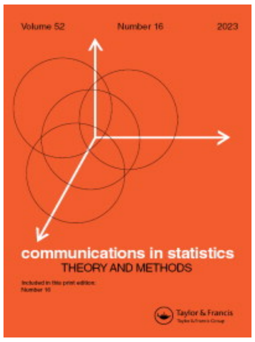


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# Estimation of the slope parameter in a linear regression model under a bounded loss function

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**ABSTRACT**

The estimation of the slope parameter of a simple linear regression model in the presence of nonsample prior information under the reflected normal loss function is considered. Usually, the traditional estimation methods such as the least squared (LS) error are used to estimate the slope parameter. Sometimes the researcher has information about the unknown slope parameter from experience as a point guess, the nonsample prior information. In this paper, the shrinkage pretest estimators are introduced and their risk functions are derived under the reflected normal loss function. Several methods of finding distrust coefficient of the shrinkage pretest estimators are proposed. The behavior of the estimators are compared using a simulation study. The results show that the shrinkage pretest estimator outperforms the LS estimator when nonsample prior information is close to the real value. A real data set is analyzed for illustrating the results.

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## 1. Introduction

The squared error loss (SEL) function is popularly used for estimating the unknown parameter in decision theory because of its mathematical and interpretational convenience. Due to the symmetric nature, the use of SEL function may not be appropriate, when positive and negative errors have different consequences. Varian (1975) and Zellner (1986) proposed an asymmetric loss function known as the LINEX loss function. This loss function assign unequal weights to the underestimation and overestimation by assigning an appropriate value of the shape parameter (Hoque, Wesolowski, and Hossain 2018).

The SEL and LINEX loss functions are symmetric and asymmetric loss functions, respectively, but both are unbounded. Sometimes in practice it is necessary to use bounded loss functions to estimate parameters. The first motivation for the reflected normal loss (RNL) function based estimation came from Taguchi (1986) who used a quadratic loss function to motivate and illustrate losses to society associated with departures from the target in industrial applications. Spiring (1993) modified this loss function approach using an inverted normal probability density function in an attempt to

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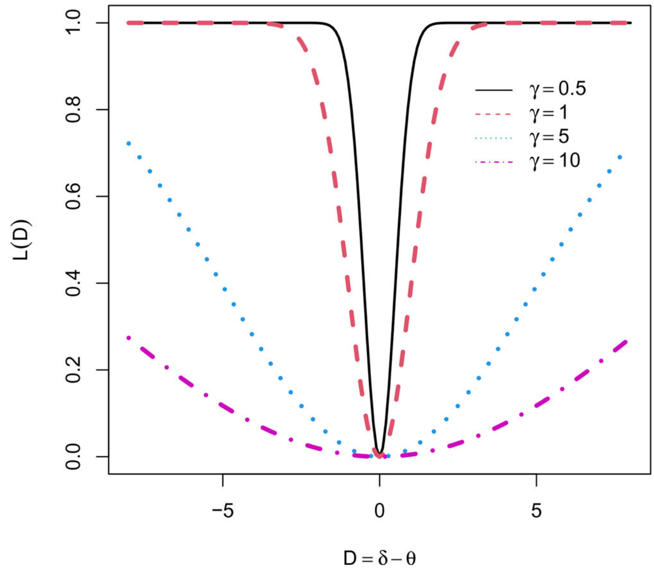


**Figure 1.** Plot of the reflected normal loss function for different values $\gamma = 0.5, 1, 5, 10$.

provide a more reasonable assessment of economic loss. The RNL function provides an alternative representation of the loss associated with the quality characteristic. An interesting aspect of the RNL function is its shape parameter, which is used much like the intersection point in the modified quadratic loss function that permits a smooth function rather than a piecewise function. Also, The SEL and LINEX loss functions have an infinite maximum value which isn't always appropriate. So we used the reflected normal loss function to estimate parameters, which is the bounded loss function. Spiring (1993) and Spiring and Yeung (1998) proposed the reflected normal loss (RNL) function of the form

$$L(D) = k\left(1 - e^{-\frac{D^2}{2\gamma^2}}\right), \tag{1}$$

where $D = \delta - \theta$, $\delta$ is an estimator of parameter $\theta$, $k > 0$ is the maximum loss and $\gamma > 0$ is the shape parameter. The loss function (1) is bounded function of $D$, and is essentially a normal density flipped upside down, whence its name. Figure 1 shows the reflected normal loss function for $k = 1$ and the different values of $\gamma = 0.5, 1, 5, 10$. The features of the loss function (1) are:

1. $L(D)$ is boundary function.
2. $L(D)$ is infinitely derivative and therefore continuous.
3. $L(D)$ has a minimum of zero in $D = 0$ and in the interval $(-\infty, 0)$ is an descending function and in the interval $(0, \infty)$ is a ascending function. Also $\lim_{D\to\pm\infty} L(D) = k$.
4. Loss function (1) is special mode of Hellinger distance by

$$H^2\big(f(\cdot \mid \theta), (\cdot \mid \delta)\big) = \frac{1}{2}E\left\{\left(\frac{\sqrt{f(x|\theta)} - \sqrt{f(x|\delta)}}{\sqrt{f(x|\theta)}}\right)^2\right\}, \tag{2}$$

where $k = 1$ and $\gamma = 2\sigma$ and $f(\cdot|\theta)$ is the probability density function (p.d.f) of $N(\theta, \sigma^2)$. The loss function (1) is investigated by Towhidi and Behoodian (2001), Naghizadeh

Qomi, Nematollahi, and Parsian (2012), and Naghizadeh Qomi and Kiapour (2016) for various problems of estimation. Clearly the value of $k > 0$ does not have any influence on our results, therefore without loss of generality, we shall set $k = 1$ in the rest of the article.

In some situations, the experimenter has some prior information about the parameter of interest in the form of a point guess value. To utilize this guess value, some shrunken techniques are proposed. For some works in estimation of unknown parameter, see Khan and Saleh (2001), Prakash and Singh (2008), and Kiapour and Naghizadeh Qomi (2016), among others. In the problem of the parameter estimation in linear regression model using nonsample information, Khan, Hoque, and Saleh (2002, 2005) investigate the estimation of the slope and intercept parameters using shrinkage pretest estimator (SPE) under the SEL function. Hoque, Khan, and Wesolowski (2009) and Hoque and Hossain (2012) studied the performance of the preliminary test estimator (PTE) of the slope and intercept parameters under the LINEX loss function, respectively. Hoque, Wesolowski, and Hossain (2018) studied the performance of the shrinkage estimators under the LINEX loss function.

In this paper, we use the RNL function for estimation of the slope parameter of a simple linear regression model. The paper is organized as follows. The RNL is discussed in Section 1. The model and preliminaries are presented in Section 2. The SPE of the slope parameter is presented in Section 3. Various methods are applied to obtain SPEs in Section 4. The risk functions of the proposed estimators are calculated under the RNL function in Section 5. The numerical comparison of the risk functions of the estimators is performed using a simulation study in Section 6. A real data set is used for illustration of the results in Section 7. Finally, some concluding remarks and discussions are presented in Section 8.

