of duration $\geq n$ years. This is given by

$$\begin{aligned} Q_n(n) &= \sum_{k=n}^{\infty} P_n = (1-p)^2 \sum_{k=n}^{\infty} kp^k \\ &= (1-p)^2 \left\{ \sum_{k=1}^{\infty} kp^k - \sum_{k=1}^{n-1} kp^k \right\} \\ &= (1-p)^2 \left\{ S - S_{n-1} \right\}, \end{aligned} \tag{2}$$

where $S = \sum_{k=1}^{\infty} kp^k$ and $S_{n-1} = \sum_{k=1}^{n-1} kp^k$. In Appendix 1 we show that $S = \frac{p}{(1-p)^2}$ (from Eq. (13)) and $S_{n-1} = p\left[\frac{1-p^n-np^{n-1}(1-p)}{(1-p)^2}\right]$ (Eq. (14)).

Using (13) and (14) in (2) then gives

$$Q_n(n) = np^n + (1-n)p^{n+1}. \tag{3}$$

Note that $Q_n(n)$ is also the proportion of years that are in a UIS(p) drought of duration $k \geq n$ years.

Table 1 Summary of the key formulae for the average value of the statistics mentioned for droughts consisting of uninterrupted sequences (UIS) of dry years (Rows 1–7), where a dry year is a year in which precipitation falls below a given threshold percentile

Ref. No.	Statistic and formula	Description	Eq. No.
1	$P_n = np^n(1-p)^2$	Probability that a randomly selected year is part of an n-year uninterrupted sequence (UIS) drought with threshold p; Fraction of years that are part of a drought of duration n years	(1)
2	$Q_n(n) = np^n + (1-n)p^{n+1}$	Probability that a randomly selected year is part of a UIS(p) drought of duration $k \geq n$=fraction of years that are part of a UIS(p) drought of duration $k \geq n$ years	(3)
3	$P_n^* = (1-p)p^{n-1}$	Proportion of all UIS(p) droughts that are of duration n years	(5)
4	$Q_n^* = p^{n-1}$	Proportion of all UIS(p) droughts that have duration $k \geq n$	(7)
5	$N_D(n) = Np^n(1-p)^2$	Number of UIS(p) droughts of duration n years in sample of size N years	(8)
6	$N_D(k \geq n) = N(1-p)p^n$	Number of UIS(p) droughts of duration $n \geq k$	(9)
7	$DD(k \geq n) = n - 1 + \frac{1}{1-p}$	Duration of UIS(p) droughts in a sample of N years	(11)

3.3 $\hat{P}_n$ = proportion of droughts that have duration n years

It is also of interest to know what proportion of droughts have duration n years. As the proportion of years that are in a drought of duration n years is given by P_n, the average number of droughts in samples of size N years is approximated by NP_n/n, and the average number of droughts (of any size) is approximated by $N\sum_{n=1}^{\infty} P_n/n$. So the proportion of all droughts that are of duration n years, $\hat{P}_n$ say, is given by

$$\hat{P}_n = \frac{NP_n/n}{N\sum_{n=1}^{\infty} P_n/n} = \frac{P_n/n}{\sum_{n=1}^{\infty} P_n/n} = \frac{p^n(1-p)^2}{(1-p)^2\sum_{n=1}^{\infty} p^n} = \frac{p^n}{\sum_{n=1}^{\infty} p^n} = \frac{p^n}{T}, \tag{4}$$

where $T = \sum_{n=1}^{\infty} p^n$. And as $T = \frac{p}{1-p}$ (see e.g., Fitzpatrick 1975), then

$$\hat{P}_n = (1-p)p^{n-1}. \tag{5}$$

This formula has been provided previously (see e.g., Yevjevich 1967).

3.4 $\hat{Q}_n$ = proportion of droughts with duration $k \geq n$

In the previous subsection we examined the proportion of droughts that have duration equal to n years. It is also of interest to know the proportion of droughts that have duration $\geq n$ years. The proportion of all droughts that have duration $k \geq n$ is given by

$$\hat{Q}_n = \sum_{k=n}^{\infty} \hat{P}_k = \sum_{k=1}^{\infty} \hat{P}_k - \sum_{k=1}^{n-1} \hat{P}_k = \frac{(1-p)}{p}\left\{T - T_{n-1}\right\}, \tag{6}$$

where $T_{n-1} = \sum_{k=1}^{n-1} p^k$. Using the fact that $T_{n-1} = (p-p^n)/(1-p)$ and $T = \frac{p}{1-p}$ in (6) then gives

$$\hat{Q}_n = p^{n-1}. \tag{7}$$

3.5 $N_D(n)$ = the average number of droughts of duration n years

The average number of droughts of duration n years in samples of size N years is approximated by

$$N_D(n) = N\frac{P_n}{n} = \frac{nNp^n(1-p)^2}{n} = Np^n(1-p)^2. \tag{8}$$

3.6 $N_D(k \geq n)$ = number of droughts of duration $k \geq n$

The number of droughts of duration $k \geq n$ years is given by

$$\begin{aligned} N_D(k \geq n) &= \sum_{k=n}^{\infty} N_D(n) = N(1-p)^2 \sum_{k=n}^{\infty} p^n \\ &= N(1-p)^2\left\{\sum_{k=1}^{\infty} p^n - \sum_{k=1}^{n-1} p^n\right\} = N(1-p)^2\left\{T - T_{n-1}\right\} \\ &= N(1-p)^2\left\{\frac{p}{1-p} - \frac{p-p^n}{1-p}\right) = N(1-p)p^n. \end{aligned} \tag{9}$$

It can be shown that the formulae for $N_D(n)$ and $N_D(k \geq n)$ are the same as the average values of the asymptotic distributions obtained by Mood (1940) in his analysis of "runs". Our derivations are much simpler, and we provide formulae for the mean of additional statistics that Mood (1940) does not.

3.7 $DD(k \geq n)$ = average duration of droughts of duration $k \geq n$

The average duration of droughts in a sample of N years is given by.

$$\begin{aligned} DD(k \geq n) &= \frac{\text{sum of all the } DD(k) \text{ in the sample with duration } \geq n \text{ years}}{\text{Total number of droughts in the sample with duration } \geq n \text{ years}} \\ &= \frac{N\sum_{k=n}^{N} P_n}{N_D(k \geq n)} = N(1-p)^2\frac{\sum_{k=n}^{N} kp^k}{N(1-p)p^n} = (1-p)\frac{\sum_{k=n}^{N} kp^k}{p^n} \\ &= \frac{(1-p)}{p^n}\left\{\sum_{k=1}^{N} kp^k - \sum_{k=1}^{n-1} kp^k\right\} \\ &\approx \frac{(1-p)}{p^n}\left\{\sum_{k=1}^{\infty} kp^k - \sum_{k=1}^{n-1} kp^k\right\} = \frac{(1-p)}{p^n}\left\{S - S_{n-1}\right\} \text{for large N.} \end{aligned} \tag{10}$$

Using Eqs. (13) and (14) in (10) then gives

$$\begin{aligned} DD(k \geq n) &= \frac{(1-p)}{p^n}\left\{\frac{p}{(1-p)^2} - p[\frac{1-p^n-np^{n-1}(1-p)}{(1-p)^2}\right\} \\ &= n - 1 + \frac{1}{1-p}, \end{aligned} \tag{11}$$

and so the average duration of all MYDs is

$$DD(k \geq 2) = 1 + 1/(1-p). \tag{12}$$

Note that the average total number of days in droughts of duration $\geq n$ years is given by the product of $N_D(k \geq n)$ and $DD(k \geq n)$, i.e., $N_D(k \geq n)DD(k \geq n) = Np^N\left[n(1-p)+p\right]$.

4 Presentation, analysis and accuracy of the formulae

4.1 Solutions with zero autocorrelation

4.1.1 Solutions for $n = 1$ and $n = 2$

Solutions for (a) P_1, P_2, Q_1, Q_2, (b) $P_1^*, P_2^*, Q_1^*, Q_2^*$, (c) $DD(k \geq 1), DD(k \geq 2)$, and (d) $N_D(k \geq 1), N_D(k \geq 2)$ are all presented in Fig. 1 as a function of p. Recall (and as indicated in Table 1) that P_j is the proportion of years in a j-year drought, Q_j is the proportion of years in a drought of duration $\geq j$ years, P_j^* is the proportion of droughts that have duration j years, Q_j^* is the proportion of droughts that have duration $\geq j$ years, $DD(k \geq j)$ is the average duration of droughts with duration $\geq j$ years, $N_D(k \geq j)$ is the average number of droughts in a 200 year period that have duration $\geq j$ years, and p is the probability of a given year being dry.

If $n = 1$ then $Q_1 = p$, while Q_2 is a monotonically increasing cubic ($= 2p^2 + p^3$). Both P_1 and P_2, on the other hand, exhibit turning points given by $\frac{dP_n}{dp} = 0$, which gives maxima at $p = \frac{n}{n+2} = 1/3$ for P_1, and 1/2 for P_2. In other words, the proportion of years in a one-year drought is a maximum if $p = 0.3$, while the proportion of years in a two-year drought is a maximum if $p = 1/2$. At $p = 1/2$ both P_1 and $P_2 = 1/8$ at, while $Q_1 = 1/2$ and $Q_2 = 3/8$.

If $p = 1/2$ then the values of P_2, Q_2, P_2^*, Q_2^*, (c) $DD(k \geq 2)$, and (d) $N_D(k \geq 2)$ are 0.125, 0.375, 0.25, 0.5, 3 years, and 25 respectively.

Figure 1b indicates that $Q_1^* = 1$ for all p. This is because $Q_1^*(p)$ is the proportion of droughts that have duration ≥ one year, i.e., all droughts. Less trivially, $Q_2^* = p$ and $P_1^* = 1 - p$ both monotonically increase with p, while $P_2^* = p(1 - p)$ peaks at $p = 1/2$. This indicates that the proportion of two-year droughts is a maximum if $p = 1/2$.

Figure 1c indicates that both $DD(k \geq 1)$ and $DD(k \geq 2)$ increase monotonically as p increases with $DD(k \geq 1) = 2$ and $DD(k \geq 2) = 3$ at $p = 1/2$.

Figure 1d indicates that both $N_D(k \geq 1), N_D(k \geq 2)$ exhibit maxima. Maxima occur where $\frac{dN_D(k \geq n)}{dp} = 0$, which gives $p = n/(n+1)$, which = 1/2 for $N_D(k \geq 1)$ and 2/3 for $N_D(k \geq 1)$. This analysis and the curve in Fig. 1d collectively indicate that $N_D(k \geq 2)$, the number of droughts of duration ≥ 2 years in a 200-year sample, increases from $p = 0.1$, reaches a maximum value of $4N/27 = 29.6$ years at $p = 2/3$, and then declines as p increases further.

