Low-dimensional Boundary-layer Model of Turbulent Dispersion in a Channel

Dmitry V. Strunin *, Anthony J. Roberts †

Abstract—**We analyse dispersion of contaminants in turbulent boundary layer using centre manifold technique. The method describes long-term asymptotics of the contaminant concentration as it becomes spread across the entire layer and is weakly distorted by the velocity shear. The dispersion is investigated in two cases: (a) logarithmic and (b) power velocity profile across the layer according to a traditional and a more recent theory respectively. We deduce an advection-diffusion equation for the depth-average concentration in each case. The equation represents a leading approximation of the dynamics and can be extended to include higher-order derivatives for better precision.**

Keywords: Dispersion, channel, turbulent boundary layer

1 Introduction

Dispersion of contaminants in channels under velocity shear and diffusion can be described by low-dimensional models written in terms of depth-average concentration as a function of time, t, and downstream coordinate, x. In the 1950s Taylor [1] formulated the advection-diffusion equation for such concentration. Mercer and Roberts [2] deduced the same equation from more rigorous principles of centre manifold theory and showed that the equation is only a leading approximation of a more accurate description involving higher-order spatial derivatives. Chikwendu and Ojiakor [3] subdivided the flow into two zones, the fast zone near the surface and the slow zone near the bottom and empirically constructed a system of coupled equations for the average concentrations in each zone. Roberts and Strunin [4] applied the technique [2] to build a two-zone model on the firm foundation of the centre manifold theory. A more extensive list of publications on the shear dispersion in channels can be found in the above mentioned papers.

The models [2, 4] are best applicable to laminar flows; turbulent flows are more difficult to model. We point to a principle difference between the two cases: for the laminar flows, cross-flow (molecular) diffusion coefficient is independent of the advection, with or without shear, whereas the turbulent diffusion coefficient is essentially linked to the shear because the turbulence is caused by the shear. A possible approach is to use one of averaged models of turbulence, for example the k-ε model, in conjunction with the advection-diffusion equation for the contaminant. In this approach, the fields of the turbulent energy and its dissipation rate would be part of solution alongside with the contaminant concentration. Mei et al. [5] and Georgiev et al. [6, 7] used the k-ε, k-ω and other models for low-dimensional modelling of floods. However, in such an approach a number of empirical coefficients appear both in the dynamical equations and boundary conditions bringing uncertainty into the modelling. But most critically, the very form of the boundary conditions appears not to be unique.

In this paper we assume that the flow is given, steady and the velocity profile has one of the forms characteristic of the turbulent boundary layer. Traditionally a logarithmic profile is adopted for the boundary layer [8]. However, Barenblatt et al. [9] argued that close examination of the experimental data rather provides evidence in favour of power profile, an important feature of which is dependence on the Reynolds number. Without offering a discussion of the two types of the profile, we separately use them in our low-dimensional modelling of dispersion. As an approximation we extrapolate the velocity profiles beyond the boundary layer and adopt them for the entire open channel flow.

For the power profile, Barenblatt [10] studied three cases of dispersion in the boundary layer with sources of contaminant located on the bottom. He assumed that entire amount of the contaminant remained inside the boundary layer, which was reflected in the boundary condition of zero concentration at $y = \infty$ (y is the distance from the bottom). Hence, this formulation represents asymptotics of the dispersion at small times, specifically before the contaminant reaches outer regions of the boundary layer. In the present paper we investigate the opposite extreme case — the dispersion at large times, that is after the contaminant reaches the free surface and consequently becomes distributed over the entire cross-section of the flow. Our centre manifold approach produces a dynamic equation for the depth-average concentration, which can

*Department of Mathematics and Computing, University of Southern Queensland, Toowoomba, QLD 4350, Australia Tel/Fax: +61-7-4631-5541/5550 Email: strunin@usq.edu.au

†School of Mathematical Sciences, University of Adelaide, SA 5005, Australia Tel: +61-8-8303-5077, Email: anthony.roberts@adelaide.edu.au

be considered in conjunction with various boundary and initial conditions depending on the concrete problem of interest. While the solutions of the equation are not necessarily self-similar as in [10], most importantly, the equation itself represents attractive dynamics, that is a universal dynamical law of dispersion.