## 2. The model and preliminaries

Consider the simple linear regression model

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, 2, ..., n, \tag{3}$$

where $y_i$ is the response variable, $x_i$ is predictor, $\beta_0$ and $\beta_1$ are the unknown intercept and slope parameters respectively, and $\varepsilon_i$'s are i.i.d. random variables distributed as $N(0, \sigma^2)$. Classical methods such as the least squared error method are usually estimate regression coefficients using exclusive sample information. The least square (LS) estimator of $\beta_1$ is $\tilde{\beta}_1 = S_{xx}^{-1} S_{xy}$, where $S_{xx} = \Sigma_{i=1}^n (x_i - \bar{x})^2$ and $S_{xy} = \Sigma_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$. $\tilde{\beta}_1$ is an unbiased estimator of $\beta_1$ and the sampling distribution of $\tilde{\beta}_1$ is normal with mean $\beta_1$ and variance $E(\tilde{\beta}_1 - \beta_1)^2 = \frac{\sigma^2}{S_{xx}}$. Assume that uncertain nonsample prior information about the value of the slope, $\beta_{10}$, is available either from previous study or from practical experience of researchers or experts. Nonsample prior information can be tested by testing the null hypothesis $H_0 : \beta_1 = \beta_{10}$ against $H_1 : \beta_1 \neq \beta_{10}$ to remove uncertainty. The likelihood ratio test (LRT) for testing the null hypothesis is given by the test statistic

$$T_\nu = \frac{\sqrt{S_{xx}}(\tilde{\beta}_1 - \beta_{10})}{S_n}, \tag{4}$$

where $S_n^2 = \frac{1}{n-2}\Sigma_{i=1}^n (y_i - \hat{y}_i)^2$ is an unbiased estimator of $\sigma^2$ in which $\hat{y}_i = \tilde{\beta}_0 + \tilde{\beta}_1 x_i$ and $\tilde{\beta}_0 = \frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}$ is the least square (LS) estimator of $\beta_0$ and $T_\nu$ has a Student's t distribution with $\nu = n-2$ degrees of freedom. Under $H_1$, it follows a noncentral Student's t distribution with $\nu$ degrees of freedom and the non-centrality parameter $\frac{1}{2}\Delta^2$, where

$$\Delta^2 = \frac{S_{xx}(\beta_1 - \beta_{10})^2}{\sigma^2}. \tag{5}$$

## 3. Shrinkage pretest estimator of the slope

Following Thompson (1968) we consider a point shrinkage estimator of $\beta_1$ as

$$\hat{\beta}_1(d) = d\tilde{\beta}_1 + (1-d)\beta_{10}, \tag{6}$$

where $0 \le d \le 1$ is the shrinkage factor representing the degree of distrust in the null hypothesis, $H_0 : \beta_1 = \beta_{10}$ which is specified by the experimenter according to his belief in $\beta_{10}$. If $d = 1$ (complete distrust), then we use the sample data, while for $d = 0$, we use the nonsample information only. The value $(1-d)$ is proportional to the experimenters confidence in $\beta_{10}$. If $0 \le d \le 1$, the degree of distrust is an intermediate value which results in an interpolated value between $\beta_{10}$ and $\tilde{\beta}_1$.

Following Bancroft (1944) we consider a SPE of the slope parameter as

$$\begin{aligned} \hat{\beta}_1^{SPE}(d) &= \begin{cases} \hat{\beta}_1(d) & F_{1,\nu} < F_\alpha \\ \tilde{\beta}_1 & F_{1,\nu} > F_\alpha \end{cases} \\ &= \tilde{\beta}_1 - (1-d)(\tilde{\beta}_1 - \beta_{10}) I(F_{1,\nu} < F_\alpha), \end{aligned} \tag{7}$$

where $F_{1,\nu} = T_\nu^2$ follows a $F$ distribution with $(1,\nu)$ degrees of freedom, $F_\alpha$ is the $\alpha$ quantile of a central F distribution with $(1,\nu)$ degrees of freedom and $I(.)$ is the indicator function assigning 0 or 1. For $d = 0$, the SPE reduces to PTE proposed by Hoque, Khan, and Wesolowski (2009). Note that $T_\nu^2$ under $H_1$, follows a non-central F distribution with $(1,\nu)$ degrees of freedom and the non-centrality parameter $\frac{1}{2}\Delta^2$.

## 4. Different SPEs of $\beta_1$

An important issue for the pretest estimator is the proper selection of the distrust coefficient, $d$. In this section, we apply several methods for finding the distrust coefficient to obtain SPEs.

The first SPE of $\beta_1$ is $\hat{\beta}_{11}^{SPE} = \hat{\beta}_1^{SPE}(d_1)$, where $d_1$ is found by minimizing the risk function of SPE, i.e.,

$$d_1 = \arg\min_{d \in [0,1]} R(\hat{\beta}_1^{SPE}(d), \beta_1). \tag{8}$$

The critical region for testing the null hypothesis $H_0 : \beta_1 = \beta_{10}$ versus $H_1 : \beta_1 \ne \beta_{10}$ is $|t| \ge t_{\alpha/2}$, where $t = \frac{(\tilde{\beta}_1 - \beta_{10})\sqrt{s_{xx}}}{s_n}$ is the observed value of $T_\nu$ under the null hypothesis. The significance value ($p$-value) of this test indicates how much the null hypothesis is supported by the data. A large value of the $p$-value indicates that $\beta_1$ is close to $\beta_{10}$. Also the two-sided $p$-value is calculated as

$$\begin{aligned} p-\text{value} &= 2\min\{P(T_\nu > t|H_0), P(T_\nu < t|H_0)\} \\ &= 2\min\{H_\nu(t), 1-H_\nu(t)\}, \end{aligned} \tag{9}$$

where $H_\nu(.)$ is the cumulative distribution function (c.d.f) of the Student's t distribution with $\nu$ degrees of freedom. The second SPE is $\hat{\beta}_{12}^{SPE} = \hat{\beta}_1^{SPE}(d_2)$ where $d_2 = 1 - p -$ value. Notice that $p$-value tends to zero with increasing $n$. In other words, $d_2$ is close to 1. The idea of using this method is to put more weight on $\tilde{\beta}_1$ when more data is available.

We can use the square root of $p$-value for stronger support of $\beta_{10}$ and find the third SPE as $\hat{\beta}_{13}^{SPE} = \hat{\beta}_1^{SPE}(d_3)$, where $d_3 = 1 - \sqrt{p - \text{value}}$.