4.1.2 Theoretical solutions for p = 0.3, 0.5 and 0.7

Solutions for (a) P_n, (b) Q_n, (c) P_n^*, (d) Q_n^*, (e) $N_D(k \geq n)$ and (f) $N_D(n)$ for $n = 1$ to 10 are presented in Fig. 2 as curves. Each panel provides solutions for three different values of p: 0.3 (rust), 0.5 (black), and 0.7 (blue).

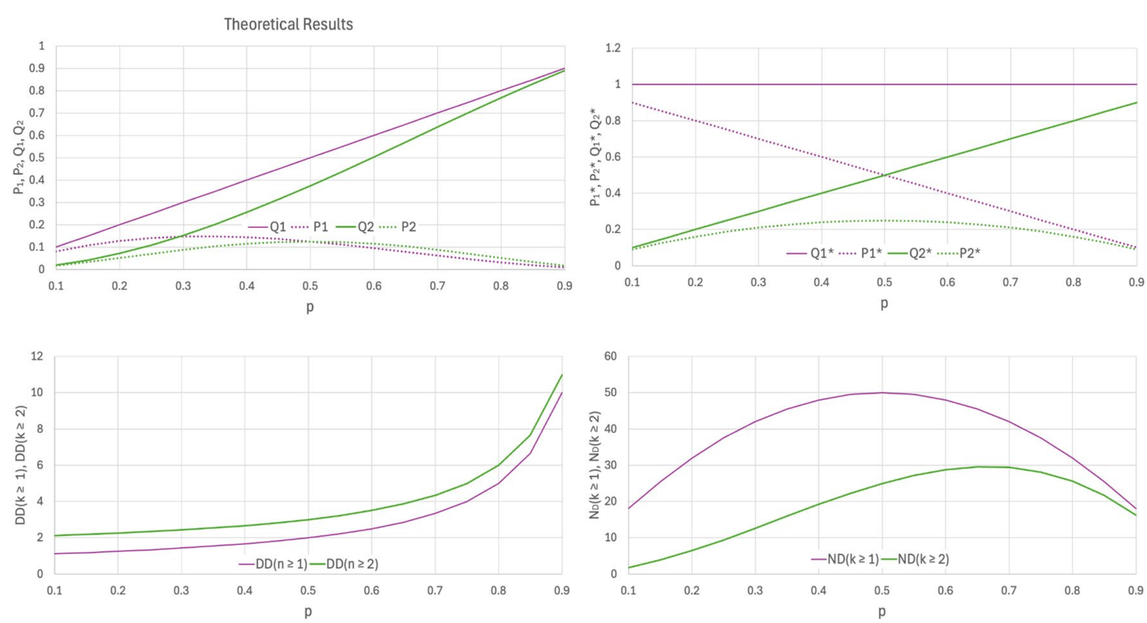

Fig. 1 Theoretical Solutions for **a** P_1, P_2, Q_1, Q_2, **b** $P_1^*, P_2^*, Q_1^*, Q_2^*$, **c** $DD(k \geq 1), DD(k \geq 2)$, and **d** $N_D(k \geq 1), N_D(k \geq 2)$. Note that $N_D(k \geq 1)$ is the average, total number of droughts, while $N_D(k \geq 2)$ is the average, total number of MYDs

Figure 2a indicates that: $P_n(0.7)$ is greater than $P_n(0.5)$ for $n \geq 4$, but less than $P_n(0.5)$ for $n \leq 3$, while $P_n(0.3)$ is less than $P_n(0.5)$ for $n \geq 3$.

Figure 2b and Table 2 indicate that $Q_n(0.7)$ is greater than $P_n(0.5)$ for all n, while $Q_n(0.5)$ is greater than $Q_n(0.3)$ for all n, and the proportion of years in MYD is 0.64, 0.38, and 0.15 for p = 0.7, 0.5 and 0.3 respectively, while the proportion of droughts that are MYDs (i.e., Q_2^*; Fig. 2d) is equal to p.

The proportion of droughts that have duration ≥ 5 years (Fig. 2d; Table 2), i.e., $Q_n^* = p^{n-1} = 0.24$, 0.063 and 0.0081 for p = 0.7, 0.5 and 0.3 respectively, while the equivalent numbers for droughts that have duration ≥ 10 years are 0.04, 0.002, 0.00002. This indicates that the proportion of droughts with duration ≥ 10 years is tiny (0.2%) if $p = 0.5$, and exceptionally small (≤ 0.002%) if $p \leq 0.3$. These and other results using *Theory I* are summarised in Table S2.

The proportion of MYDs that have duration ≥ 5 years, i.e., $Q_5^*/Q_2 = p^{5-1}/p^1 = p^3 = 0.343$, 0.125 and 0.027 for p = 0.7, 0.5 and 0.3 respectively, while the equivalent numbers for droughts that have duration ≥ 10 years (i.e., p^8) are 0.058, 0.004, and 0.00007. This indicates that the percentage of MYDs that have duration ≥ 10 years, is tiny (0.4%) if $p = 0.5$, and extremely low (≤ 0.007%) if $p \leq 0.3$.

The number of MYDs that have duration ≥ 5 years in a 200-year sample (Fig. 2e; Table 2) is, on average, 10.1, 3.1, and 0.34 for p = 0.7, 0.5 and 0.3 respectively, and the equivalent numbers for droughts that have duration ≥ 10 years are 1.7, 0.1, and 0.0008. The latter indicates that if e.g., $p = 0.5$ then we'd expect a MYD of duration ≥ 10 years once every 2,000 years on average.

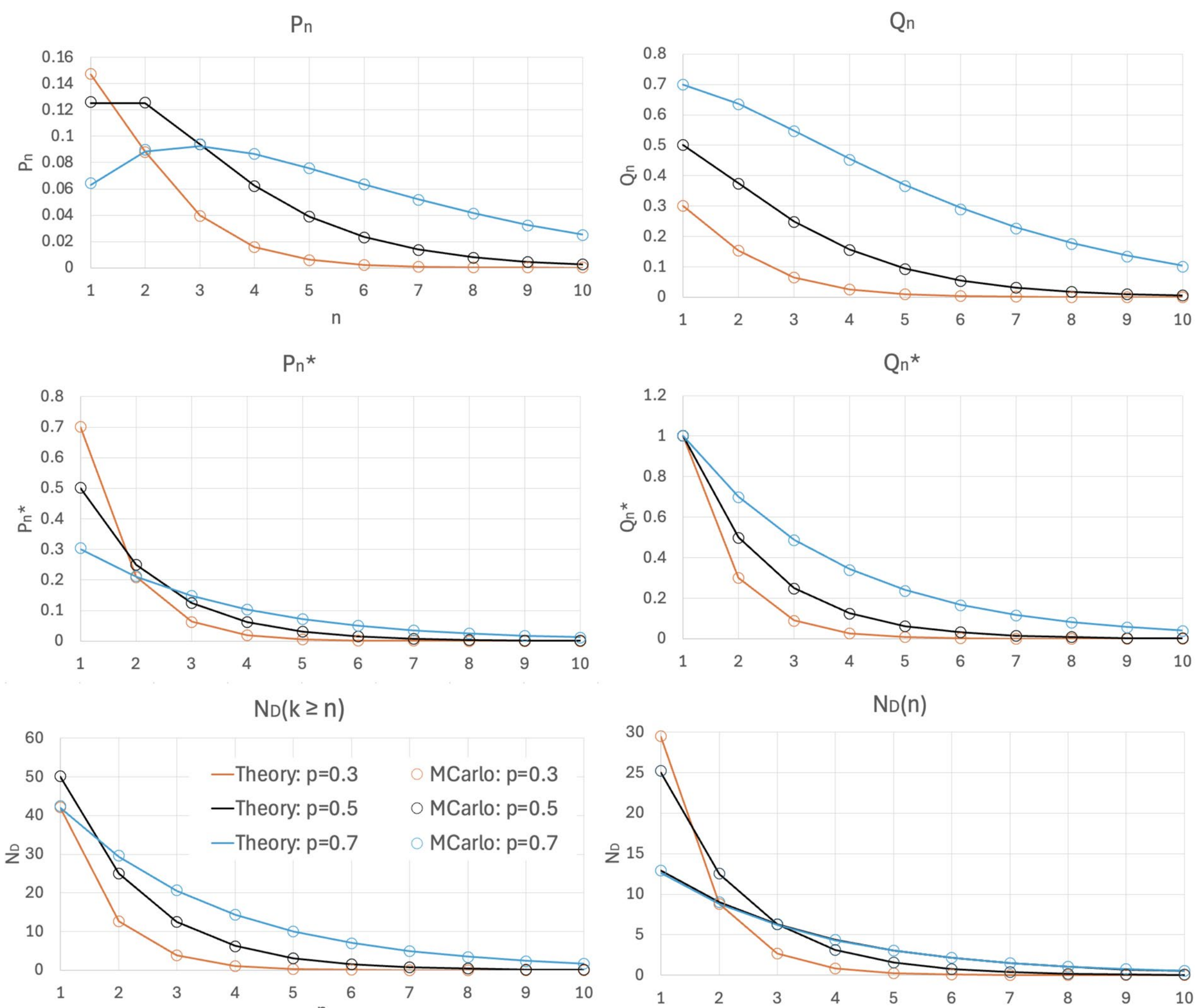

Fig. 2 *Theory I* solutions for **a** P_n, **b**Q_n, **c** P_n^*, **d** Q_n^*, **e** $N_D(k \geq n)$ and **f** $N_D(n)$. Analytic values are shown as lines, Monte Carlo simulations (see text below) are shown as circles

4.1.3 Accuracy of the theoretical solutions

Monte Carlo solutions for (a) P_n, (b) Q_n, (c) P^*_n, (d) Q^*_n, (e) $N_D(k \geq n)$ and (f) $N_D(n)$, are also presented in Fig. 2 as open circles, again for three different values of p: 0.3 (red circles), 0.5 (black circles), and 0.7 (blue circles). In every case the Monte Carlo estimates are very close to the theoretical estimates.

5 Statistical properties of MYDs in climate models

In the previous section we derived formulae for MYDs defined in terms of p. We can, if we wish, also define MYDs in terms of the mean, rather than a percentile. In this case the MYDs are UISs of years with below average precipitation. The corresponding analytic estimates are then given by the formulae in Table 2, but with p set to the value that corresponds to the mean (μ), i.e., $p = p(\mu)$. If the data are symmetric then $p(\mu) = 0.5$. In general, however, the annual precipitation data are skewed, so that $p(\mu) \neq 0.5$ (Fig. 3).

Over virtually all land locations the mean exceeds the median (Fig. 3c) and $p(\mu) > 0.5$ (Fig. 3d), indicating that annual precipitation distribution tends to be skewed to the right, so that minimum precipitation values tend to be closer to the median than the highest precipitation values. The largest values of $p(\mu)$ occur over North Africa, the Middle East (especially in parts of the Arabian Peninsula, specifically, eastern Yemen, Oman, the United Arab Emirates, and southern Saudi Arabia), northwest India and southeast Pakistan. There are only a few places where $MMM(p(\mu)) < 0.5$ (in western Papua, Papua New Guinea, eastern Borneo and small scattered regions in parts of South America), but these regions are not stippled.

