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Review

Tests on a Subset of Regression Parameters for Factorial Experimental Data with Uncertain Higher Order Interactions

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ABSTRACT

The data generated by many factorial experiments are analyzed by linear regression models. Often the higher order interaction terms of such models are negligible (e.g., R. Mead, The Design of Experiments, Cambridge University Press, Cambridge, 1988, p. 368) although there is uncertainty around it. This kind of nonsample prior information (NSPI) can be presented by null hypotheses (cf. T.A. Bancroft, Ann. Math. Stat. 15 (1944), 190–204), and its uncertainty removed through appropriate statistical test. Depending on the level of the NSPI the unrestricted, restricted, and pretest (PTT) tests are defined. The sampling distributions of test statistics and power functions of the three tests are derived. The graphical and analytical comparisons of powers reveal that the PTT dominates over the other tests.

1. INTRODUCTION

In many real-life applications, the data of factorial experiments are analyzed using linear regression models. Unlike the classical and cell mean models, the regression model based method has the advantage of fitting the model in the presence of missing values or even with unbalanced data. The regression model for the response, Y of a 2^3 factorial experiment without any replication can be written as

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} 4 x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{123} x_1 x_2 x_3 + \epsilon, \tag{1}$$

where β's are unknown regression parameters and x_1, x_2, and x_3 represent the coded level of factors 1, 2, and 3, respectively, each assuming value -1 or 1 for the absence and presence of the factor. It is commonly assumed that the error term $\epsilon \sim N\left(0, \sigma^2\right)$, where $\sigma^2 > 0$ is a unknown spread parameter.

Mead [1] and Hinkelmann and Kempthorne [2] discussed how the higher order interactions of factorial experiments are believed to be negligible. Kabaila and Tesseri [3] reinforced that this kind of believe on the higher order interactions is the basis for fractional factorial experiments. To make valid inference on the remaining parameters, the uncertainty in the assumption of negligible interactions of any order can be represented by a hypothesis and conduct an appropriate test to remove the uncertainty (cf. Bancroft [4]). Any such assumptions can be considered as the nonsample prior information (NSPI) and used in the formal inferences on the remaining parameters of the model. Hodges and Lehmann [5] discussed the use of prior information from previous experience in reaching statistical decisions. Kabaila and Dharmarathne [6] compared Bayesian and frequentist interval estimators in regression utilizing uncertain prior information.

In the classical approach inferences about, unknown population parameters are drawn exclusively from the sample data. This is true for both estimation of parameters and hypothesis tests. Use of reliable NSPI from trusted sources (cf. Bancroft [4]), in addition to the sample data, is likely to improved the quality of estimation and test. The use of NSPI has also been demonstrated by Kempthorne [7,8], Bickel [9], Khan [10–12], Khan and Saleh [13–15], Khan *et al.* [38] and Saleh [16].

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Such NSPI is usually available from previous studies or experts in the field or practical experience of the researchers, and is independent of the sample data under study. The main purpose of inclusion of NSPI is to improve the quality of statistical inference. In reality, NSPI on the value of any parameter may or may not be close to the unknown true value of the parameter, and hence there is always an element of uncertainty. But the uncertain NSPI can be expressed by a null hypothesis and an appropriate statistical test can be used to remove the uncertainty. The purpose of the preliminary test (pretest) on the uncertain NSPI in the hypothesis testing or estimation is to improve the quality of the inference (cf. Khan [17]; Saleh [16]; Yunus [18]). Kabaila and Dharmarathne [6] and Kabaila and Tissera [3] used NSPI to construct confidence intervals for regression parameters. In this paper, we express the data from a factorial experiment as a linear model (see (1)) in order to test the coefficients of the main effects (and lower order interactions) when there is uncertain NSPI on the coefficients of higher level interactions.

The uncertain NSPI can be any of the following types: (i) *unknown* (unspecified)—NSPI is not available, (ii) *known* (certain or specified)—exact value is the same as the parameter, and (iii) *uncertain*—suspected value is unsure. In the estimation regime, to cater for the three different scenarios, the following three different estimators are appropriate: (i) unrestricted estimator (UE), (ii) restricted estimator (RE), and (iii) preliminary test estimator (PTE) (see eg Judge and Bock [19]; Saleh [16]).

Almost all of the works in this area are on the estimation of parameter(s). These include Bancroft [4,20], Han and Bancroft [21], and Judge and Bock [19] introduced the preliminary test estimation method to estimate the parameters of a model with uncertain NSPI. Later Khan [10–12], Khan and Saleh [14], and Khan and Hoque [22] covered various work in the area of improved estimation.