The value of $d$ can be calculated using mathematical expectation of $p$-value. We consider the significant value of the hypothesis test as $p = 2min\{H(t), 1-H(t)\}$. Since in repeated random sampling the $p$-value is a random variable, so the observed value $p$ can be considered to be a value of a random variable

$$P = 2min\{H(T), 1-H(T)\}, \tag{10}$$

where $T = \frac{(\tilde{\beta}_1 - \beta_{10})\sqrt{S_{xx}}}{S_n} = \Delta + \frac{(\tilde{\beta}_1 - \beta_1)\sqrt{S_{xx}}}{S_n}$, and $T - \Delta \sim t_\nu$. To find the p.d.f of $P$ we first note the c.d.f.

$$\begin{aligned} G_{\beta_1}(p) &= P_{\beta_1}(P \le p) = P_{\beta_1}\left(2\min\{H_\nu(t), 1-H_\nu(t)\} \le p\right) \\ &= 1 - P_{\beta_1}\left(2\min\{H_\nu(t), 1-H_\nu(t)\} \ge p\right) \\ &= 1 - P_{\beta_1}\left(\frac{p}{2} \le H_\nu(t) \le 1 - \frac{p}{2}\right) \\ &= 1 - \left[P_{\beta_1}\left(H_\nu(t) \le 1 - \frac{p}{2}\right) - P_{\beta_1}\left(H_\nu(t) \le \frac{p}{2}\right)\right] \\ &= 1 + P_{\beta_1}\left(T \le H_\nu^{-1}\left(\frac{p}{2}\right)\right) - P_{\beta_1}\left(T \le H_\nu^{-1}\left(1 - \frac{p}{2}\right)\right) \\ &= 1 + P_{\beta_1}\left(T - \Delta \le H_\nu^{-1}\left(\frac{p}{2}\right) - \Delta\right) - P_{\beta_1}\left(T - \Delta \le H_\nu^{-1}\left(1 - \frac{p}{2}\right) - \Delta\right) \\ &= 1 + H_\nu\left(H_\nu^{-1}\left(\frac{p}{2}\right) - \Delta\right) - H_\nu\left(H_\nu^{-1}\left(1 - \frac{p}{2}\right) - \Delta\right). \end{aligned}$$

Therefore, the p.d.f of $P$ is given by

$$g_{\beta_1}(p) = h_\nu\left(H_\nu^{-1}\left(\frac{p}{2}\right) - \Delta\right)\left(\frac{1/2}{H_\nu^{-1}\left(\frac{p}{2}\right)}\right) + h_\nu\left(H_\nu^{-1}\left(1 - \frac{p}{2}\right) - \Delta\right)\left(\frac{1/2}{H_\nu^{-1}\left(1 - \frac{p}{2}\right)}\right),$$

where $h_\nu(.)$ is the pdf of a Student's t distribution with $\nu$ degrees of freedom. Then the mathematical expectation of the random $p$-value is

$$E(P) = \int_0^1 p\ g_\beta(p)dp.$$

So, the fourth SPE is $\hat{\beta}_{14}^{SPE} = \hat{\beta}_1^{SPE}(d_4)$, where $d_4 = 1 - E(P)$.

Another form of estimator is obtained using the square root of mathematical expectation of the random $p$-value, say $d_5 = 1 - \sqrt{E(P)}$. Therefore, the fifth SPE is $\hat{\beta}_{15}^{SPE} = \hat{\beta}_1^{SPE}(d_5)$.

## 5. Risk of estimators

In this section, we calculate the risk functions of the LS and SPE under the RNL function. To derive the risk functions we need the following lemmas.

**Lemma 1.** *If $Z \sim N(0,1)$, and $Z$ and $S \sim \chi_k^2$ are independent, then for any Borel measurable function $\phi : \Re \times (0,\infty) \to \Re$ and for any $c < 0.5$,*

$$E\left[\exp(cZ^2)\phi(Z,S)\right] = \frac{1}{\sqrt{1-2c}} E\left[\phi\left(\frac{Z}{\sqrt{1-2c}}, S\right)\right]. \tag{11}$$

*Proof.* By definition,

$$\begin{aligned}
E\left[\exp(cZ^2)\phi(Z,S)\right] &= E\left[E\left[\exp(cZ^2)\phi(Z,S)|S\right]\right] \\
&= E\left[\int_{-\infty}^{+\infty} e^{cz^2}\phi(z,s)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\right] \\
&= E\left[\int_{-\infty}^{+\infty} \phi(z,s)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}(1-2c)}dz\right] \\
&= E\left[\int_{-\infty}^{+\infty} \phi(z,s)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z\sqrt{1-2c})^2}dz\right] \\
&= \frac{1}{\sqrt{1-2c}}E\left[\int_{-\infty}^{+\infty} \phi\left(\frac{u}{\sqrt{1-2c}}, s\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du\right],
\end{aligned}$$

where $U = Z\sqrt{1-2c}$. The Jacobian of the transformation is $|J| = \frac{1}{\sqrt{1-2c}}$. Therefore,

$$E\left[\exp(cZ^2\phi(Z,S))\right] = \frac{1}{\sqrt{1-2c}} E\left[\phi\left(\frac{Z}{\sqrt{1-2c}}, S\right)\right].$$

**Lemma 2.** *If $Z \sim N(0,1)$, and $Z$ and $S \sim \chi_k^2$ are independent, then for any Borel measurable function $\phi : \Re \times (0,\infty) \to \Re$ and for any $c \in \Re$,*

$$E[\exp(cZ)\phi(Z,S)] = e^{\frac{c^2}{2}}E[\phi(Z+c,S)]. \tag{12}$$

*Proof.* The proof of Lemma 2 is given by Hoque and Hossain (2012).

The risk function of the $\tilde{\beta}_1$ under the loss function (1) is

$$R(\tilde{\beta}_1, \beta_1) = E\left[L(\tilde{\beta}_1, \beta_1)\right] = 1 - E\left[e^{-\frac{1}{2\gamma^2}(\tilde{\beta}_1-\beta_1)^2}\right],$$

using the fact $Z = \sqrt{S_{xx}}(\tilde{\beta}_1 - \beta_1)\sigma^{-1} \sim N(0,1)$, we have $(\tilde{\beta}_1 - \beta_1)^2 = \frac{Z^2\sigma^2}{S_{xx}}$. Therefore,

$$R(\tilde{\beta}_1, \beta_1) = 1 - E\left[e^{-\frac{1}{2\gamma^2}\left(\frac{Z^2\sigma^2}{S_{xx}}\right)}\right] = 1 - E\left[e^{-aZ^2}\right],$$

where $a = \frac{\sigma^2}{2\gamma^2 S_{xx}}$. Applying Lemma 1 with $\phi = 1$ and $c = -a$, we get

$$R(\tilde{\beta}_1, \beta_1) = 1 - \frac{1}{\sqrt{1+2a}} = 1 - (1+2a)^{-\frac{1}{2}}.$$