The transport equation for the ensemble-average concentration, $c(x, y, t)$, has the form

$$\partial_t c + u(y)\partial_x c = \partial_y [D(y)\partial_y c] \,, \tag{1}$$

where $u(y)$ is the velocity and $D(y)$ is the turbulent diffusion coefficient. The boundary conditions stipulate that the flux through the surface, $y = h$, and the bottom, $y = 0$, is zero:

$$D\partial_y c|_{y=0} = D\partial_y c|_{y=h} = 0 \,. \tag{2}$$

2 Centre manifold approach

Suppose for a moment that $u(y)$ and $D(y)$ in (1) are independent of each other. If it was not for the advection, or, more precisely, the velocity shear, the concentration would quickly become constant across the channel because of the diffusion. The concentration can have any value and such a state is neutrally stable. The centre manifold approach treats the velocity shear as a perturbation which, loosely speaking, makes the system drift from one neutral state to another. The mechanism can be illustrated by the following simple example from [11]:

$$\begin{aligned} da/dt &= -ab \,, \\ db/dt &= -b + a^2 - 2b^2 \,. \end{aligned} \tag{3}$$

The linearised system (3), $da/dt = 0$, $db/dt = -b$, is characterised by zero eigenvalue for the slow variable a and a negative eigenvalue for the fast variable b. It can be shown that all trajectories of (3) are attracted to a single curve

$$b = a^2 \tag{4}$$

called the centre manifold. If it was not for the nonlinear terms, the variable b would quickly fall onto the equilibrium state $b = 0$ while a would stay in the neutral state a =const. However, for the full system (3) this does not occur; instead, the trajectories drop onto the manifold (attractor) (4), on which the perturbation, $a^2 - 2b^2$, is comparable to the linear term, $-b$. On the manifold the motion is slow and, taking into account (4), is described by $da/dt = -ab = -a^3$. Observe that on the manifold the variable b depends on t not independently, but through a to which it is connected algebraically, (4).

Performing the Fourier transformation of (1) we get

$$\partial_t \hat{c} = L[\hat{c}] - iku(y)\hat{c} \,, \tag{5}$$

where $\hat{c}(k, t)$ is the Fourier transform defined as $1/(2\pi)\int_{-\infty}^{\infty} \exp(-ikx)c\,dx$ and the linear operator $L[\hat{c}] = \partial_y[D(y)\partial_y \hat{c}]$ expresses the cross-flow turbulent diffusion and has a discrete spectrum of eigenvalues. One of them is equal to zero, corresponding to the neutral eigenmode $\hat{c}$ =const, that is any constant concentration across the channel. All the other eigenvalues are negative. This expresses the fact that the diffusion makes non-uniformities of the concentration across the channel decay under the zero-flux boundary conditions.

We intend to reformulate our dispersion problem in a way that makes it similar to (3). After sufficiently long time, variations of the concentration along the channel, that is in x direction, become slow; so we suppose that the wave number k is small. Adjoin to (5) the trivial equation $\partial_t k = 0$ in order to formally treat the wave number k as a variable and the term $k\hat{c}$ as a "nonlinear" term. As governed by (5) and $\partial_t k = 0$, the dynamics exponentially quickly evolve to a low-dimensional state, when each of the decaying (fast) modes of $\hat{c}$ depends on t via the slow neutral mode. As a measure of the "amplitude" of the neutral mode we choose the depth-average concentration, $\hat{C}$. As a result, we have

$$\hat{c} = \hat{c}(\hat{C}, y) \quad \text{such that} \quad \partial_t \hat{C} = G(\hat{C}) \,. \tag{6}$$

With (6) taken into account, equation (5) becomes

$$L[\hat{c}] = \frac{\partial \hat{c}}{\partial \hat{C}} G + iku\hat{c} \,. \tag{7}$$

Since the problem is linear, we assume linear asymptotics

$$\hat{c} = \sum_{n=0}^{\infty} c_n(y)(ik)^n \hat{C} \,, \quad G = \sum_{n=1}^{\infty} g_n(ik)^n \hat{C} \,. \tag{8}$$

The definition of $\hat{C}$ as the depth-average implies the conditions

$$\frac{1}{h}\int_0^h c_0\,dy = 1 \,, \quad \int_0^h c_n\,dy = 0 \quad \text{for } n = 1, 2, \ldots \,. \tag{9}$$

Substituting (8) into (7) and collecting same order terms on the small parameter k we obtain a sequence of equations for the unknown functions $c_n(y)$ and coefficients g_n:

$$L[c_0] = 0 \,, \tag{10}$$

$$L[c_n] = \sum_{m=1}^{n} c_{n-m} g_m + u(y)c_{n-1} \quad \text{for } n = 1, 2, \ldots \,. \tag{11}$$