Given that $p(\mu)$ varies spatially we would therefore expect that the key quantities (i.e., $Q_2(\mu)$ etc.) to also vary spatially. This is indeed the case (Fig. 4).

There is a strong resemblance between the maps for all four quantities in Fig. 4 and the corresponding theoretical estimates in Fig. S1, with corresponding spatial (Pearson) correlation coefficients (using land points only) of 0.98, 0.88, 0.93, and 0.84 for $Q_2(\mu), Q^*_2(\mu), DD(k \geq 2;\mu)$, and $N_D(k \geq 2;\mu)$ respectively.

Maps of the differences between the fields in Figs. 4 and S1 (given in Fig. S2) indicate that the MMM (climate model) values of $Q_2(\mu), Q^*_2(\mu)$ and $DD(k \geq 2;\mu)$ tend to be somewhat larger than the corresponding theoretical estimates, i.e., models tend to simulate more MYDs and the average duration of MYDs tends to be longer than expected from *Theory I*. In contrast the corresponding values of $N_D(k \geq 2;\mu)$ tend to be somewhat less than corresponding *Theory I* values, indicating that the models tend to simulate fewer MYDs than expected from *Theory I*. Note, however, that the degree of agreement on differences among the climate models is limited to very few regions. The close resemblance between the MMM (Fig. 4) and *Theory I* values (Fig. S1), and the high spatial correlation coefficients between them, nevertheless indicates that regional contrasts in the value of the various statistics primarily arise from regional contrasts in the percentile of the mean, $p(\mu)$.

To further assess the degree to which the theoretical formulae apply to the climate model results we will now examine the way in which the key quantities vary as a function of p over a wide range of p values. To obtain information on a wider range of p values, we look at the values of the key quantities over the globe using three different sets of thresholds: $p = 0.5 * p(\mu)$, $p = p(\mu)$, and $p = 1.5 * p(\mu)$. In other words, the threshold used to define a dry year was set to half the mean (red dots), the mean

Table 2 Summary of the key formulae for the average value of statistical properties of UIS droughts, taking the possibility of nonzero autocorrelation into account. While the probability that a randomly selected year is *dry* (i.e., precipitation falls below a given threshold) is p, the probability that a year immediately following a dry year is also dry is p_D, and n is the length of the drought in years. See Appendix 2 for further details

Ref. No.	Statistic and formulae	Description	Equation No.
1	$\widehat{P_n} = \frac{p}{p_D} n p_D^n (1-p_D)^2 = \frac{p}{p_D} P_n(p_D)$	Probability that a randomly selected year is part of an n-year UIS(p) drought	(18)
2	$\widehat{Q_n}(n) = \frac{p}{p_D}[n p_D^n + (1-n)p_D^{n+1}] = \frac{p}{p_D} Q_n(p_D)$	Probability that a randomly selected year is part of a UIS(p) drought of duration $k \geq n$ = fraction of years that are part of a UIS(p) drought of duration $k \geq n$ years	(19)
3	$\widehat{P^*_n} = (1-p_D)p_D^{n-1} = P^*_n(p_D)$	Proportion of all UIS(p) droughts that are of duration n years	(20)
4	$\widehat{Q^*_n} = p_D^{n-1} = Q^*_n(p_D)$	Proportion of all UIS(p) droughts that have duration $k \geq n$	(21)
5	$\widehat{N_D}(n) = \frac{p}{p_D} N p_D^n (1-p_D)^2 = \frac{p}{p_D} N_D(n;p_D)$	Number of UIS(p) droughts of duration n years in sample of size N years	(22)
6	$\widehat{N_D}(k \geq n) = \frac{p}{p_D} N(1-p_D)p_D^n = \frac{p}{p_D} N_D(k \geq n;p_D)$	Number of UIS(p) droughts of duration $n \geq k$	(23)
7	$\widehat{DD}(k \geq n) = n - 1 + 1/(1-p_D) = DD(k \geq n;p_D)$	Duration of UIS(p) droughts in a sample of N years	(24)

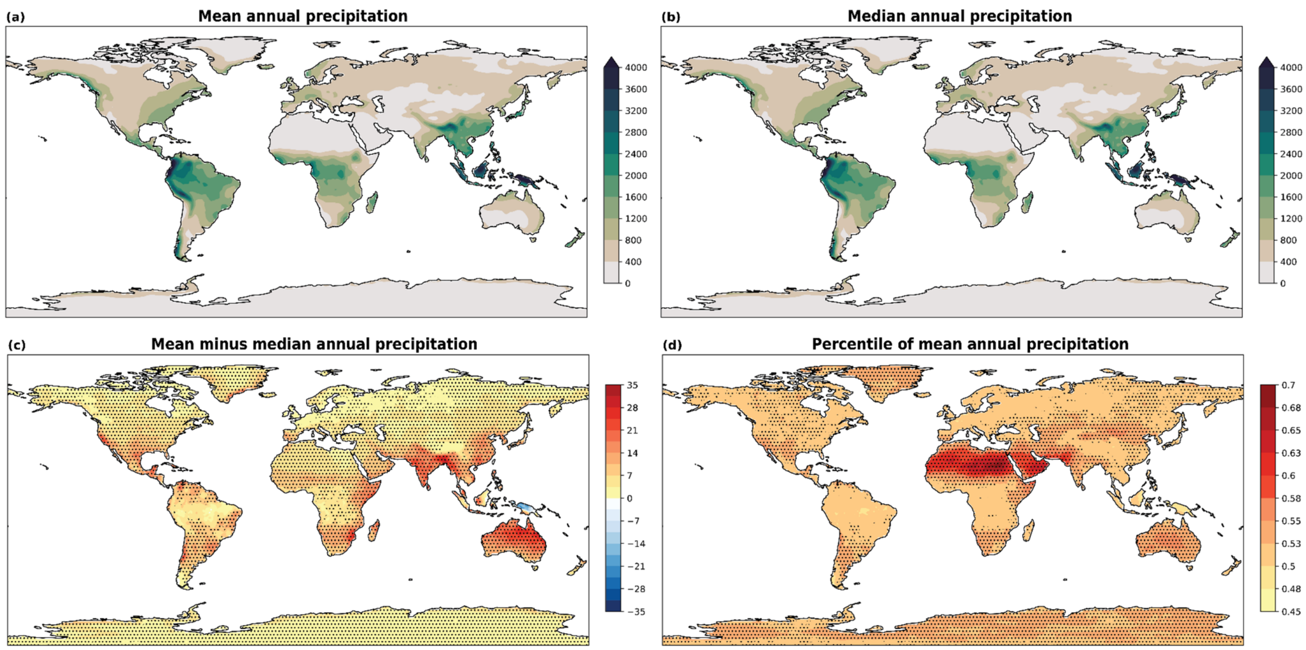

Fig. 3 Maps showing MMM values of **a** mean precipitation (μ;mm), **b** median precipitation (mm), **c** mean-median precipitation (mm), and **d** the percentile of the mean, i.e., $p(\mu)$. Stippling indicates that > 70% of models agree on the sign of the difference: (c) between the mean and median values of annual precipitation; (d) $p(\mu) - 0.5$. Results are based on simulations from 38 CMIP6 models under preindustrial control conditions

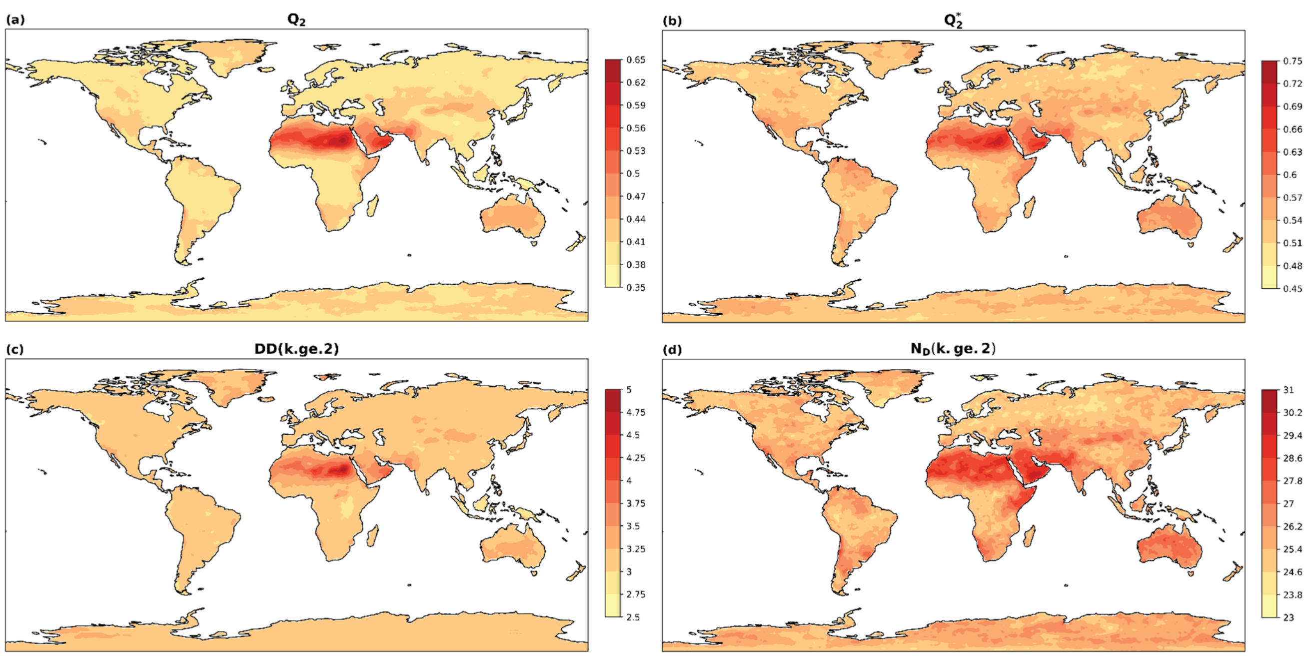

Fig. 4 MMM values of **a** $Q_2(\mu)$, **b** $Q_2^*(\mu)$, **c** $DD(k \geq 2;\mu)$ and **d** $N_D(k \geq 2;\mu)$ over land. $Q_2(\mu)$ is the proportion of years that are part of MYDs, $Q_2^*(\mu)$ is the proportion of droughts that are MYDs, $DD(k \geq 2;\mu)$ is the average duration of MYDs, and $N_D(k \geq 2;\mu)$ is the average number of MYDs in a 200-year sample; a given year is defined here to be "dry" if the precipitation falls below the mean, μ

(black dots) and 1.5 times the mean (blue dots); Fig. 5). Each scatter point is taken from an individual grid point in an individual model.

