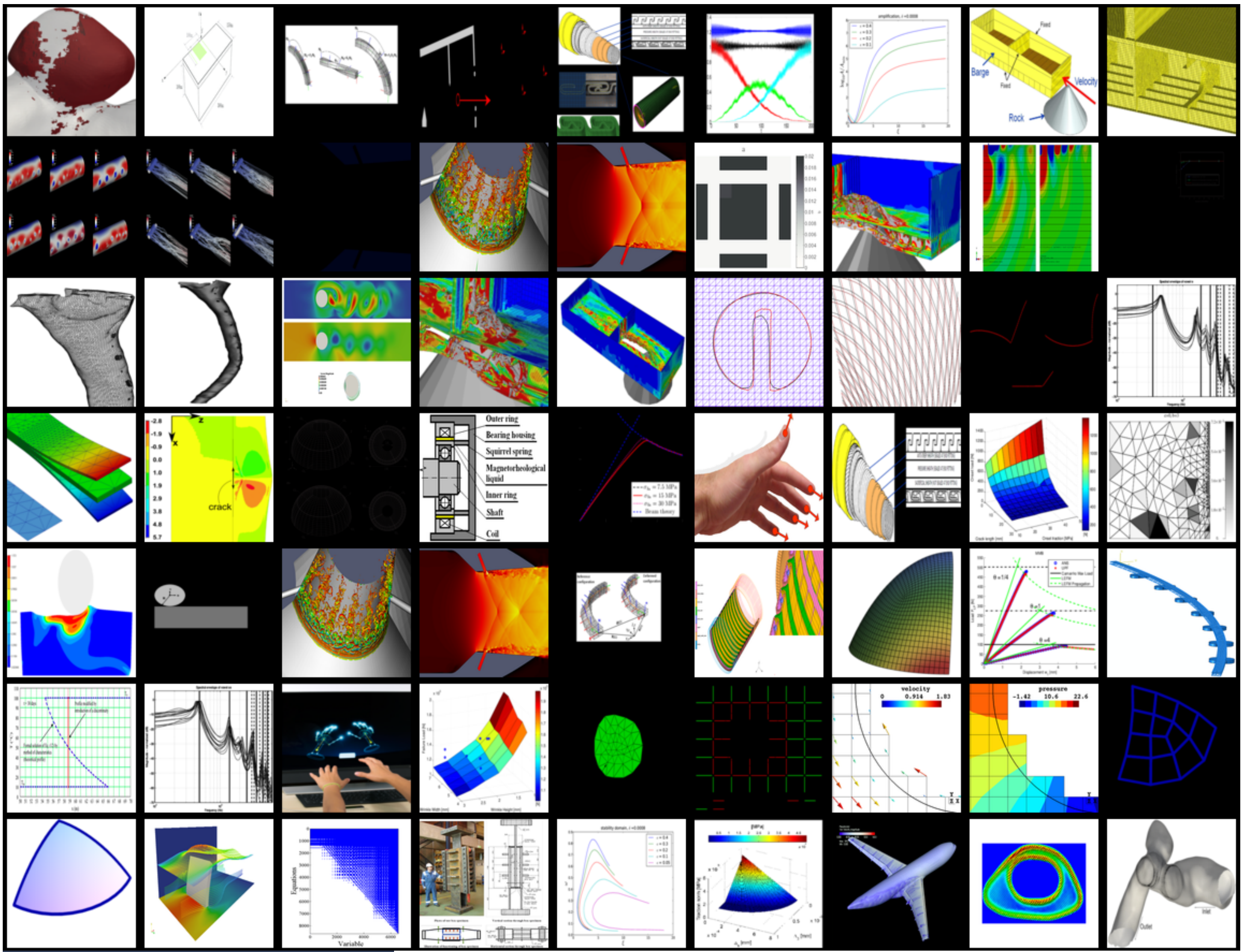

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Center for Biomedical Computing
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Stress-strained state and the stability of a spherical segment under the influence of a load with a flat base.

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Summary. In this paper the problem of the buckling of the transversal-isotropic segment of spherical shell with the different thicknesses under the influence of the load with a flat base is studied. The spherical segment has a rigid support on the edge and previously has been loaded by internal pressure. The solution of this problem is based on the theory of the shell of moderate thickness by Paly-Spiro. This theory takes into account the influence of the cross section shear and change of the shell thickness. For modelling such large deformations the method of consequent loading is used. In this method, due to the use of linear physical relations, it is possible to trace the non-linear problem at each separate stage to the solution of a linear system. The comparison of the results which were obtained with the use of the method of linearization of non-linear equilibrium equations and the method of minimization of elastic potential of the shell has been done. The problems of stress-strain state of soft and close to soft shells that are under the influence of a load with a flat base are important for analyzing the data related to measuring a very important in ophthalmology characteristic of intraocular pressure.