Now, the risk function of the shrinkage estimator, $\hat{\beta}_1(d)$ under the loss function (1) is

$$\begin{aligned}
R(\hat{\beta}_1(d), \beta_1) &= E\left[1 - e^{-\frac{1}{2\gamma^2}(\hat{\beta}_1(d)-\beta_1)^2}\right] \\
&= 1 - E\left[e^{-\frac{1}{2\gamma^2}(d\tilde{\beta}_1 + (1-d)\beta_{10} - \beta_1)^2}\right] \\
&= 1 - E\left[e^{-\frac{1}{2\gamma^2}(d(\tilde{\beta}_1 - \beta_1 + \beta_1 - \beta_{10}) + (\beta_{10} - \beta_1))^2}\right] \\
&= 1 - E\left[e^{-\frac{1}{2\gamma^2}(d(\tilde{\beta}_1 - \beta_1) - (1-d)(\beta_1 - \beta_{10}))^2}\right] \\
&= 1 - E\left[e^{-\frac{1}{2\gamma^2}(\frac{d\sigma}{\sqrt{S_{xx}}}Z - \frac{(1-d)\Delta\sigma}{\sqrt{S_{xx}}})^2}\right] \\
&= 1 - e^{-(1-d)^2 a\Delta^2} E\left[e^{-ad^2 Z^2} e^{(2ad(1-d)\Delta)Z}\right].
\end{aligned}$$

Applying Lemma 1 with $c = -ad^2$ and $\phi(Z, S) = e^{(2ad(1-d)\Delta)Z}$, we get

$$\begin{aligned}
R\left(\hat{\beta}_1(d), \beta_1\right) &= 1 - \frac{1}{\sqrt{1+2ad^2}} e^{-(1-d)^2 a\Delta^2} E\left[e^{\frac{2ad(1-d)}{\sqrt{1+2ad^2}}Z}\right] \\
&= 1 - \frac{1}{\sqrt{1+2ad^2}} e^{-(1-d)^2 a\Delta^2} M_Z\left(\frac{2ad(1-d)\Delta}{\sqrt{1+2ad^2}}\right) \\
&= 1 - \frac{1}{\sqrt{1+2ad^2}} e^{-(1-d)^2 a\Delta^2\left(1 - \frac{2ad^2}{1+2ad^2}\right)},
\end{aligned}$$

where $M_Z(\cdot)$ is the moment generating function. Note that, the risk function of the two estimators is a function of $\Delta$. The efficiency of the estimators change with the change in the value of $\Delta$. Under the null hypothesis, the risk function of $\hat{\beta}_1(d)$ is equal to

$$R(\hat{\beta}_1(d), \Delta = 0) = 1 - \frac{1}{\sqrt{1+2ad^2}} < R(\tilde{\beta}_1) = 1 - \frac{1}{\sqrt{1+2a}}.$$

Thus, at $\Delta = 0$, $\hat{\beta}_1(d)$ is better than $\tilde{\beta}_1$.

The risk function of $\hat{\beta}_1^{SPE}$ under the RNL function is given by

$$\begin{aligned}
R(\hat{\beta}_1^{SPE}(d), \beta_1) &= E\left[1 - e^{-\frac{1}{2\gamma^2}(\hat{\beta}_1^{SPE}(d) - \beta_1)^2}\right] \\
&= 1 - E\left[e^{-\frac{1}{2\gamma^2}(\hat{\beta}_1(d)-\beta_1)^2} I(F_{1,\nu} < F_\alpha)\right] \qquad (13) \\
&\quad - E\left[e^{-\frac{1}{2\gamma^2}(\tilde{\beta}_1 - \beta_1)^2} I(F_{1,\nu} > F_\alpha)\right].
\end{aligned}$$

The first expectation of the left hand side can be written as

$$
\begin{aligned}
E\left[e^{-\frac{1}{2\gamma^2}\left(\hat{\beta}_1(d)-\beta_1\right)^2} I(F_{1,\nu} < F_\alpha)\right] &= E\left[e^{-\frac{1}{2\gamma^2}\left(d\tilde{\beta}_1+(1-d)\beta_{10}-\beta_1\right)^2} I(F_{1,\nu} < F_\alpha)\right] \\
&= E\left[e^{-\frac{1}{2\gamma^2}\left(\frac{d\sigma}{\sqrt{S_{xx}}}Z-\frac{(1-d)\Delta\sigma}{\sqrt{S_{xx}}}\right)^2} I(F_{1,\nu} < F_\alpha)\right] \\
&= e^{-(1-d)^2 a\Delta^2} E\left[e^{-d^2aZ^2} e^{2ad(1-d)\Delta Z} I\left(\frac{(Z+\Delta)^2}{S/\nu} < F_\alpha\right)\right],
\end{aligned}
$$

where $S = \nu S_n^2/\sigma^2$. Applying Lemma 1 with $c = -ad^2$ and $\phi(Z,S) = e^{2ad(1-d)\Delta Z} I\left(\frac{(Z+\Delta)^2}{S/\nu} < F_\alpha\right)$, we get

$$
\begin{aligned}
&E\left[e^{-\frac{1}{2\gamma^2}\left(\hat{\beta}_1(d)-\beta_1\right)^2} I(F_{1,\nu} < F_\alpha)\right] \\
&= \frac{e^{-(1-d)^2 a\Delta^2}}{\sqrt{1+2ad^2}} E\left[e^{\frac{2ad(1-d)\Delta}{\sqrt{1+2ad^2}}Z} I\left(\frac{\left(\frac{Z}{\sqrt{1+2ad^2}}+\Delta\right)^2}{S/\nu} < F_\alpha\right)\right] \\
&= \frac{e^{-(1-d)^2 a\Delta^2}}{\sqrt{1+2ad^2}} E\left[e^{\frac{2ad(1-d)\Delta}{\sqrt{1+2ad^2}}Z} I\left(\frac{\left(Z+\sqrt{1+2ad^2}\Delta\right)^2}{S/\nu} < (1+2ad^2)F_\alpha\right)\right].
\end{aligned}
$$

Using Lemma 2 with $c = \frac{2ad(1-d)\Delta}{\sqrt{1+2ad^2}}$ and $\phi(Z,S) = I(\frac{\left(Z+\sqrt{1+2ad^2}\Delta\right)^2}{S/\nu} < (1+2ad^2)F_\alpha)$, we obtain

$$
\begin{aligned}
&E\left[e^{-\frac{1}{2\gamma^2}\left(\hat{\beta}_1(d)-\beta_1\right)^2} I(F_{1,\nu} < F_\alpha)\right] \\
&= \frac{e^{-(1-d)^2 a\Delta^2}}{\sqrt{1+2ad^2}} e^{\frac{c^2}{2}} E\left[I\left(\frac{\left(Z+c+\sqrt{1+2ad^2}\Delta\right)^2}{S/\nu} < (1+2ad^2)F_\alpha\right)\right] \\
&= \frac{e^{-(1-d)^2 a\Delta^2+c^2/2}}{\sqrt{1+2ad^2}} P\left(\frac{\left(Z+c+\sqrt{1+2ad^2}\Delta\right)^2}{S/\nu} < (1+2ad^2)F_\alpha\right) \\
&= \frac{e^{-(1-d)^2 a\Delta^2+c^2/2}}{\sqrt{1+2ad^2}} G_{1,\nu}\left((1+2ad^2)F_\alpha, \left(c+\sqrt{1+2ad^2}\Delta\right)^2\right),
\end{aligned}
$$

where $G_{i,j}(q,\theta)$ is the c.d.f. of a noncentral F distribution with $(i,j)$ degrees of freedom and non-centrality parameter $\theta$, and evaluated at $q$. The second expectation of Equation (13) can be calculated as

$$
\begin{aligned}
E\left[e^{-\frac{1}{2\gamma^2}(\tilde{\beta}_1-\beta_1)^2} I(F_{1,\nu} > F_\alpha)\right] &= E\left[e^{-\frac{\sigma^2}{2\gamma^2 S_{xx}}Z^2} I(F_{1,\nu} > F_\alpha)\right] \\
&= E\left[e^{-aZ^2} I\left(\frac{(Z+\Delta)^2}{S/\nu} > F_\alpha\right)\right].
\end{aligned} \tag{14}
$$

