STABILITY OF PLANE POISEUILLE FLOW OF A FLUID WITH PRESSURE-DEPENDENT VISCOSITY

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Summary We study the linear stability of a plane Poiseuille flow of an incompressible fluid whose viscosity depends linearly on the pressure. It is shown that the local critical Reynolds number is a sensitive function of the applied pressure gradient and that it decreases along the channel. While in the limit of small pressure gradients conventional results for a pressure-independent Newtonian fluid are recovered, a significant stabilisation of the flow and an elongation of the critical disturbance wavelength are observed when the longitudinal pressure gradient is increased. These features drastically distinguish the stability characteristics of a piezo-viscous flow from its pressure-independent Newtonian counterpart.

PROBLEM DEFINITION AND GOVERNING EQUATIONS

We consider a flow of an incompressible fluid with a pressure-dependent viscosity. Such fluids are found, for example, in geological, planetary and lubrication applications. It was shown in [1, 2] that steady unidirectional plane flows can exist only when the fluid viscosity is a linear function of the pressure. Therefore here we consider the flow of such a fluid between two parallel horizontal plates separated by distance $2L$. The x and y axes have the left-to-right and upward positive directions, respectively. The horizontal centre-plane of a channel is located at $y = 0$. The negative pressure gradient $\nabla\pi$ is applied along the channel. With the gravity neglected the governing equations are [1]

$$\rho\frac{d\mathbf{u}}{dt} = -\nabla\pi + \nabla\cdot(2\mu(\pi)\mathbf{D})\ , \quad \nabla\cdot\mathbf{u} = 0 \tag{1}$$

which are complemented with the constitutive equations for density ρ and viscosity μ

$$\rho = \text{const.}\,, \quad \mu = a\pi > 0 \tag{2}$$

and the no-slip/no-penetration boundary conditions

$$(u, v) = (0, 0) \text{ at } y = \pm L\,. \tag{3}$$

Here the velocity field $\mathbf{u} = (u(\mathbf{x}), v(\mathbf{x}))$, the pressure $\pi = \pi(\mathbf{x})$, the coordinate vector $\mathbf{x} = (x, y)$, and $\mathbf{D} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$. Following [3], in order to capture the piezo-viscous nature of the flow in the most explicit way we non-dimensionalise these equations using the pressure π^* evaluated at $(x, y) = (0, 0)$ at time $t = 0$, the characteristic speed $u^* = (\pi^*/\rho)^{1/2}$ and time $t^* = L(\rho/\pi^*)^{1/2}$ as the scales for length, pressure, velocity and time, respectively, to obtain

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla\pi + \alpha\nabla\cdot(2\pi\mathbf{D})\,, \quad \nabla\cdot\mathbf{u} = 0\,, \tag{4}$$

$$(u, v) = (0, 0) \quad \text{at} \quad y = \pm 1\,, \quad \pi = 1 \quad \text{at} \quad (x, y, t) = (0, 0, 0)\,, \tag{5}$$

where parameter $\alpha = a\left(\frac{\pi^*}{\rho L^2}\right)^{\frac{1}{2}} = \frac{a\pi^*}{\rho(\pi^*/\rho)^{1/2}L} = \frac{\mu^*}{\rho u^* L}$ plays the role of effective inverse Reynolds number. Note that all unstarred symbols now denote the corresponding non-dimensional quantities.

BASIC FLOW

The governing equations (4), (5) admit a steady unidirectional solution for the velocity and a two-dimensional solution for the pressure which were first reported in [1] and are given by

$$U(y) = \frac{2}{\alpha C_0}\ln\frac{\cosh\frac{C_0 y}{2}}{\cosh\frac{C_0}{2}}\,, \quad \Pi(\mathbf{x}) = \frac{1 + e^{-C_0 y}}{2}e^{C_0(x+y)/2}\,. \tag{6}$$

Parameter $C_0 < 0$ plays a dual role. Firstly, it characterises the strength of the applied pressure gradient. Secondly, it measures the piezo-viscous effects which distinguish the considered fluid from its conventional Newtonian counterpart. This is best seen in the limit of $C_0 \to 0$ when the series expansion of (6) leads to

$$U(y) \approx \frac{y^2 - 1}{4\alpha}C_0 - \frac{y^4 - 1}{96\alpha}C_0^3\,, \quad \Pi(\mathbf{x}) \approx 1 + \frac{x}{2}C_0 + (x^2 + y^2)\frac{C_0^2}{8}\,. \tag{7}$$

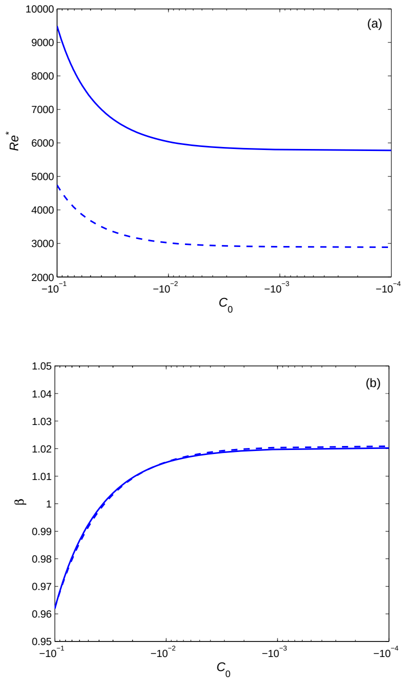

Figure 1. The critical values of Reynolds number (a) and wavenumber (b) as functions of the pressure parameter C_0 for two locations along the channel corresponding to $E = 1$ (solid lines) and $E = 1/2$ (dashed lines). The critical wavenumber curves for these spatial locations overlap in plot (b).