For all four variables (i.e., $Q_2(p), Q_2^*(\mu), DD(k \geq 2)$, and $N_D(k \geq 2)$), the theoretical values provide a reasonable fit to the data presented.

5.1 Longer MYDs

MMM values of $Q_5(\mu), Q_5^*(\mu), DD(k \geq 5;\mu)$, and $N_D(k \geq 5;\mu)$ over land are presented in Fig. S3 (first column). The largest occur in North Africa, North Africa, the Middle East, and a small region incorporating parts of India, Pakistan, Afghanistan, Uzbekistan and Tajikistan. The four maps using *Theory I* (second column) are very similar to their MMM counterparts. The MMM values of $Q_5(\mu), Q_5^*(\mu)$, and $DD(k \geq 5;\mu)$ tend to be somewhat larger than their *Theory I* counterparts, whereas the MMM values of $N_D(k \geq 5;\mu)$ tend to be somewhat smaller than their *Theory I* counterparts.

Scatter plots for (a) $Q_5(p)$, (b) $Q_5^*(p)$, (c) $DD(k \geq 5;p)$ and (d) $N_D(k \geq 5;p)$ are presented in Fig. S4. Each scatter point is again taken from an individual grid point in an individual model. The theoretical estimate of the way each quantity depends on p again indicates that the theory outlined

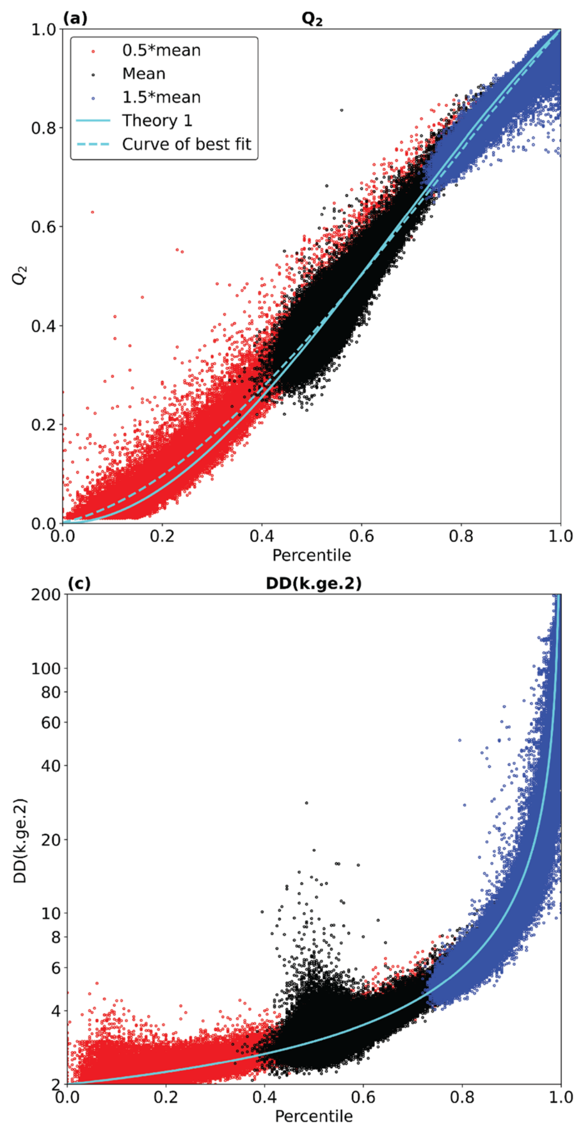

Fig. 5 Scatter plots for **a** $Q_2(p)$, **b** $Q_2^*(p)$, **c** $DD(k \geq 2;p)$ and **d** $N_D(k \geq 2;p)$. Each scatter point is a value taken from an individual land grid point in an individual model on the interpolated 1° grid. Theoretical values (Sect. 3; solid aqua curve) and curves of best fit [dashed aqua curves; cubic for (**a**, **d**), linear for (**b**)] are shown. We were unable to obtain a sensible curve of best for (c) using either polynomial or exponential splines

in Sect. 3 does a reasonable job of estimating the climate model results.

6 Formulae with non-zero autocorrelation—Theory II

Figure 6 shows the MMM autocorrelation coefficient at one year lag. The vast majority of land points (approximately 90%) have small, positive values, with a global average of 0.03. Larger MMM values up to 0.25 are evident in some locations.

It is therefore worth knowing what the impact of non-zero temporal autocorrelation in precipitation has on the theoretical estimates of the statistical properties of MYDs. So, in this section we provide mathematical formulae that are similar to those provided in the previous section, but they are modified to allow for the possibility that there is nonzero autocorrelation. The statistics, the associated formulae, and their equation numbers are summarized in Table 2. The derivations are given in Appendix 2.

Theoretical results taking autocorrelation into account are presented in Fig. 7, which shows (a) $\hat{P}_1$, (b) $\hat{P}_2$, (c) $\hat{Q}_2$, (d) $\widehat{Q_n^*}$, (e) $\widehat{N_D}(k \geq 1)$, (f) $\widehat{N_D}(k \geq 2)$, (g) $\widehat{DD}(k \geq 1)$, and (h) $\widehat{DD}(k \geq 2)$. Five different functions for the likelihood of a dry year occurring immediately after a dry year are considered. In two cases the likelihood is reduced if $p_D = p - 0.2$ (blue lines), $p_D = p - 0.1$ (light blue-colored lines)), in the second the likelihood does not depend on what happened the year before so that $p_D = p$ (black lines), while in the remaining two cases the likelihood is increased ($p_D = p + 0.1$ (light rust-colored lines),$p_D = p + 0.1$ (light rust-colored lines)).

Impacts of nonzero autocorrelation on the key statistics include the following:

- For all values of p, the probability of a single-year drought ($\hat{P}_1$; Fig. 7a) decreases as as p_D declines below p, whereas the probability increases as p_D increases above p. As the number of dry years stays the same, the proportion of years in MYDs (i.e., $\hat{Q}_2$; Fig. 7c) is therefore greater if $p_D > p$ (i.e., if the autocorrelation > 0), whereas the proportion is reduced if $p_D < p$
- The proportion of droughts of duration $k \geq n$ years is given by $\widehat{Q_n^*} = p_D^{n-1}$ (Table 2), and so $\widehat{Q_2^*} = p_D$. And since in every case considered p_D is a linear function of p, $\widehat{Q_2^*}$ is also a linear function of p (Fig. 7d). The results also show that $\widehat{Q_2^*}$ increases if $p_D > p$ (i.e., the proportion of droughts that are MY increases as p_D or the $a(1)$ increase), and that $\widehat{Q_2^*}$ increases if $p_D < p$ (Fig. 7d)
- The total number of droughts (i.e., $\widehat{N_D}(k \geq 1)$) decreases if $p_D > p$, and increases if $p_D < p$ (Fig. 7e)
- The impact of nonzero autocorrelation on the number of MYDs (i.e., $\widehat{N_D}(k \geq 2)$; Fig. 7f) is more complicated. For example, if $p_D > p$ then at $p = 0.3$ the average number of MYDs marginally increases from its $p_D = p = 0.3$ value of approximately 14 to approximately 15 if $p_D = 0.4$ or 0.5, whereas if $p = 0.7$ then the number of MYDs drops markedly from the $p_D = p = 0.7$ value of approximately 30 to approximately 16 ($p_D = 0.8$) or 13 ($p_D = 0.9$)
- The average duration of droughts (Fig. 7g), and the average duration of MYDs (Fig. 7h), increase if $p_D > p$, and decrease if $p_D < p$. In other words, persistence increases the average duration of droughts.

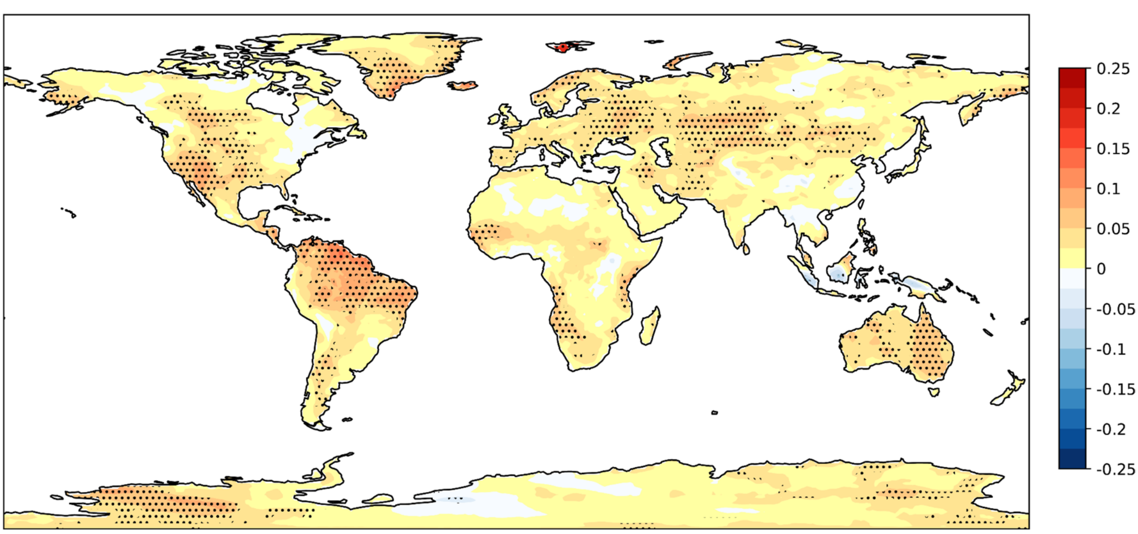

Fig. 6 The autocorrelation coefficient of annual precipitation at a 1-year lag. Stippling indicates that > 70% of models agree on the sign of the autocorrelation coefficient