Using Lemma 1 with $c = -a$ and $\phi(Z,S) = I\left(\frac{(Z+\Delta)^2}{S/\nu} > F_\alpha\right)$, we get

$$E\left[e^{-\frac{1}{2\gamma^2}(\tilde{\beta}_1-\beta_1)^2} I(F_{1,\nu} > F_\alpha)\right] = \frac{1}{\sqrt{1+2a}} E\left[I\left(\frac{\left(\frac{Z}{\sqrt{1+2a}}+\Delta\right)^2}{S/\nu} > F_\alpha\right)\right]$$
$$= \frac{1}{\sqrt{1+2a}} P\left(\frac{\left(Z+\sqrt{1+2a}\Delta\right)^2}{S/\nu} > (1+2a)F_{1-\alpha}\right) \tag{15}$$
$$= \frac{1}{\sqrt{1+2a}}\left(1 - G_{1,\nu}\left((1+2a)F_\alpha, (1+2a)\Delta^2\right)\right).$$

Combining Equations (14) and (15), the risk of $\hat{\beta}_1^{SPE}(d)$ is

$$R\left(\hat{\beta}_1^{SPE}(d)\right) = 1 - \frac{e^{-(1-d)^2 a\Delta^2 + c^2/2}}{\sqrt{1+2ad^2}} G_{1,\nu}\left((1+2ad^2)F_\alpha, \left(c+\sqrt{1+2ad^2}\Delta\right)^2\right)$$
$$- \frac{1}{\sqrt{1+2a}}\left(1 - G_{1,\nu}\left((1+2a)F_\alpha, (1+2a)\Delta^2\right)\right). \tag{16}$$

Under the null hypothesis $\Delta = 0$, we have

$$R\left(\hat{\beta}_1^{SPE}(d), \Delta = 0\right)$$
$$= 1 - \frac{1}{\sqrt{1+2a}} - \frac{1}{\sqrt{1+2ad^2}} G_{1,\nu}\left((1+2ad^2)F_\alpha, 0\right) + \frac{1}{\sqrt{1+2a}} G_{1,\nu}\left((1+2a)F_\alpha, 0\right)$$
$$= R(\tilde{\beta}_1) - \frac{1}{\sqrt{1+2ad^2}} G_{1,\nu}\left((1+2ad^2)F_\alpha, 0\right) + \frac{1}{\sqrt{1+2a}} G_{1,\nu}\left((1+2a)F_\alpha, 0\right)$$
$$< R(\tilde{\beta}_1).$$

Therefore, at $\Delta = 0$ the risk of $\hat{\beta}_1^{SPE}(d)$ is less than of LS, $\tilde{\beta}_1$. As $\alpha \to 0$, $G_{1,\nu}\left((1+2ad^2)F_\alpha, 0\right) \to 1$ and $G_{1,\nu}\left((1+2a)F_\alpha, 0\right) \to 1$, then

$$R\left(\hat{\beta}_1^{SPE}(d), \Delta = 0\right) \to 1 - \frac{1}{\sqrt{1+2ad^2}},$$

which is the risk of shrinkage estimator, $\hat{\beta}_1(d)$ at $\Delta = 0$. On the other hand, if $F_\alpha \to 0$, then

$$R\left(\hat{\beta}_1^{SPE}(d), \Delta = 0\right) \to 1 - \frac{1}{\sqrt{1+2a}} = R(\tilde{\beta}_1),$$

which is the risk of LS, $\tilde{\beta}_1$. As $\Delta \to \infty$, $G_{1,\nu}(\cdot, \Delta) \to 0$ then $R\left(\hat{\beta}_1^{STE}(d)\right) \to R(\tilde{\beta}_1)$.

## 6. Simulation study

In this section, we conduct a simulation study for comparison of SPEs and the LS. The dependent variable ($y_i$) is computed from the equation $y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, for $\beta_0 = 3$ and $\beta_1 = 2$ where $x_i$ were generated from $U(0,\ 1)$ and the errors ($\varepsilon_i$) are generated from $N(0,\ 1)$ for sample sizes $n = 10, 25, 40, 60$. Let $\hat{\beta}_1^k$, $k = 1, 2, 3, 4, 5, 6$ stands for $\hat{\beta}_{1i}^{SPE}$, $i = 1, 2, 3, 4, 5$ and LS ($\tilde{\beta}_1$), respectively. For $\alpha = 0.05$ and $\gamma = 2$, repeat these tasks 10,000 times and calculate the value of Estimated Risk (ER) using the following formula