Integrating (11) across the channel and using the zero-flux boundary conditions, we get

$$g_n = -\overline{u(y)c_{n-1}} \quad \text{for } n = 1, 2, \ldots \,, \tag{12}$$

where overline denotes averaging across the channel. Successively one can calculate g_n and c_n for any n. Retaining only two leading terms in the G series in (8) we get

$$\partial_t \hat{C} = g_1(ik)\hat{C} + g_2(ik)^2\hat{C} \,. \tag{13}$$

Now, taking the inverse Fourier transform of (13) , we obtain the advection-diffusion equation for the depth-average concentration

$$\partial_t C = g_1 \partial_x C + g_2 \partial_x^2 C \,. \tag{14}$$

3 Logarithmic velocity profile

Consider the traditional logarithmic velocity profile,

$$u = \frac{u_*}{\kappa} \ln\left(\frac{u_* y}{\nu}\right) + E \,, \tag{15}$$

where u_* is the friction velocity, κ is the von Karman constant, ν is the kinematic molecular viscosity and E is constant. The law (15) gives negative infinity at $y = 0$, but within viscous sublayer, $0 < y < h_1$, where roughly $h_1 \sim 70\nu/u_*$, formula (15) is not applicable. Therefore, we exclude the region $0 < y < h_1$ from consideration and apply the zero-flux boundary condition at $y = h_1$ instead of $y = 0$:

$$D\partial_y c|_{y=h_1} = D\partial_y c|_{y=h} = 0 \,. \tag{16}$$

The turbulent diffusion coefficient is

$$D(y) = K(Sc)\frac{u_*^2}{\partial_y u} \,, \tag{17}$$

where the non-dimensional coefficient $K(Sc)$ is positive and generally depends on the Schmidt number. Substituting (15) into (17) we get

$$D(y) = \kappa K u_* y \,. \tag{18}$$

Now we insert (15) and (18) into the basic equation (1) and non-dimensionalise using h/u_* as the time scale and h as the length scale. As a result, we obtain in non-dimensional form (keeping the old notations for convenience):

$$\partial_t c + u(y)\partial_x c = \kappa K \partial_y (y \partial_y c) \,, \tag{19}$$

where

$$u(y) = \frac{1}{\kappa}\ln(Ry) + \frac{E}{u_*} \quad \text{for } \varepsilon < y < 1 \,, \ \varepsilon = h_1/h \,, \tag{20}$$

where

$$R = \frac{u_* h}{\nu} \tag{21}$$

is the Reynolds number. We note that

$$\varepsilon = h_1/h \sim 70\nu/(u_* h) = 70/R \,. \tag{22}$$

At large Reynolds numbers ε is small, so our final results in this section are obtained by taking the limit $\varepsilon \to 0$. In non-dimensional form the boundary conditions (16) become

$$y\partial_y c|_{y=\varepsilon} = y\partial_y c|_{y=1} = 0 \,. \tag{23}$$

Calculating c_0 from (10) under the boundary conditions (23) and satisfying (9) in non-dimensional form,

$$\int_0^1 c_0 \, dy = 1 \,,$$

we readily find

$$c_0 = 1 \,. \tag{24}$$

Using (12) for $n = 1$ and using (20) and (24) we have

$$g_1 = -\overline{u(y)c_0} = -\frac{1}{1-\varepsilon}\int_\varepsilon^1 \left[\frac{1}{\kappa}\ln(Ry) + \frac{E}{u_*}\right] dy$$

$$= -\frac{1}{\kappa}(\ln R - 1) - E/u_* + \frac{1}{\kappa}\frac{\varepsilon}{1-\varepsilon}\ln\varepsilon \,.$$

In the limit $\varepsilon \to 0$, taking into account that $\varepsilon \ln \varepsilon \to 0$, we get

$$g_1 = -\frac{1}{\kappa}(\ln R - 1) - E/u_* \,. \tag{25}$$

Using (25) and (24) we can determine c_1 from (11) at $n = 1$,

$$L[c_1] = c_0 g_1 + u(y) c_0 \,, \tag{26}$$

giving

$$\partial_y (y \partial_y c_1) = -\frac{1}{K\kappa^2}\left[\ln R - 1 + \ln(Ry)\right] \,. \tag{27}$$