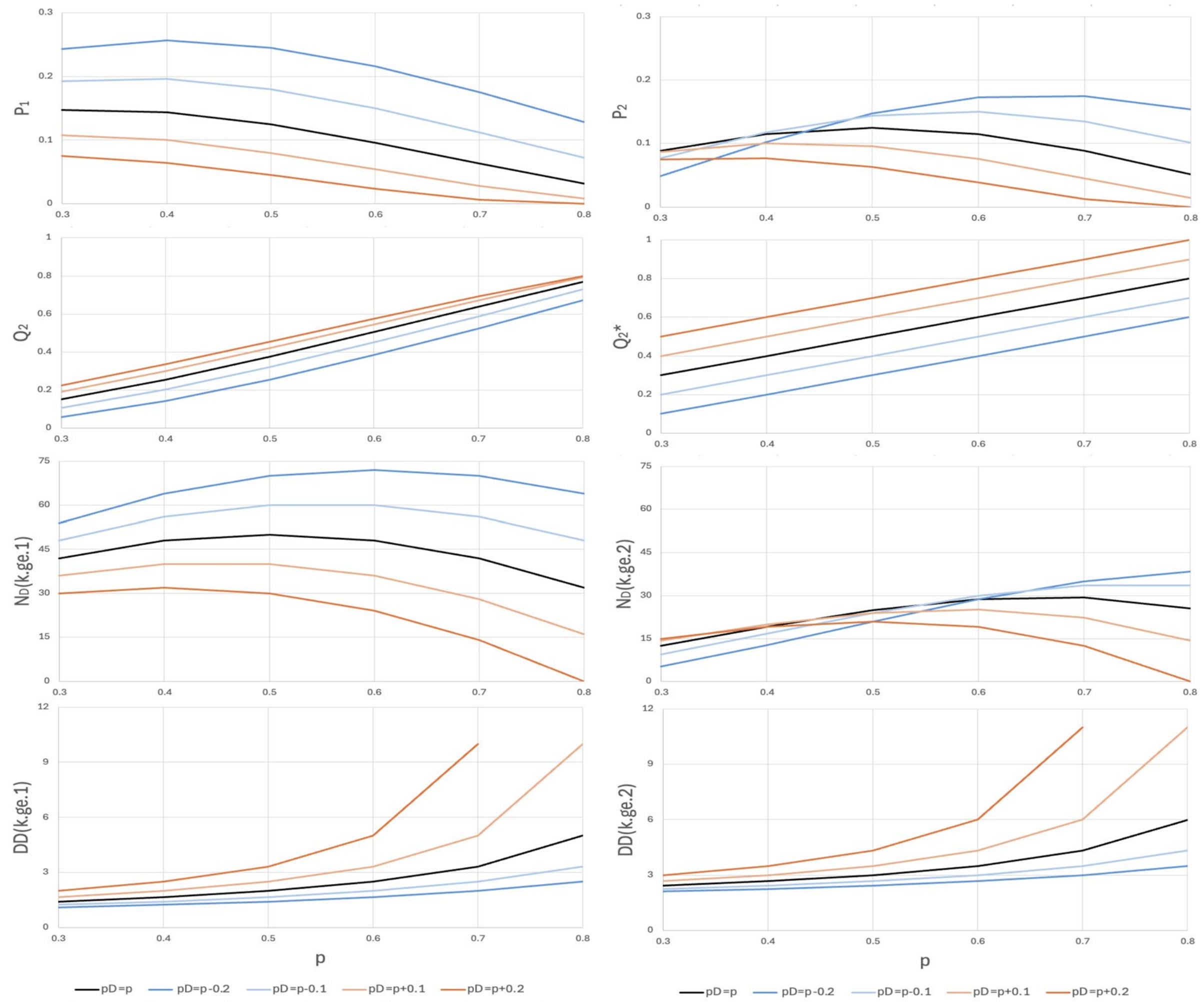

Fig. 7 Theoretical results taking autocorrelation into account: **a** $\hat{P}_1$, **b** $\hat{P}_2$, **c** $\hat{Q}_2$, **d** $\widehat{Q_n^*}$, **e** $\widehat{N_D}(k \geq 1)$, **f** $\widehat{N_D}(k \geq 2)$, **g** $\widehat{DD}(k \geq 1)$, and **h** $\widehat{DD}(k \geq 2)$. Four situations are considered in each panel: $p_D = p - 0.2$ (blue lines), $p - 0.1$ (light blue lines), p (black lines), $p + 0.1$ (light rust lines), and $p + 0.2$ (rust)

Figure 7 also indicates that neglecting persistence for values of p at or near 0.5 would tend to underestimate MMM values of Q_2, Q_2^*, and $DD(k \geq 2)$, while overestimating MMM values of $N_D(k \geq 2)$. This is indeed what happens at most locations across the globe (Fig. S2).

Maps showing the differences *Theory II—Theory I* (Fig. S5), based on the values obtained using the MMM value of $p(\mu)$ at each grid point, tend to be positive for $\hat{Q}_2(\mu) - Q_2(\mu), \hat{Q}_2^*(\mu) - Q_2^*(\mu), \widehat{DD}(k \geq 2;\mu) - DD(k \geq 2;\mu)$, but negative for $\hat{N}_D(k \geq 2;\mu) - N_D(k \geq 2;\mu)$, as expected from Fig. 7.

Maps showing $Q_2(\mu)$, $Q_2^*(\mu)$, $DD(k \geq 2;\mu)$ and $N_D(k \geq 2;\mu)$ over land (Fig. 8) show that *Theory II* values of $\hat{Q}_2(\mu), \widehat{Q_n^*}(\mu), \widehat{DD}(k \geq 2;\mu)$ and $\hat{N}_D(k \geq 2;\mu)$(second column from the left) are similar to the corresponding MMM (climate model) values of $Q_2(\mu), Q_2^*(\mu), DD(k \geq 2;\mu)$ and $N_D(k \geq 2;\mu)$ (left column), while the differences MMM-*Theory II* (third column) tend to be smaller than the MMM-*Theory I* differences (Fig. S2). The exception is $N_D(k \geq 2;\mu)$. In some places *Theory I* produces values closer to the MMM value, while in other places *Theory II* does. The differences in both cases (i.e., the differences between the MMM values and both *Theory I* and *Theory II* values) tend to be small ($<5\%$ of the MMM values).

The theoretical values of the statistics P_2, (b)Q_2, P_2^*, Q_2^*, $N_D(k \geq 2)$, and $DD(k \geq 2)$ are given in Table S3 in the case where $p = 0.5$ and where $p_D = 0.3, 0.5$, and 0.7.

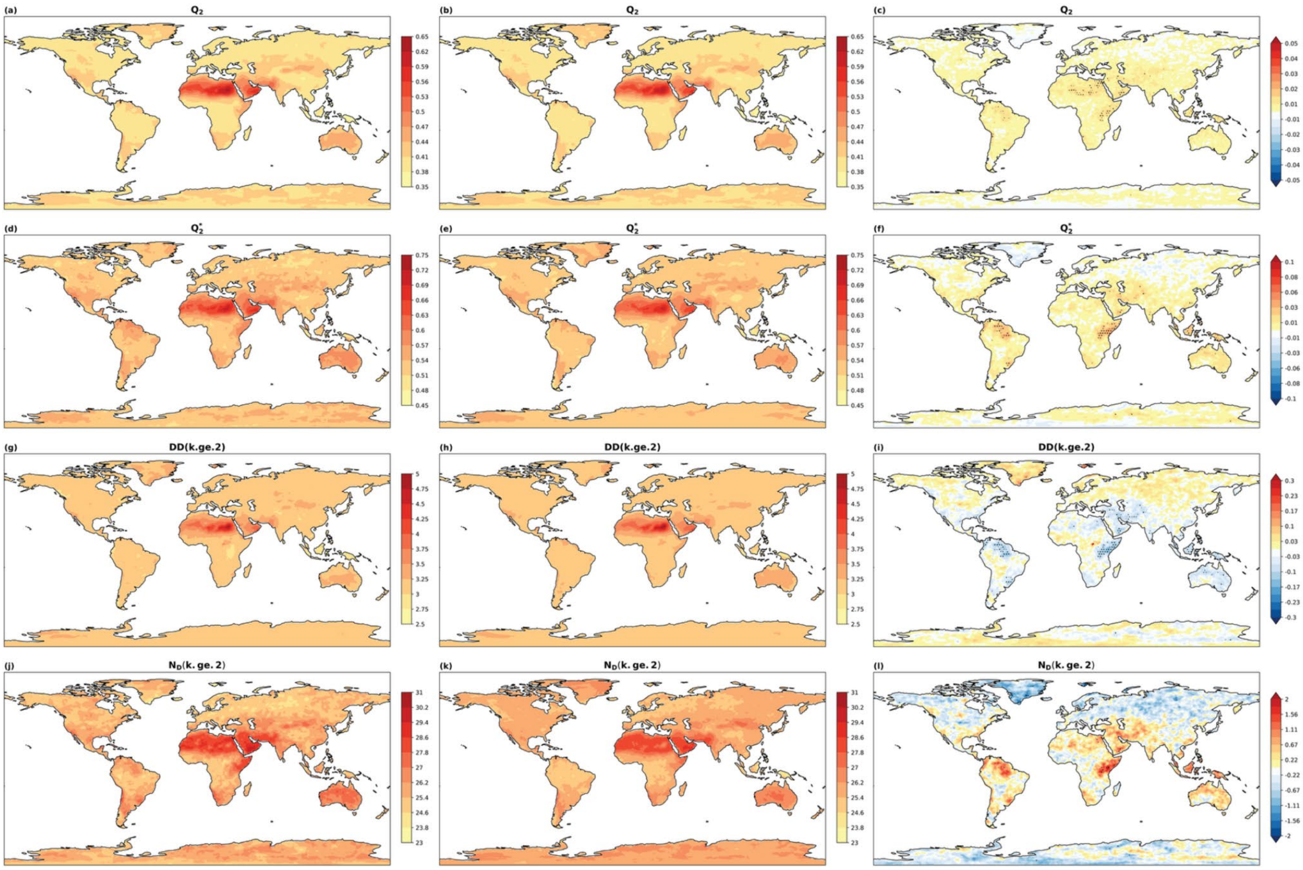

Fig. 8 MMM (climate model) values of **a** $Q_2(\mu)$, **b** $Q_2^*(\mu)$, **c** $DD(k \geq 2;\mu)$, **d** $N_D(k \geq 2;\mu)$ (left column), *Theory II* values of $\hat{Q}_2(\mu)$, $\widehat{Q_n^*}(\mu)$, $\widehat{DD}(k \geq 2;\mu)$ and $\hat{N}_D(k \geq 2;\mu)$ (middle column), and the differences MMM–*Theory II* (right column)

7 MYDs in the observations, theory, and models

In this section we examine observational estimates of the MYD statistics and compare observational estimates of the main MYD statistics with their corresponding theoretical and climate model estimates.

While the modelled and theoretical results strictly apply to preindustrial conditions, it is nevertheless of interest to see how the model and theoretical results compare with observations during the historical period. A direct comparison between MYDS in historically-forced runs with observations and theory will be made in a future study.

Observational estimates of (a) $Q_2(\mu)$, (b) $Q_2^*(\mu)$, (c) $DD(k \geq 2;\mu)$, (d) $N_D(k \geq 2;\mu)$ for 1900–2014 are given in Fig. 9.

The observational values of these statistics vary from region to region around values of approximately 0.5, 0.6, 3.75 and 27.

A comparison between observational and corresponding theoretical estimates is given in Fig. 10.

While *Theory I* (aqua curves) appears to do reasonably well in estimating the observational value of the MYD statistics (small circles) if $p \approx 0.5$, *Theory I* seems less appropriate for some other values of p. For example, *Theory I* tends to give lower than observed values for lower values of p for all four MYD statistics, while *Theory I* tends to give higher than observed values for higher values of p for all metrics.