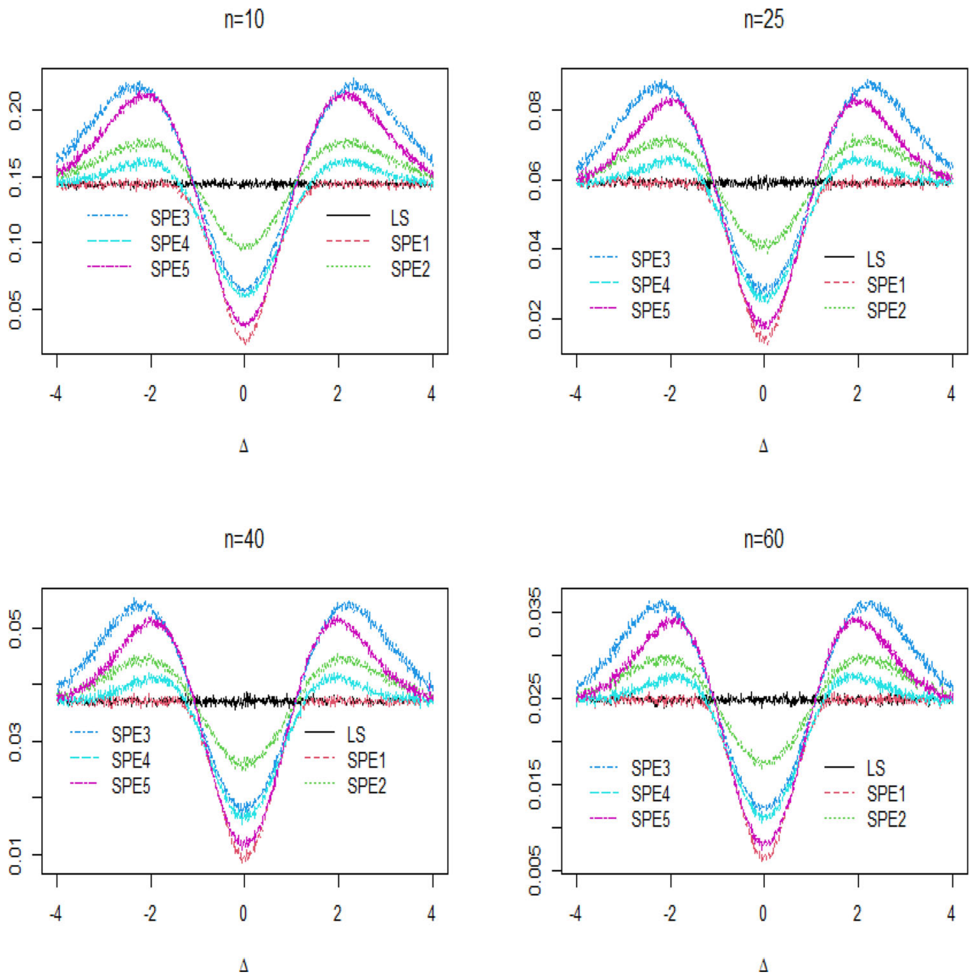


**Figure 2.** Plots of estimated risk of $SPEi = \hat{\beta}_{1i}^{SPE}$, $i = 1, 2, 3, 4, 5$, and LS for different values of $n = 10, 25, 40, 60$, $\alpha = 0.05$, and $\gamma = 2$.

$$ER\left(\hat{\beta}_1^k\right) = \frac{1}{10000}\sum_{i=1}^{10000}\left(1 - e^{-\frac{1}{2\gamma^2}\left(\hat{\beta}_{1i}^k - \beta_1\right)^2}\right). \tag{17}$$

The graphs for the risk function of the estimators are plotted in Figure 2 as a function of $\Delta$. We observe from Figure 2 that the estimators $\hat{\beta}_{11}^{SPE}$ and $\hat{\beta}_{15}^{SPE}$ have smaller risk than other estimators in neighborhood of null hypothesis. All SPEs are better than LS in the neighborhood of null hypothesis. Also, the risk of the estimators decreases as the sample size increases.

For any nonzero value of $\Delta$, the risk function of the SPE of $\beta_1$ can be written as

$$R\left[\hat{\beta}_1^{SPE}(d), \beta_1\right] = R\left[\tilde{\beta}_1, \beta_1\right] + g(\Delta), \tag{18}$$

where

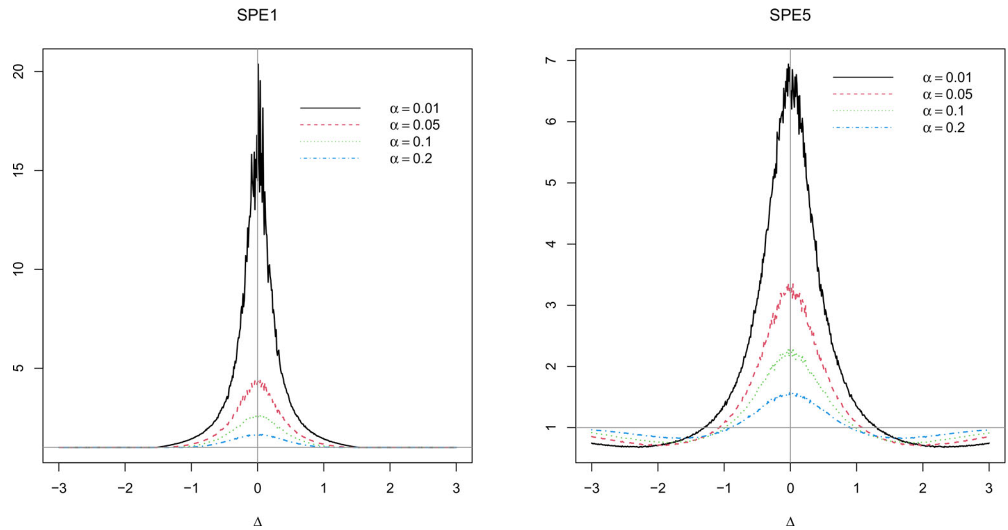


**Figure 3.** Plots of relative efficiency of $\hat{\beta}_{11}^{SPE}$ and $\hat{\beta}_{15}^{SPE}$ relative to LS for different values of $\alpha$, and $n = 25$, $\gamma = 2$.

$$g(\Delta) = \frac{1}{\sqrt{1+2a}} G_{1,\nu}\big((1+2a)F_\alpha, (1+2a)\Delta^2\big) - \frac{e^{-(1-d)^2 a\Delta^2 + c^2/2}}{\sqrt{1+2ad^2}} G_{1,\nu}\Big((1+2ad^2)F_\alpha, \big(c+\sqrt{1+2ad^2}\Delta\big)^2\Big).$$

Therefore, the relative efficiency between SPE and LS can be written as

$$\text{Eff}\left[\hat{\beta}_1^{SPE}(d), \tilde{\beta}_1\right] = \frac{R[\tilde{\beta}_1, \beta_1]}{R\left[\hat{\beta}_1^{SPE}(d), \beta_1\right]} = \frac{1-(1+2a)^{-\frac{1}{2}}}{1-(1+2a)^{-\frac{1}{2}} + g(\Delta)}. \tag{19}$$

Under $H_0 : \Delta = 0$ we have

$$g(\Delta) = \frac{1}{\sqrt{1+2a}} G_{1,\nu}((1+2a)F_\alpha, 0) - \frac{1}{\sqrt{1+2ad^2}} G_{1,\nu}\big((1+2ad^2)F_\alpha, 0\big) < 0.$$

Therefore, at $\Delta = 0$ the SPE is more efficient than the LS.

Figure 3 shows the relative efficiency between $\hat{\beta}_{11}^{SPE}$ and $\hat{\beta}_{15}^{SPE}$ relative to $\tilde{\beta}_1$ against $\Delta$ for selected values of $\alpha = 0.01, 0.05, 0.1, 0.2$ and $n = 25$. The SPEs are more efficient than $\tilde{\beta}_1$ in neighborhood of null hypothesis ($\Delta = 0$). Also, the SPE with smaller level of significance has higher efficiency.

The effective intervals (the range of $\Delta$ values in which $\hat{\beta}_1^{SPE}(d)$ is more efficient than $\tilde{\beta}_1$) are summarized in Table 1 for selected values of $\alpha = 0.01, 0.05, 0.1, 0.2$, $d = 0.2, 0.4, 0.6, 0.8$, $n = 10, 20, 30, 50$ and $\gamma = 2$.