Integrating (27) once under the boundary conditions (23) we get

$$\partial_y c_1 = \frac{1}{K\kappa^2}\ln y$$

and further

$$c_1 = \frac{1}{K\kappa^2}(y\ln y - y) + A \,.$$

The integration constant A is determined from the condition (9),

$$\int_\varepsilon^1 c_1 \, dy = 0 \,.$$

As a result,

$$c_1 = \frac{1}{K\kappa^2}(y\ln y - y) + \frac{3}{4K\kappa^2} \,. \tag{28}$$

Using (28) and (12) for $n = 2$ we determine g_2:

$$g_2 = -\overline{u(y)c_1(y)} = -\frac{1}{1-\varepsilon}\int_\varepsilon^1 \left[\frac{1}{\kappa}\ln(Ry) + E/u_*\right]$$

$$\times \left[\frac{1}{K\kappa^2}(y\ln y - y) + \frac{3}{4K\kappa^2}\right] dy$$

$$= -\frac{1}{1-\varepsilon}\left\{\frac{1}{K\kappa^3}\left[\ln R\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right) + \frac{y^2}{2}(\ln y)^2 - \frac{y^2}{2}\ln y + \frac{y^2}{4}\right]\right.$$

$$+\frac{E/u_*}{K\kappa^2}\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right) - \frac{1}{K\kappa^3}\left(\frac{y^2}{2}\ln R + \frac{y^2}{2}\ln y - \frac{y^2}{4}\right)$$

$$\left. -\frac{E/u_*}{K\kappa^2}\frac{y^2}{2} + \frac{3}{4K\kappa^3}\left[y\ln(Ry) - y\right] + \frac{3E/u_*}{4K\kappa^2}\, y\right\}_\varepsilon^1 \,. \tag{29}$$

Taking the limit $\varepsilon \to 0$ in (29) (note that $\varepsilon \ln \varepsilon \to 0$ and $\varepsilon^2 \ln \varepsilon \to 0$) we get the result in the simple form

$$g_2 = \frac{1}{4K\kappa^3} .$$

See that g_2 does not depend on R and hence on u_*. As we observe from the advection-diffusion equation (14), g_2 characterises the intensity of diffusion along the flow. Thus, this intensity is independent of the friction velocity u_* measuring how fast the fluid flows. However, u_* is naturally present in the expression (25) for g_1 characterising advection.

4 Power velocity profile

The power velocity profile in the turbulent boundary layer is proposed in [9] as an alternative to the logarithmic profile (15). In dimensional form

$$u(y) = u_*(1/\sqrt{3}\ln Re + 5/2)\left(\frac{u_* y}{\nu}\right)^\alpha , \qquad (30)$$

where

$$\alpha = \frac{3}{2\ln Re}$$

and Re is the Reynolds number. It is different from conventional definitions of the Reynolds number such as (21). The number Re is not defined by an explicit formula, see for details [9]. However, Re is connected to the traditional Reynolds number Re_θ based on the momentum displacement thickness. The tables in [9] show how to transform one into the other. Further, Re_θ can be linked to R using experimental data on turbulent flows in channels. However, at this stage we leave this issue out of our attention and focus on the low-dimensional modelling.

Substituting (30) into the expression for the diffusion coefficient

$$D(y) = K(Re, Sc)\frac{u_*^2}{\partial_y u}$$

we find

$$D(y) = \frac{u_*^{1-\alpha} K \nu^\alpha}{(1/\sqrt{3}\ln Re + 5/2)\alpha}\, y^{1-\alpha} . \qquad (31)$$

The boundary condition on the bottom is set at $y = 0$, see (2). In non-dimensional form (hereafter all the quantities are non-dimensional)

$$y^{1-\alpha}\partial_y c\big|_{y=0} = y^{1-\alpha}\partial_y c\big|_{y=1} = 0 . \qquad (32)$$

Substituting (30) and (31) into the basic equation (1) and non-dimensionalising we obtain

$$\begin{aligned} &\partial_t c + R^\alpha(1/\sqrt{3}\ln Re + 5/2)y^\alpha \partial_x c \\ &= \frac{K}{(1/\sqrt{3}\ln Re + 5/2)\alpha R^\alpha}\partial_y[y^{1-\alpha}\partial_y c] . \end{aligned} \qquad (33)$$

The velocity in (33) is

$$u(y) = R^\alpha(1/\sqrt{3}\ln Re + 5/2)y^\alpha . \qquad (34)$$

Calculating c_0 from (10) under the boundary conditions (32) we trivially get

$$c_0 = 1 . \qquad (35)$$

Using (35) and (34) in (12) at $n = 1$, we find

$$g_1 = -\overline{u(y)c_0} = -R^\alpha\frac{(1/\sqrt{3}\ln Re + 5/2)}{\alpha + 1} . \qquad (36)$$