Theory II (small crosses) more accurately simulates the observational results, providing a reasonably good fit in all cases. For example, in the majority of cases the observationally estimated mean value (black circles) lies within or very near the 95% confidence interval of the *Theory II* values. *Theory II* is least accurate for Q_2^*, $DD(k \geq 2)$ and $N_D(k \geq 2)$ at $p = 0.8$, where the observational/*Theory II* estimates of these statistics are approximately 0.6/0.8, 0.75/0.55, and 18/25 respectively.

Simulations of (a) $Q_2(p(\mu))$, (b) $Q_2^*(p(\mu))$, (c) $DD(k \geq 2;p(\mu))$, (d) $N_D(k \geq 2;p(\mu))$ from individual models are compared with their observational counterparts in

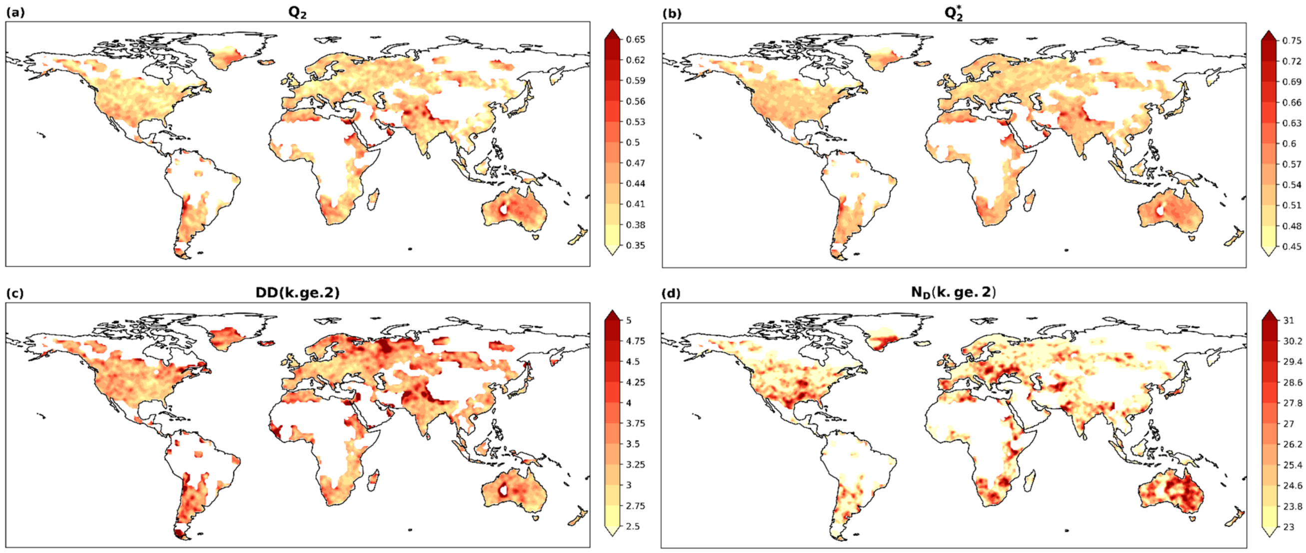

Fig. 9 Observational estimates of (a) $Q_2(\mu)$, (b) $Q_2^*(\mu)$, (c) $DD(k \geq 2;\mu)$, (d) $N_D(k \geq 2;\mu)$. Results are only shown over land where data (GPCC) is available in all years 1900–2014

Fig. 11. Each bar represents an areal average over the region for which observational data is available in all years during the period 1900–2014 (illustrated in Fig. 9).

In all cases the models give results that are similar to the observations and there is not much model-to-model variability in the statistics. While similar, systematic biases are evident: most models marginally underestimate Q_2^* and nearly all models tend to marginally underestimate Q_2, while $DD(k \geq 2)$ and $N_D(k \geq 2)$ tend to be overestimated. Figure 7 indicates that the sign of these biases tends to occur under *Theory II* if the models underestimate the magnitude of a positive $a(1)$. This is in fact the case: the value of $a(1)$ averaged over the same region used in Fig. 11 is given in Fig. 12. It shows that 37 out of the 38 models have an area-averaged $a(1)$ less than the observed value, and the vast majority of models have values that are much smaller than the observed value. On this metric, the two best performing models are CMCC-CM2-SR5 and CMCC-ESM2.

In summary, the climate models do not display enough temporal persistence, and this causes systematic biases in MYD statistics. Specifically, Q_2 and Q_2^* tend to be marginally too low, while $DD(k \geq 2)$ tends to be approximately 20% too low and $N_D(k \geq 2)$ tends to be approximately 10% too high, i.e., MYDs in models tend to occur too often and they tend to be too brief relative to historical data.

8 Sampling issues

The theoretical estimates derived above are for population averages, whereas the climate model and observational results are derived from finite samples. The latter will therefore typically be subject to sampling error.

While this paper is primarily focused on the average value of statistics, in this section we use Monte Carlo experiments to illustrate sampling error by estimating its magnitude in idealised cases. To keep things simple, we model precipitation as a Gaussian process with population mean μ and standard deviation σ. Five thousand samples each N years long are obtained using $p = 0.5$ as the threshold defining dry years, so that if the simulated precipitation < the median (= μ for Gaussian data) the year is considered dry. Three cases are considered: (i) $\mu = 500mm$, $\sigma = 50mm$ and $N = 200$, (ii) $\mu = 500mm$, $\sigma = 100mm$ and $N = 200, and (iii) \mu = 500mm$, $\sigma = 50mm$ and $N = 50$.

The 95% range for the sample mean values of the median precipitation, Q_2, Q_2^*, $DD(k \geq 2)$, and $N_D(k \geq 2)$ estimated from the simulations are approximately $\pm1.7\%, \pm11.3\%, \pm23.9\%$, 16.1% and $\pm19.7\%$ respectively in Case (i), $\pm3.5\%, \pm11.5\%, \pm24\%, \pm15.6\%$ and $\pm19.1\%$ respectively in Case (ii), and $\pm3.4\%, \pm23\%, \pm47\%, \pm34\%$ and $\pm39\%$ respectively in Case (iii). Here "%" is the percentage of the respective *Theory I* mean-value.

While uncertainty associated with the median precipitation is relatively small, uncertainty in the MYD statistics is not. Even with 200-year-long runs, the 95% ranges for Q_2^*, $DD(k \geq 2)$, and $N_D(k \geq 2)$ all exceed 15% of the population

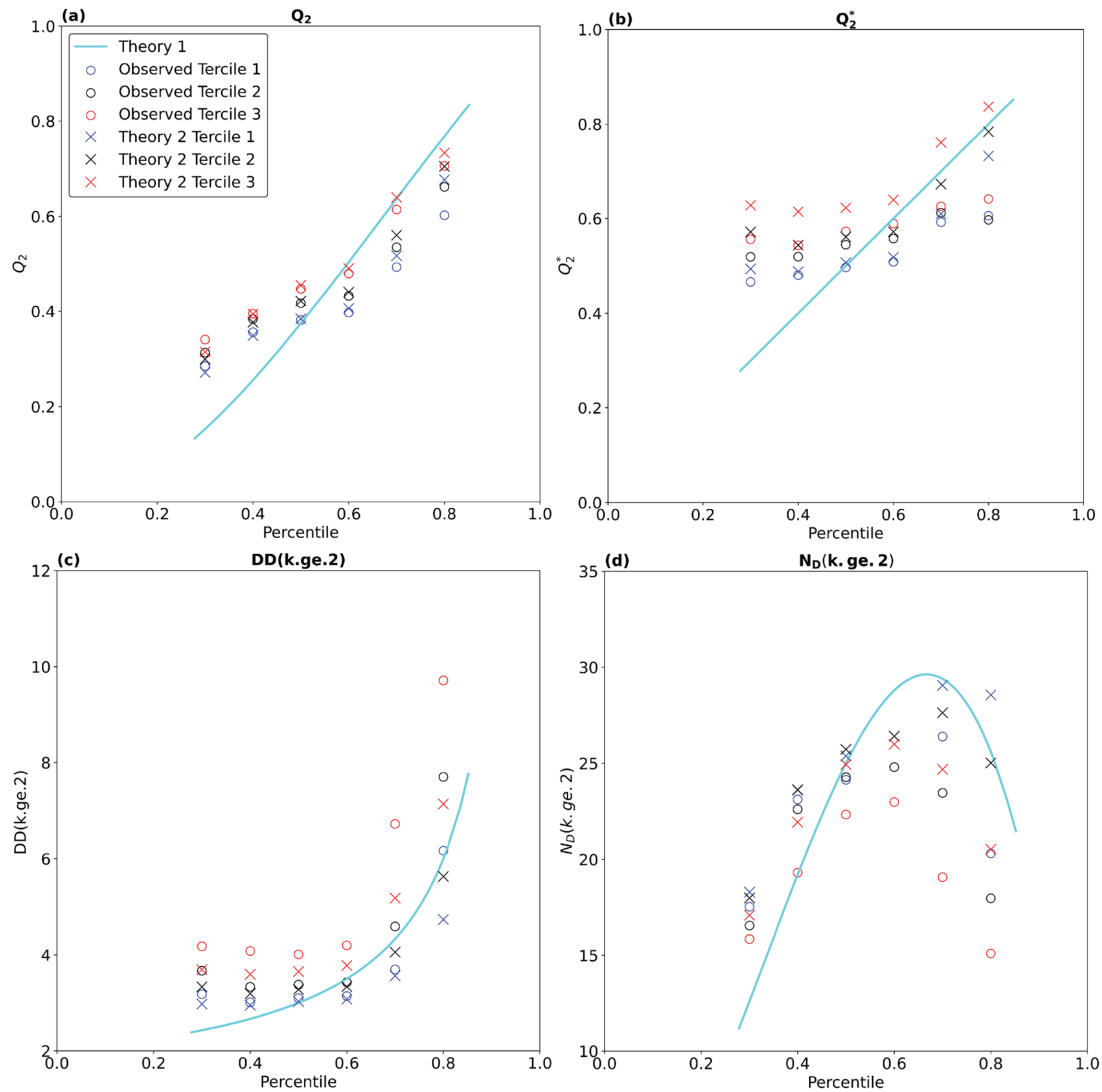

Fig. 10 **a** Q_2, **b** Q_2^*, **c** $DD(k \geq 2)$, **d** $N_D(k \geq 2)$ vs p. Each panel shows estimates of a MYD statistic using observations (small open circles), *Theory II* (crosses) and *Theory I* (aqua curves). The observational and *Theory II* data for $p=0.3$, for example, is based on MYD data at locations where $p(\mu)$ is in the range [0.2, 0.4). This subset of the data was further subdivided according to the value of p_D. One smaller subset consists of MYD values for which p_D is in the bottom tercile of the values of p_D evident in the larger subset and is represented by the blue circles and crosses. Another smaller subset consists of the corresponding cases in which p_D is in the middle tercile (black circles and crosses) and the third and final smaller subset consists of the corresponding values of the MYD statistic at locations where p_D is in the top tercile (red circles and crosses). The values presented are the corresponding average values of the MYD statistic in each smaller subset. The *Theory II* values were estimated using values of $p(\mu)$ (and $p_D(\mu)$ for *Theory II*) obtained from the observations

means (i.e., the mean corresponding to an infinitely large sample) in all cases. The impact of doubling precipitation variability (compare Case (ii) with Case (i)) has only a marginal impact on the 95% range of the sample means. Reducing the number of years to 50 (compare Case (iii) with Case (i)), on the other hand, approximately doubles uncertainty in all the MYD statistics. We hope to investigate these issues in more detail in a future study.