The risk function of SPE depends on the level of significance $\alpha$ of the test of null hypothesis, the departure parameter $\Delta$ and shape parameter of the RNL function, $\gamma$. We are interested in knowing the amount of $\alpha$ that is used for constructing the SPEs. For this purpose, we consider the relative efficiency given in Equation (19) as a function of

**Table 1.** A range of $\Delta$ values where $\hat{\beta}_1^{SPE}(d)$ is more efficient than $\tilde{\beta}_1$.

| | | $\alpha$ | | | |
|---|---|---|---|---|---|
| $n$ | $d$ | 0.01 | 0.05 | 0.1 | 0.2 |
| 10 | 0.2 | $(-1.1240, 1.1240)$ | $(-1.0096, 1.0096)$ | $(-0.9455, 0.9455)$ | $(-0.8781, 0.8781)$ |
| | 0.4 | $(-1.3397, 1.3397)$ | $(-1.1584, 1.1584)$ | $(-1.0653, 1.0653)$ | $(-0.9722, 0.9722)$ |
| | 0.6 | $(-1.6087, 1.6087)$ | $(-1.3235, 1.3235)$ | $(-1.1919, 1.1919)$ | $(-1.0667, 1.0667)$ |
| | 0.8 | $(-1.9561, 1.9561)$ | $(-1.5104, 1.5104)$ | $(-1.3273, 1.3273)$ | $(-1.1624, 1.1624)$ |
| 20 | 0.2 | $(-1.0919, 1.0919)$ | $(-0.9787, 0.9787)$ | $(-0.9213, 0.9213)$ | $(-0.8635, 0.8635)$ |
| | 0.4 | $(-1.2799, 1.2799)$ | $(-1.1103, 1.1103)$ | $(-1.0299, 1.0299)$ | $(-0.9518, 0.9518)$ |
| | 0.6 | $(-1.4984, 1.4984)$ | $(-1.2507, 1.2507)$ | $(-1.1415, 1.1415)$ | $(-1.0392, 1.0392)$ |
| | 0.8 | $(-1.7555, 1.7555)$ | $(-1.4024, 1.4024)$ | $(-1.2575, 1.2575)$ | $(-1.1264, 1.1264)$ |
| 30 | 0.2 | $(-1.0805, 1.0805)$ | $(-0.9694, 0.9694)$ | $(-0.9143, 0.9143)$ | $(-0.8593, 0.8593)$ |
| | 0.4 | $(-1.2601, 1.2601)$ | $(-1.0962, 1.0962)$ | $(-1.0198, 1.0198)$ | $(-0.9462, 0.9462)$ |
| | 0.6 | $(-1.4648, 1.4648)$ | $(-1.2301, 1.2301)$ | $(-1.1275, 1.1275)$ | $(-1.0316, 1.0316)$ |
| | 0.8 | $(-1.7000, 1.7000)$ | $(-1.3732, 1.3732)$ | $(-1.2386, 1.2386)$ | $(-1.1166, 1.1166)$ |
| 50 | 0.2 | $(-1.0715, 1.0715)$ | $(-0.9648, 0.9648)$ | $(-0.9109, 0.9109)$ | $(-0.8574, 0.8574)$ |
| | 0.4 | $(-1.2446, 1.2446)$ | $(-1.0895, 1.0895)$ | $(-1.0151, 1.0151)$ | $(-0.9435, 0.9435)$ |
| | 0.6 | $(-1.4392, 1.4392)$ | $(-1.2205, 1.2205)$ | $(-1.1210, 1.1210)$ | $(-1.0281, 1.0281)$ |
| | 0.8 | $(-1.6591, 1.6591)$ | $(-1.3596, 1.3596)$ | $(-1.2298, 1.2298)$ | $(-1.1121, 1.1121)$ |

$\alpha$ and $\gamma$ as $\text{Eff}\left[\hat{\beta}_1^{SPE}(d); \alpha, \Delta\right]$. From the analyze of the relative efficiency function of the SPE, it is evident that the SPE does not have uniform domination over the LS for all values of $\Delta$. Also, the value of $\Delta$ is usually unknown for the experimenter. Thus, we preassign a value of the relative efficiency, say $\text{Eff}_0$, that we are willing to accept. Then, consider the set

$$A_\alpha = \left(\alpha \mid \text{Eff}\left[\hat{\beta}_1^{SPE}(d); \alpha, \Delta\right]\right) \geq \text{Eff}_0). \tag{20}$$

An estimator $\hat{\beta}_1^{SPE}(d)$ is chosen which maximizes $\text{Eff}\left[\hat{\beta}_1^{SPE}(d); \alpha, \Delta\right]$, for all $\alpha \in A_\alpha$ and $\Delta$. The solution of the following equation for $\alpha$

$$\max_{\alpha} \min_{\Delta} \text{Eff}\left[\hat{\beta}_1^{SPE}(d); \alpha, \Delta\right] = \text{Eff}_0, \tag{21}$$

provides the maximum and minimum relative efficiencies of the SPE relative to the LS, for selected values of $n$ and $\Delta$. The maximum relative efficiency $(\text{Eff}^*)$ and minimum relative efficiency $(\text{Eff}_0)$ of the SPE relative to the LS, and the value of $\Delta(\Delta_0)$ at which $\text{Eff}_0$ occurs are calculated and summarized in Table 2 for selected values of $n = 10, 20, 30, 40, 50$, $\alpha = 0.05, 0.1, 0.5, 0.2, 0.3$, $d = 0.5$ and $\gamma = 2$. For example, if $\gamma = 2$, $d = 0.5$ and $n = 20$, and the experimenter wishes to achieve the minimum relative efficiency 0.7711 of the SPE of $\beta_1$, the recommended value of $\alpha$ is 0.15. This minimum relative efficiency attains at $\Delta_0 = 2.0674$.

## 7. A numerical example

In this section, a numerical example is provided to illustrate the proposed estimators. Data related to the propulsion system of a rocket motor from Montgomery et al. (2012) that the dependence of shear strength as a response variable on the age in weeks of the batch of propellant as a predictor variable was investigated. Twenty observations on shear strength (SS) and the age of propellant (AP) have been collected and are shown