Key words: nonlinear shell theory, stability, load with a flat base

The problem statement

Let us consider the problem of state-strain state and the loss of stability of the transversally isotropic segment of a spherical shell under the influence of the load with a flat base, see figure 1. The spherical segment has a rigid support on the edge and previously has been loaded by the internal pressure.

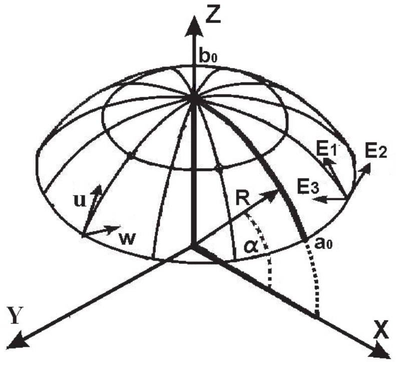

Figure 1. The dual structure of continuum mechanics.

The deformation and loss of stability are considered to be axisymmetric therefore we can take into account only the half of the arc created by the vertical cross section. On the pole point and the edge the boundary conditions of the symmetry and rigid support were introduced. Thus all the values depend only on one spherical coordinate, $\alpha \in [a_0, \pi/2]$, where a_0 – characterizes the angle of segment opening.

The elastic moduli of the considered shell differ by an order, therefore the theory of anisotropic shells of moderate thicknesses by Paly-Spiro [1] is used. This theory takes into account the influence of cross section shear and the deformation into the direction of the normal to the middle surface and is based on the following hypotheses:

1) a rectilinear element normal to the middle surface of a shell remains rectilinear after the deformation;

2) the cosine of the angle of inclination of the shell of these fibers to the middle surface of the deformed shell equals to the averaged angle of the cross displacement.

The mathematical formulation of the accepted hypotheses resolves to the following equations:

$$\begin{aligned} &u_1 = u + \phi \cdot z, \quad u_3 = w + F(\alpha, z), \\ &\phi = \gamma_1 + \phi_0, \qquad \phi_0 = -\frac{1}{A_1}\frac{\partial w}{\partial \alpha} + k_1 u, \end{aligned} \tag{1}$$

where u_1 and u_3 – tangential and normal displacements of the shell, u and w – displacements of the middle surface, φ – turning angle of the normal in plane (α, z); φ_0 – turning angle of the normal to the medial surface; γ_1 – shear angle. The function F(α, z) characterizes length change of the normal to the medial surface.

The methods of solution

The deformations which appear under the influence of the load with a flat base are large and we need the geometrical non-linear shell theory to describe them. However, the construction of the solution of the non-linear theory equations presents a significant difficulty [2, 3]. Therefore the solution of this problem is based on the method of consequent loading.

In this method pressure P is presented as the sum of monotonous consequent loadings:

$$P = \Delta P_1 + \Delta P_2 + ... + \Delta P_n \qquad (\Delta P > 0, n >> 1). \tag{2}$$

Thus the geometrically non-linear problem is reduced to a consequent solution of linear problems for a previously loaded shell of revolution. The original stress strain state of the shell was defined by the results of the previous loadings $\sum_{i=1}^{N} \Delta P_i$.

At each stage only the part of all the loading ΔP_i is applied to the shell so that the deformations should stay small. We take into account the fact that at previous stage of the loading each point of the original surface changed its own position and thickness. It needs recalculating Lame coefficients A_1, A_2, curvatures k_1, k_2, the new law of distribution of function of thicknesses h. Besides, the values of the stress strain state of the shell obtained at the previous stages of the loading are included in the resulting equation. The adding loads ΔP_1, ΔP_2, ΔP_3 must be small in comparison with that values which the upper critical load corresponds to.

In this work the method of consequent loading is presented in two ways. The first of them is the method of linearization of non-linear equilibrium equations [1] and the second - the method of minimization of elastic potential of the shell [4]. These ways of the problem solution give us the different approaches to the estimation of the critical load.

In the first method the system of differential equations of the equilibrium of shell is solved at the each separate stage of the loading. The right part of the system except the next addition of the load ΔP_i takes into account the forces and deformations from the previous step of loading. We presume that the critical load corresponds to the case when ΔP_i=0 the system ceases to be an identical. Non-trivial solutions appear due to the influence of the internal forces.