9 Summary and discussion

We derived theoretical estimates based on simple stochastic models for a range of key characteristics of droughts, including multi-year droughts (MYDs). The MYDs considered here consist of uninterrupted sequences (UISs) of years with annual precipitation below a given threshold. The solutions are given in terms of the percentile of the threshold, p. We

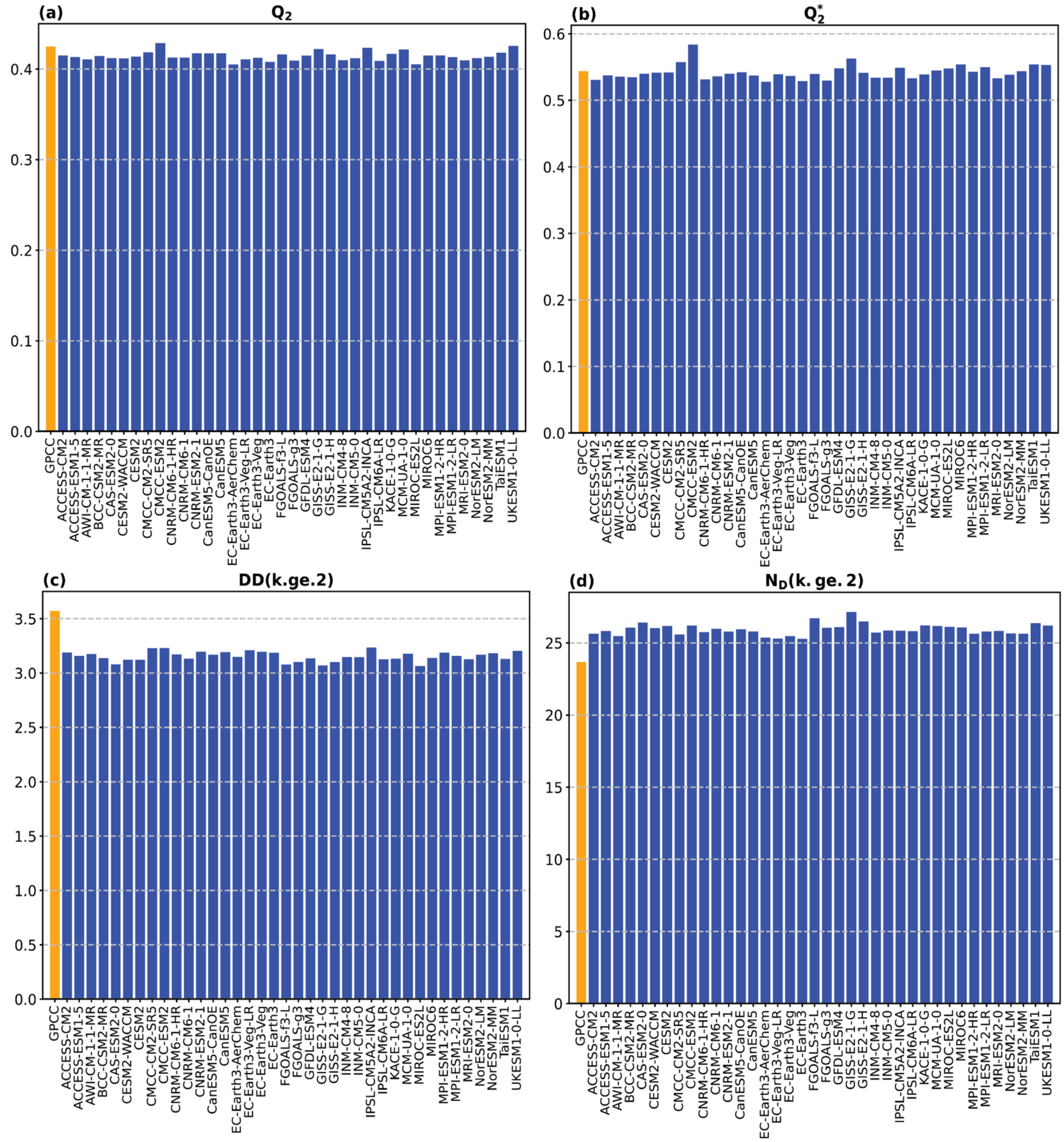

Fig. 11 Observed and individual model values of **a** $Q_2(p(\mu))$, **b** $Q_2^*(p(\mu))$, **c** $DD(k \geq 2;p(\mu))$, **d** $N_D(k \geq 2;p(\mu))$. $p(\mu)$ is the percentile of the mean. Each bar represents an areal average over the entire region in which observational data is available in all years during the period 1900–2014 (illustrated in Fig. 9)

also examined the way in which the solutions depend on n, the duration of the drought.

Formulae were obtained for eight quantities: the proportion of years that are part of a MYD of (i) duration (a) n years (P_n) or (b) $\geq n$ years (Q_n), (ii) the proportion of droughts that have a duration of (a) n years $\left(P_n^*\right)$ or (b) $\geq n$ years $\left(Q_n^*\right)$, and both (ii) the number (N_D) and (iv) the average length of MYDs of duration (a) n years or (b) $\geq n$ years ($DD(k \geq n)$).

Two sets of formulae were obtained. The first set (*Theory I*; white noise; Sect. 3) assumes that the temporal autocorrelation in precipitation from year to year is zero, whereas the second set (*Theory II*; Sect. 6) is derived taking the possibility of non-zero temporal autocorrelation ($a(1)$) into account. Cases where $a(1) > 0$ correspond to red noise processes. The statistical properties and the corresponding formulae (with corresponding Equation numbers) are summarized in Table 1 (*Theory I*) and Table 2 (*Theory II*).

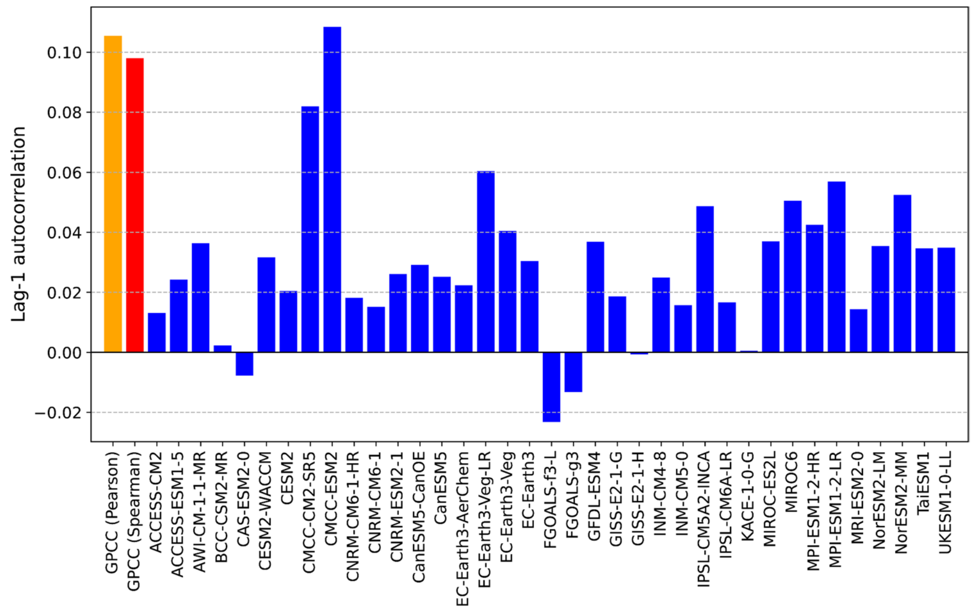

Fig. 12 Observed (GPCC; rust and red) and individual model values (blue) of $a(1)$ averaged over the region in which observational data is available in the GPCC data set in all years 1900–2014. The observational value is calculated using Pearson (rust) and Spearman (red) formulae. The individual model values were obtained using the Pearson formula only

If the autocorrelation is zero, then the functional dependence of the statistics of MYDs on p for $n = 2$ years are as follows:

Q_2^*: linear; monotonic increasing; $Q_2^*(p = 1/2) = 0.5$.

Q_2: weakly nonlinear; monotonic increasing; $Q_2(p = 1/2) = 0.375$.

$DD(k \geq 2)$: strongly nonlinear; monotonic increasing; $DD(k \geq 2) = 3$ years.

$N_D(k \geq 2)$: strongly nonlinear; monotonic increasing for $p < 2/3$, monotonic decreasing for $p > 2/3$; $N_D(k \geq 2) = 25$ (if $N = 200$ years).

We found that at most locations over land the MMM autocorrelation coefficient at one year lag (i.e., $a(1)$) is positive, so that the mean tends to be greater than the median. If $a(1)$ is positive then *Theory II* shows that Q_2, Q_2^* increase relative to their *Theory I* values. This makes intuitive sense as given a dry year the likelihood of a multi-year drought is clearly higher given rainfall persistence. Similarly, $DD(k \geq 2)$ increases. This too makes sense as given a dry sequence the likelihood of an even longer sequence increases given rainfall persistence. The situation for $N_D(k \geq 2)$ is more complicated. For example, if $a(1) > 0$ then $N_D(k \geq 2)$ gets smaller if $p \geq 0.45$, but get marginally larger if $p < 0.4$.

The value of each statistic was estimated for droughts in climate models (estimated using single 200-year runs from 38 CMIP6 models under preindustrial control conditions) that consist of UISs of years in which annual precipitation is below average. We found that the modeled values tended to agree well with the theoretical results, with *Theory II* tending to provide somewhat closer agreement with the climate model values than does *Theory I*. The theoretical values were obtained by choosing a threshold $p = p(\mu)$, where μ is the mean. We showed that $p = p(\mu)$ tends to be somewhat greater than 0.5 over the globe, and that regional variations in the value of the eight statistics tend to be well-explained by the formulae provided that regional variations in $p(\mu)$ are considered.

We also examined MYDs in observational data and compared the observational results with theoretical and climate model results. We found that *Theory II* tends to give a much better fit to the observational MYD statistics than does *Theory I*. This contrasted with the result that *Theory II* only gives a marginally better fit to the MMM results. This contrast arises because the observational precipitation tends to display larger temporal persistence than the models and this results in MYDs in the models that tend to be too brief and occur more often than they do in the observations. The extent to which these differences arise because we have compared pre-industrial runs with historical data will be the subject of a future study.