The maximum flow speed achieved at $y = 0$ in this limit is

$$U_{\max} \approx \frac{|C_0|}{4\alpha}\left(1 - \frac{C_0^2}{24}\right). \tag{8}$$

In the above expressions, the terms linear in C_0 correspond to the flow of a Newtonian fluid with pressure-independent viscosity while the higher order in C_0 terms describe piezo-viscous effects. For a more straightforward comparison of our results with the conventionally non-dimensionalised solutions for a Newtonian fluid we introduce Reynolds number based on the maximum speed

$$Re^* = \frac{\rho U_{\max} u^* L}{\mu^*} = \frac{\rho |C_0| u^* L}{4\alpha\mu^*} = \frac{|C_0|}{4\alpha^2}. \tag{9}$$

STABILITY RESULTS

Equations (4) and (5) are linearised about the basic flow solution assuming the disturbed flow in the form

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{U}(y) + \mathbf{u}'(\mathbf{x}, t), \quad \pi(\mathbf{x}, t) = \Pi(\mathbf{x}) + \pi'(\mathbf{x}, t), \tag{10}$$

where $\mathbf{U}$, $\mathbf{u}'$, Π, π' are the basic and disturbance velocity and the basic and disturbance pressure, respectively. The disturbance quantities then are written in a normal mode form $(\mathbf{u}'(\mathbf{x}, t), \pi'(\mathbf{x}, t)) = (\mathbf{u}'(y), \pi'(y))e^{\sigma t + i\beta x}$. Upon discretisation using Chebyshev pseudo-spectral method with 100 collocation points, the resulting algebraic generalized eigenvalue problem is solved for the complex amplification rate σ over a range of wavenumbers β and Reynolds numbers Re^*. Note that since the basic flow pressure and thus the fluid viscosity depend on x the above normal mode expansion is local in its nature. It is only valid if the characteristic length over which the pressure changes significantly is much longer than the disturbance wavelength, i.e. if $|C_0| \ll \beta$. This condition is safely satisfied in the current analysis, see Figure 1(b). The locality of the solution is parametrised by $E = e^{C_0 x/2}$.

As expected from equations (7) and (8), when $C_0 \to 0$, the basic velocity profile reduces to that of a conventional Poiseuille flow and we recover the critical values of Reynolds and wavenumber $(Re_c^*, \beta_c) \approx (5772, 1.02)$ for a Newtonian fluid. However increasing the pressure gradient parameter $|C_0|$ gives rise to a significant stabilisation of the flow, see Figure 1(a). This behaviour is completely opposite to that observed in flows of fluids with pressure-independent viscosity. Physically, the larger values of $|C_0|$ correspond to a larger pressure difference between the channel ends. In order to increase the pressure gradient the pressure upstream has to be raised. For tested rheological model this leads to the increase of the fluid's viscosity and, subsequently, to the decrease of the maximum speed of the flow (see the expression for $U_{\max}$ above). Both these effects result in the observed stabilisation. It is found that the flow remains stable near $x = 0$ regardless of the strength of the applied pressure gradient for the values of $|C_0| \geq 0.35$. This can be traced back to the channel-choking effect discovered in [3], an essentially piezo-viscous effect when increasing the pressure gradient leads to the proportional increase in the fluid viscosity so that the flow maximum speed remains constant.

At the same time, the flow is destabilised downstream where the local pressure and the fluid viscosity decrease, see the dashed line in Figure 1(a). The critical Reynolds number drops below the classical value of 5772 even in the limit of $C_0 \to 0$. This signifies the essential differences between piezo-viscous and Newtonian fluids. The present results suggest that in a sufficiently long channel the instability will develop near the channel exit regardless of the entrance flow conditions. This instability will destroy a unidirectional flow before the fluid reaches the channel end. Such a finding provides a possible resolution of the concern expressed in [4]. There the authors noted that due to the exponential decrease of the pressure along the channel the fluid becomes essentially inviscid, which is physically unlikely. Therefore according to [4] the steady plane unidirectional solutions for piezo-viscous flows might have limited physical relevance. The current study shows that such flows can exist at least in a relatively short channel. They never become fully inviscid because the instability inevitably sets once the viscosity reaches a sufficiently low level and then the developing pressure disturbances guarantee (via the constitutive law (2)) that the viscosity remains non-zero.

References

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