**Table 2.** Maximum and minimum efficiencies of SPE relative to the LS.

| | | $n$ | | | | |
|---|---|---|---|---|---|---|
| $\alpha$ | | 10 | 20 | 30 | 40 | 50 |
| 0.05 | Eff* | 2.3698 | 2.2678 | 2.2378 | 2.2235 | 2.2152 |
| | $\text{Eff}_0$ | 0.5860 | 0.6204 | 0.6299 | 0.6343 | 0.6369 |
| | $\Delta_0$ | 2.6582 | 2.440535 | 2.3839 | 2.3578 | 2.342912 |
| 0.10 | Eff* | 1.8586 | 1.7875 | 1.7672 | 1.7576 | 1.7520 |
| | $\text{Eff}_0$ | 0.6854 | 0.7105 | 0.7175 | 0.7208 | 0.7227 |
| | $\Delta_0$ | 2.3429 | 2.2015 | 2.1641 | 1.7576 | 2.1368 |
| 0.15 | Eff* | 1.5905 | 1.5401 | 1.5258 | 1.5191 | 1.5152 |
| | $\text{Eff}_0$ | 0.7513 | 0.7711 | 0.7766 | 0.7792 | 0.7807 |
| | $\Delta_0$ | 2.1705 | 2.0674 | 2.0398 | 2.0270 | 2.0197 |
| 0.20 | Eff* | 1.4243 | 1.3876 | 1.3773 | 1.3725 | 1.3696 |
| | $\text{Eff}_0$ | 0.8014 | 0.8173 | 0.8218 | 0.8239 | 0.8251 |
| | $\Delta_0$ | 2.0557 | −1.9768 | 1.9556 | 1.9458 | 1.9401 |
| 0.30 | Eff* | 1.2316 | 1.2113 | 1.2057 | 1.2030 | 1.2015 |
| | $\text{Eff}_0$ | 0.8743 | 0.8847 | 0.8876 | 0.8890 | 0.8898 |
| | $\Delta_0$ | −1.9081 | −1.8591 | 1.8458 | 1.8397 | 1.8361 |

**Table 3.** Data for numerical examples.

| Observation $i$ | SS $y_i$ | AP $x_i$ | Observation $i$ | SS $y_i$ | AP $x_i$ |
|---|---|---|---|---|---|
| 1 | 2158.70 | 15.50 | 11 | 2156.20 | 13.00 |
| 2 | 1678.15 | 23.75 | 12 | 2399.55 | 3.75 |
| 3 | 2316.00 | 8.00 | 13 | 1779.82 | 25.00 |
| 4 | 2061.30 | 17.00 | 14 | 2336.75 | 9.75 |
| 5 | 2207.50 | 5.5 | 15 | 1765.30 | 22.00 |
| 6 | 1708.30 | 19.00 | 16 | 2053.50 | 18.00 |
| 7 | 1784.70 | 24.00 | 17 | 2414.40 | 6.00 |
| 8 | 2575.00 | 2.50 | 18 | 2200.50 | 12.50 |
| 9 | 2357.90 | 7.50 | 19 | 2654.20 | 2.00 |
| 10 | 2265.70 | 11.00 | 20 | 1753.70 | 21.50 |

in Table 3. The reviewed results confirm the appropriate assumptions of the model. The least squares estimation of the slope parameter is $\tilde{\beta}_1 = -37.15$. Shrinkage pretest estimators $\hat{\beta}_{1i}^{SPE}$, $i = 1, 2, 3, 4, 5$ are computed for selected values of guess value $\beta_{10}$ and summarized in Table 4.

For example, in the first row of Table 4, we consider the estimation of $\beta_1$ when the guessed value is $\beta_{10} = -30$. For testing the null hypothesis $H_0 : \beta_1 = -30$ against $H_1 : \beta_1 \neq -30$, the test statistic is

$$F_{1,18} = \frac{S_{xx}\left(\tilde{\beta}_1 - \beta_{10}\right)^2}{S_n^2} = 6.124. \tag{22}$$

Since $F_{1,18} > F_{0.05,1,18} = 4.413$, then the null hypothesis is rejected at the 0.05 level, therefore the SPE is equal to $\hat{\beta}_{1i}^{SPE} = \tilde{\beta}_1 = -37.15, i = 1, ..., 5$. If the guessed value is $\beta_{10} = -35$ then $F_{1,18} = 0.5537 < F_{0.05,1,18} = 4.413$, therefore, the null hypothesis is accepted. We obtain the value of $d_1 = 0.531$ by minimizing the risk function of SPE, $d_2 = 1 - p - \text{value} = 0.533$, $d_3 = 1 - \sqrt{p - \text{value}} = 0.317$, $d_4 = E(P) = 0.422$, the mathematical expectation of the random $p$-value and $d_5 = 1 - \sqrt{E(P)} = 0.350$. So the SPE's are obtained as

**Table 4.** Estimated values of the slope parameter.

| $\beta_{10}$ | $\tilde{\beta}_1$ | $\hat{\beta}_{11}^{SPE}$ | $\hat{\beta}_{12}^{SPE}$ | $\hat{\beta}_{13}^{SPE}$ | $\hat{\beta}_{14}^{SPE}$ | $\hat{\beta}_{15}^{SPE}$ |
|---|---|---|---|---|---|---|
| −30 | −37.15 | −37.15 | −37.15 | −37.15 | −37.15 | −37.15 |
| −40 | −37.15 | −37.96 | −38.11 | −38.80 | −38.94 | −38.89 |
| −35 | −37.15 | −36.14 | −36.15 | −35.68 | −35.90 | −35.75 |
| −45 | −37.15 | −37.15 | −37.15 | −37.15 | −37.15 | −37.15 |

$$\hat{\beta}_{11}^{SPE} = 0.531(-37.15) + (1-0.531)(-35) = -36.14,$$
$$\hat{\beta}_{12}^{SPE} = 0.533(-37.15) + (1-0.533)(-35) = -36.15,$$
$$\hat{\beta}_{13}^{SPE} = 0.317(-37.15) + (1-0.317)(-35) = -35.68,$$
$$\hat{\beta}_{14}^{SPE} = 0.422(-37.15) + (1-0.422)(-35) = -35.90,$$
$$\hat{\beta}_{15}^{SPE} = 0.35(-37.15) + (1-0.35)(-35) = -35.75.$$

Also, according to the max-min method if $n=20$ and $d=0.5$ is considered, then to achieve a relative efficiency of at least 0.7105, from Table 2, the optimal value of the level of significance is $\alpha = 0.10$. So we test the null $H_0 : \beta_1 = -35$ against $H_1 : \beta_1 \neq -35$ at the level of significant $\alpha = 0.1$. Since $F_{1,18} = 0.5537 < F_{0.05,1,18} = 3.0069$, then the null hypothesis is accepted and the shrinkage pretest estimate is equal to

$$\hat{\beta}_1^{SPE}(0.5) = 0.5(-37.15) + (1-0.5)(-35) = -36.75.$$

## 8. Concluding remarks

In this paper, five shrinkage pretest estimators of slope parameter are proposed. The risk functions of LS and SPEs are calculated under RNL function. A simulation study is performed for comparing the proposed SPEs with the LS. The results show that the SPEs outperform the LS. In comparison between the SPEs, our findings show that the estimators $\hat{\beta}_{11}^{SPE}$ and $\hat{\beta}_{15}^{SPE}$ are better than other SPEs. Also, the range of $\Delta$ values in which $\hat{\beta}_1^{SPE}(d)$ is more efficient than $\tilde{\beta}_1$ are calculated. The optimal value of level of significance was obtained using max-min method in Table 2. Finally, a real data is analyzed for illustrating the results.

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