Note that the focus of this paper has been on long-term averages. Finite samples will of course be subject to sampling error. We highlighted this in Sect. 7, which shows that even with 200-year samples, sampling error can cause a considerable amount of uncertainty in some MYD statistics.

Our results strictly apply to MYDs consisting of UISs of years in which annual precipitation falls below a given threshold. We recognise that numerous other definitions of drought and MYD are available. While we focused on a particular type of MYD, we have established that the average value of important statistical properties of at least one kind of MYDs can be well-understood in models and observations in terms of surprisingly simple stochastic models.

Further research could be conducted into, e.g., the theory of MYDs using other definitions of drought, the extent to which the theories can be used to understand anthropogenically-forced changes in MYDs, and the sampling issues highlighted in Sect. 7.

Appendix 1: Formulae for $S = \sum_{k=1}^{\infty} kp^k$ and $S_{n-1} = \sum_{k=1}^{n-1} kp^k S_{n-1} = \sum_{k=1}^{n-1} kp^k$

While the derivations x here are readily available on the internet, they are provided for completeness.

If $U_{n-1} = 1 + p + p^2 + \dots + p^{n-1} = \sum_{k=1}^{n} p^{k-1}$ then

$$\frac{dU_{n-1}}{dp} = 0 + 1 + 2p + 3p^2 + \dots + (n-1)p^{n-2}$$
$$= \sum_{k=1}^{n-1} kp^{k-1}, \textit{ and so } \frac{dU_{n-1}}{dp}$$
$$= p + 2p^2 + \dots + (n-1)p^{n-1}$$
$$= \sum_{k=1}^{n-1} kp^k = S_{n-1},$$

, Now $pU_{n-1} = p + p^2 + p^3 + p^4 + \dots + p^{n-1} + p^n = U_{n-1} - 1 + p^n$, and so

$U_{n-1} = (1 - p^n)/(1 - p)$.

Since $d(f(x)/g(x)) = \frac{f'g - fg'}{g^2}$ then $\frac{dU_{n-1}}{dp} = \frac{1-p^n-np^{n-1}(1-p)}{(1-p)^2}$, and so

$$S_{n-1} = p\frac{dU_{n-1}}{dp} = p\left[\frac{1 - p^n - np^{n-1}(1-p)}{(1-p)^2}\right], \quad (13)$$

and as

$$n \to \infty, S_{n-1} \to S = \frac{p}{(1-p)^2}. \quad (14)$$

Appendix 2: Derivation of formulae for droughts with non-zero autocorrelation

The MMM value of $p_D - p$ is given in Fig. 13. MMM values of p_D exceed p in most locations, consistent with the generally positive values of $a(1)$ in Fig. 6. This indicates that taking autocorrelation into account should provide formulae that more closely approximate the climate model estimates. The derivations follow Fig. 13.

$\hat{P}_n$ *and* $\hat{Q}_n$.

We again assume that the probability of a randomly chosen year is dry is p. However, in order to account for nonzero autocorrelation we further suppose that the probability that a dry year occurs immediately after a dry year is p_D, and if $p_D \neq p$ then there is nonzero autocorrelation. We further suppose that that the probability of a wet year immediately after a wet year is given by q_W, and the probability of a dry year immediately after a wet year is p_W. In this case (in analogy with the derivation of P_1 in Sect. 3)

$$\hat{P}_1 = qp_W q_D = 1.q.p_W p_D^0 q_D,$$

where "^" indicates that the variable is for the case when p_D is not necessarily equal to p.

Similarly the probability of randomly selecting a year and finding that it is part of a two-year drought is $\hat{P}_2 = 2.qp_W p_D^1 q_D$, and the probability that it is part of a three-year drought is $\hat{P}_3 = 3.qp_W p_D^2 q_D$, from which we can deduce that

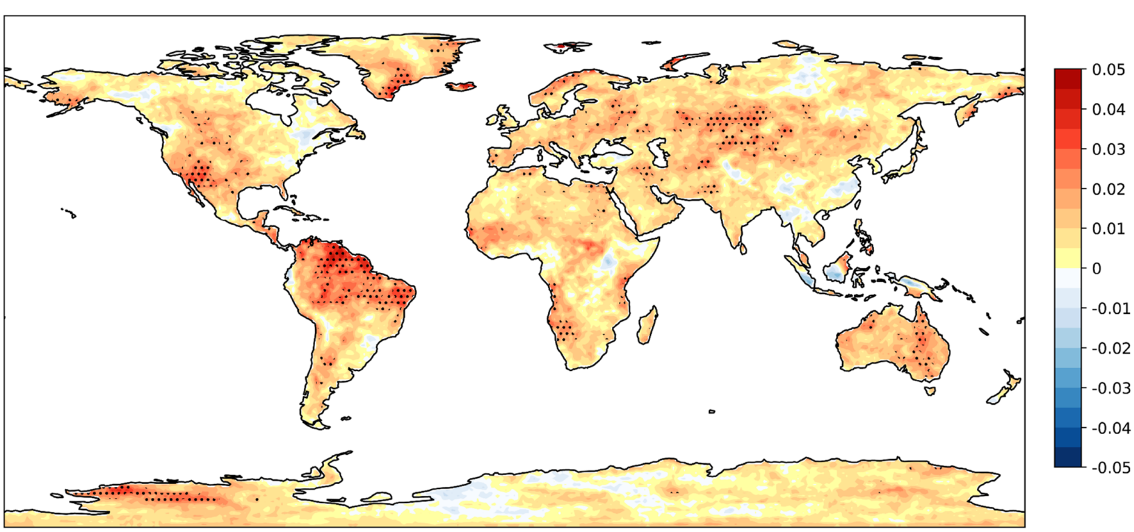

Fig. 13 MMM value of $p_D - p$. Stippling indicates that > 70% of models agree on the sign of $p_D - p$

$$\hat{P}_n = nqp_W p_D^{n-1} q_D. \tag{15}$$

We wish to obtain formulae for the average value of the various metrics in terms of p_D. Armed with the resulting formulae would then allow us to estimate the value of the various statistics given three things: n, p, and p_D. However, to accomplish this, we need to relate both p_W and q_D to p_D, as they both appear in Eq. (15).

To obtain an equation for p_W, consider the probability of randomly choosing two successive years and (a) the first year is wet and the second year is dry and (b) the first and second years are dry. Notice that (a) and (b) collectively exhaust the possibilities for the first year, so the probability of (a) or (b) is simply the probability of randomly selecting a year and finding that it is wet, i.e., q. This means that $p = qp_W + pp_D$, and so

$$p_W = p\left(1 - p_D\right)/q. \tag{16}$$

To obtain an equation for q_D, consider the probability of randomly choosing two successive years and (a) the first and second years are dry and (b) the first year is dry, but the second year is wet. Notice that (a) and (b) collectively exhaust the possibilities for the second year, so the probability of (a) or (b) is simply the probability of randomly selecting a year and finding that it is dry, i.e., p. This means that.

$p = pp_D + pq_D$, and so

$$q_D = 1 - p_D. \tag{17}$$

Using Eqs. (16) and (17) in (15) then gives

$$\begin{aligned}\hat{P}_n &= nq\left[\frac{p\left(1-p_D\right)}{q}\right]p_D^{n-1}(1-p_D)\\ &= \left(\frac{p}{p_D}\right)np_D^n(1-p_D)^2\\ &= \left(\frac{p}{p_D}\right)P_n\left(p_D\right),\end{aligned} \tag{18}$$

and given that $\hat{P}_n = \left(\frac{p}{p_D}\right)P_n\left(p_D\right)$, we can immediately write

$$\hat{Q}_n = \left(\frac{p}{p_D}\right)Q_n\left(p_D\right). \tag{19}$$

$\widehat{P_n^*}$ and $\widehat{Q_n^*}$.

In analogy with Sect. 3, the average number of droughts of duration n years is $N\hat{P}_n/n$, and so the average number of droughts of any size is $N\sum_{k=1}^{\infty}\frac{\hat{P}_k}{k}$. So the proportion of all droughts that have duration n years, $\widehat{P_n^*}$, is given by

$$\widehat{P_n^*} = \frac{N\hat{P}_n/n}{N\sum_{k=1}^{\infty}\frac{\hat{P}_k}{k}} = \left(1-p_D\right)p_D^{n-1} = P_n^*\left(p_D\right). \tag{20}$$

Since $\widehat{P_n^*} = P_n^*\left(p_D\right)$ we can immediately deduce that

$$\widehat{Q_n^*} = Q_n^*\left(p_D\right). \tag{21}$$

$\hat{N}_D(n)$ and $\hat{N}_D(k \geq n)$.

In analogy with Sect. 3,

$$\hat{N}_D(n) = N\hat{P}_n(n)/n = \frac{p}{p_D}Np_D^n(1-p_D)^2 = \left(\frac{p}{p_D}\right)N_D(n). \tag{22}$$

The number of droughts of duration $k \geq n$ is therefore given bywhich can be written as

$$\hat{N}_D(k \geq n) = \sum_{k=n}^{\infty}\hat{N}_D(n) = \frac{Np(1-p_D)^2}{p_D}\sum_{k=n}^{\infty}p_D^n = Np\left(1-p_D\right)/p_D^{n-1}$$

$$\hat{N}_D(k \geq n) = \frac{p}{p_D}N\left(1-p_D\right)p_D^n = \frac{p}{p_D}N_D\left(k \geq n;p_D\right). \tag{23}$$

$\widehat{DD}(k \geq n)$

In analogy with Sect. 3, the average duration of droughts in a sample of N years is given by

$$\begin{aligned}\widehat{DD}(k \geq n) &= \frac{\textit{sum of all drought durations in sample of duration } k \geq n}{\textit{average number of droughts of duration } k \geq n \textit{ in sample}}\\ &= \frac{N\sum_{k=n}^{N}\hat{P}_k}{\hat{N}_D(k \geq n)} = \frac{(1-p_D)}{p_D^n}\sum_{k=n}^{N}kp_D^k.\end{aligned}$$

Using the formula obtained earlier for this summation if N is large we get

$$\widehat{DD}(k \geq n) \approx n - 1 + 1/\left(1 - p_D\right) = DD\left(k \geq n;p_D\right). \tag{24}$$

Note that the average number of days in a drought of duration $\geq n$ years is given by the product of $\hat{N}_D(k \geq n)$ and $\widehat{DD}(k \geq n)$, i.e., $\hat{N}_D(k \geq n)\widehat{DD}(k \geq n)$.

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Data availability All CMIP6 data used in this study is available from https://esgf.nci.org.au/projects/esgf_nci/. Python scripts used to calculate the drought metrics are available from https://github.com/nickywright/aus-droughts-last-millennium (Wright and Falster 2024).

Declarations

Conflict of interest The authors declare that they have no (i) conflicts of interest or (ii) known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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