In the second approach the solution is obtained by minimization of elasticity potential of the shell with use of the Ritz numerical method. The displacements are represented as the functional series which satisfy the boundary conditions. Having the partial derivatives of the elasticity potential for the each member of the series of the displacement functions, we obtain the system of non-linear differential equations. To solve the obtained system of non-linear algebraic equations, the continuation method by the loading parameter ΔP_i is used. In the result all solution is reduced to the system of linear algebraic equations for the members of the series of the displacement functions. The loss of stability takes place when the resulting matrix of this system becomes degenerate. In this case the post–critical state is obtained by the change of the parameter.

The comparison of the results which were obtained with the use of these methods is done. For both methods we defined the dependence of the load contact area on the influence of the internal pressure, thickness and the curvature radius of the spherical shell segment. The distributions of the stress strain state for different values of the load are constructed. The possible appearance of exfoliation in the area of the contact with the flat base is studied. This problem may have a biomechanical application.

Numerical simulation

Let us introduce the comparison of the results which were obtained with the use of considered methods for the problem of deformation of the cornea of the eye. In the Maklakov method of tonometry a human eye is deformed by flat base load of the bar. The diameter of the contact zone with cornea is measured and the measured diameter length is used in estimating the intraocular pressure (IOP). When the intraocular pressure is not very high and the thickness of an eye shell (cornea) is small (for example, after refractive surgery) the cornea may buckle and detach from tonometer. It leads to errors in estimates of intraocular pressure[5].

The cornea average curvature radius R=8 mm, its foundation radius is constant and equal R_{os}=5.25 mm, thickness h changes linearly from h_a=1 mm at the edge of segment to h_b=0.5 mm at the pole point. For elastic modulus, cross–section shear, and poisson's ratios these values was taken [6, 7] : $E_1 = E_2 = 7 \cdot 10^4$ Pa, $E_3 = 7 \cdot 10^2$ Pa, $G = 7 \cdot 10^3$ Pa, $\nu_{21} = \nu_{31} = \nu_{12} = \nu_{32} = 0.4$, $\nu_{13} = \nu_{23} = 0.01$.

We consider that these geometrical parameters correspond to the case when the cornea is under the influence of IOP of 22 mm.Hg. (1 mm.Hg.= 133.3 Pa). Therefore firstly we consider the shell of a smaller height. It is loaded by negative IOP. The geometry of obtained form is considered to be initial. The forces which act in it are equated to zero and it is loaded by the similar but positive load. The new shell is higher than a previous one but it is taken into account the influence of internal pressure.

For modelling the influence of the bar we introduce the function which takes into account that the load acts only in straightened area on the top. In the case of appearance of displacement which leads to exfoliation of the shell from the bar the negative load starts acting. This negative load takes off the part of the overall load from the exfoliated surface [8].

Conclusions

Let us consider deformations and the distribution of normal stress in the contact area corresponding to them. It should be noted that the general views of the distribution of normal stress of reviewed methods is almost similar.

a) Under the influence of the small load 0.75 gm the main stress is concentrated in the vicinity of the pole and decreases exponentially towards the edges.

b) In the case of load increase up to 3.5 gm the normal stress redistributes and the unloaded area in the vicinity of the pole appears.

c) For the given value of load of 10 gm, we can see a slightly loaded ring area. The main surface stresses act only at the edge of the loaded area and at the pole point.

Table 1. The comparison of the radiuses of contact area .

Thickness at the pole point h_b (mm)	0.5	0.465	0.43	0.395	0.36
The method of minimization R_{out}	3.00	3.04	3.08	3.12	3.17
The method of linearization R_{out}	2.97	3.00	3.05	3.10	3.15

d) When the loading is being continued (15 gm), the distribution of the load obtains a more complex shape.

e) In the case of solving this problem with the use of the method of lineralization the shell may lose the stability as this method takes into account the force ΔT_1 evidently. Also the system corresponding to it has the higher order. The force ΔT_1 reaches significantly large values in the vicinity of the pole point. This can make the shell fall down inside – lose the stability.

The table 1 presents the values of radiuses, of the contact area of the shell with a bar of 10 gm. R_{out} for different values of the thickness in the vicinity of the pole h_b mm. curvature radiuses which are equal to R=8.5-h_b mm. These calculations may help to model the consequences of keratometric operations, because in the course of these operations a layer of tissue is cut off the top the cornea. As one can see, the reduction of the value of thickness h_b at the pole leads to the increase of the contact area R_{out}. It may contribute to error in measurement IOP. Radiuses received by method of linearization are less then radius received by method of minimization of elastic potential, which can be explained by direct accounting of effort function ΔT_1 in the first method. When the linearization method is used, unloaded or slightly loaded areas appear earlier (at a lower level of pressure), then in the case when the minimization method is used.

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