Now we need to solve (26), that is

$$L[c_1] = c_0 g_1 + u(y)c_0 ,$$

coupled with (35), (34) and (36):

$$\begin{aligned} \partial_y[y^{1-\alpha}\partial_y c_1] = &-\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K(\alpha+1)} \\ &+\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K}\, y^\alpha . \end{aligned} \qquad (37)$$

Integrating (37) and satisfying the boundary conditions (32) we get

$$\begin{aligned} c_1(y) = &-\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K(\alpha+1)^2}\, y^{\alpha+1} \\ &+\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K(\alpha+1)(2\alpha+1)}\, y^{2\alpha+1} + B . \end{aligned} \qquad (38)$$

The integration constant B is determined from the condition (9):

$$\begin{aligned} B = &\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K(\alpha+1)^2(\alpha+2)} \\ &-\frac{R^{2\alpha}(1/\sqrt{3}\ln Re + 5/2)^2\alpha}{K(\alpha+1)(2\alpha+1)(2\alpha+2)} . \end{aligned} \qquad (39)$$

Now we calculate the coefficient g_2 from (12) at $n = 2$ using (34) and (38):

$$\begin{aligned} g_2 = -\overline{u(y)c_1(y)} = &\frac{R^{3\alpha}(1/\sqrt{3}\ln Re + 5/2)^3\alpha}{K} \\ &\times\left[\frac{1}{2(\alpha+1)^3} - \frac{1}{(\alpha+1)(2\alpha+1)(3\alpha+2)}\right] \\ &-\frac{R^\alpha(1/\sqrt{3}\ln Re + 5/2)B}{\alpha+1} . \end{aligned} \qquad (40)$$

The effective diffusion coefficient, g_2, as well as the effective velocity, g_1, both depend on the Reynolds numbers.

5 Conclusion

Using centre manifold technique, we derive the advection-diffusion equation

$$\partial_t C = g_1\partial_x C + g_2\partial_x^2 C$$

for the depth-average concentration of contaminant in a channel using boundary-layer velocity profiles. Two cases are analysed separately: (a) logarithmic and (b) power velocity profile. The coefficients g_1 and g_2 responsible for the advection and downstream diffusion respectively are determined in each case in terms of flow parameters. The model can be extended to include higher-order derivatives for a more precise description.

References

[1] Taylor, G.I., "Dispersion of Soluble Matter in a Solvent Flowing Through a Tube," *Proc. Roy. Soc. London Sect. A*, V219, pp. 186-203, 1953.

[2] Mercer, G.N., Roberts, A.J., "A Centre Manifold Description of Contaminant Dispersion in Channels With Varying Flow Properties," *SIAM J. Appl. Math.*, V50, pp. 1547-1565, 1990.

[3] Chikwendu, S.C., Ojiakor, G.U., "Slow-zone Model For Longitudinal Dispersion in Two-dimensional Shear Flows," *J. Fluid Mech.*, V152, pp. 15-38, 1985.

[4] Roberts, A.J., Strunin, D.V., "Two-zone Model of Shear Dispersion in a Channel Using Centre Manifolds," *Quart. J. Mech. Appl. Math.*, V57, pp. 363-378, 2004.

[5] Mei, Z., Roberts, A.J., Zhenquan, Li, "Modelling the Dynamics of Turbulent Floods," *SIAM J. Appl. Math.*, V63, pp. 423-458, 2003.

[6] Georgiev, D.J., Roberts, A.J., Strunin, D.V., "Nonlinear Dynamics on Centre Manifolds Describing Turbulent Floods: K-Omega Model," *Discrete and Continuous Dynamical Systems - Supplement*, pp. 419-428, 2007.

[7] Georgiev, D.J., Roberts, A.J., Strunin, D.V., "The Dynamics of The Vertical Structure of Turbulence in Flood Flows," *ANZIAM Journal (E)*, V48, pp. C573-C590, 2007.

[8] Landau, L.D., Lifshits, E.M., *Fluid Mechanics*, Pergamon, 1987.

[9] Barenblatt, G.I., Chorin, A.J., Prostokishin, V.M., "Self-similar Intermediate Structures in Turbulent Boundary Layers at Large Reynolds Numbers," *J. Fluid Mech.*, V410, pp. 263-283, 2000.

[10] Barenblatt, G.I, "Transfer of a Passive Additive in a Turbulent Boundary Layer at Very Large Reynolds Numbers," *Proc. Nat. Acad. Sci.*, V100, pp. 1481-1483, 2003.

[11] Roberts, A.J., "Appropriate Initial Conditions for Asymptotic Descriptions of the Long Term Evolution of Dynamical Systems," *J. Austral. Math. Soc. Ser. B*, V31 pp. 48-75, 1989.