The testing of parameters in the presence of uncertain NSPI is relatively new. The earlier works include Tamura [23] and Saleh and Sen [24,25] in the nonparametric setup. Later Yunus and Khan [26–28] used the NSPI for testing hypothesis using nonparametric methods. The problem is yet to be explored in the parametric context. In this paper testing of hypotheses is considered on the coefficients of the main effects in the model in (1) when uncertain nonsample information on the coefficients of the higher order interactions is available.

To set up the hypotheses for the tests, let's assume that the interaction terms (e.g., last four $\beta's$) of model (1) are suspected to be zero, but not sure. Then under the three different scenarios define three different tests: (i) unrestricted test (UT), (ii) restricted test (RT), and (iii) pretest test (PTT) to test on the remaining regression parameters (first four $\beta's$) of the model. The UT uses the sample data alone but the RT and PTT use both the NSPI and the sample data. The PTT is a choice between the UT and the RT.

The regression model and hypotheses are provided in Section 2. Some useful results are discussed in Section 3. The proposed test statistics and their sampling distributions are provided in Sections 4 and 5 respectively. Section 6 derives the power function and size of the tests. An example with real data is included in Section 7. The power of the tests are compared in Section 8. Some concluding remarks are provided in Section 9.

2. THE REGRESSION MODEL AND HYPOTHESES

The regression model for the data from a 2^3 factorial experiment, as stated in (1), can be viewed as special case of the multiple regression model where each of the main effect and interaction terms are represented as the explanatory variables. For an n set of observations on the response (Y) and k explanatory/independent variables $(X_1, \cdots, X_k)$, that is, $\left(X_{ij}, Y_i\right)$, for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, k$, the linear model is given by

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_k X_{ik} + e_i, \tag{2}$$

where $\beta's$ are the regression parameters and e_i's are the error terms. The model in equation (2) can be here expressed following convenient form

$$Y = X\beta + e, \tag{3}$$

where $\beta = (\beta_0, \beta_1, \cdots, \beta_{r-1}, \beta_r, \cdots, \beta_k)'$ is a column vector of order $(k+1) = p$, $Y = \left(y_1, \cdots, y_n\right)'$ is vector of response variables of dimension $n \times 1$, X is an $n \times p$ matrix of full rank of the independent variables, and e is a vector of errors. It is assumed components e are identically and independently distributed as normal variable with mean 0 and variance σ^2, so that $e \sim N_n\left(0, \sigma^2 I_n\right)$, where I_n is an identity matrix of order n.

To formulate the testing problem, let $\delta_1 = (\beta_0, \cdots, \beta_{r-1})$ be a subset of r regression parameters and $\delta_2 = (\beta_r, \cdots, \beta_k)$ be another subset of $\left(p - r\right) = s$ regression parameters, so that $r + s = p$. The regression vector β is then partitioned as $\beta' = \left(\delta_1', \delta_2'\right)$, where δ_1' is a r dimensional sub-vector, and δ_2' is a subvector of dimension $s = p - r$. In a similar way, matrix X is partitioned as (X_1, X_2) with $X_1 = (1, x_1, \cdots, x_{r-1})$, an $n \times r$ matrix, and $X_2 = (x_r, \cdots, x_k)$, an $n \times s$ matrix. Then the multiple regression model in (3) can be written as

$$Y = X_1\delta_1 + X_2\delta_2 + e. \tag{4}$$

We wish to perform test on the subvector δ_1 (or β_1) when NSPI on the subvector δ_2 (or β_2) is available.

Depending on the nature of the NSPI on the subvector, δ_2, to be (i) unspecified, (ii) specified (fixed), or (iii) suspected to be a specific value but not sure, we define three different tests for testing the other subvector, δ_1. Let A_1 be a $q_1 \times r$ matrix of constants and A_2 be another $q_2 \times s$ matrix of constants, where $q = q_1 + q_2$ so that

$$A = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}, \tag{5}$$

that is, A is a $q \times p$ matrix and O is a matrix of zeros. The NSPI on the value of δ_2 is expressed in the form of a null hypothesis, H_0^*: $A_2\delta_2 = h_2$. Then to test the null hypothesis H_0: $A_1\delta_1 = h_1$ against H_a: $A_1\delta_1 \neq h_1$.

The hypothesis defined here, H_0: $A\beta = h$, that is,

$$H_0: A\beta = \begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

is a generalization of the test of equality of components of the regression vector and the subhypothesis

$$H_0: \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} = \begin{pmatrix} \delta_1 \\ 0 \end{pmatrix}$$

(cf. Saleh [16], pp. 340). Note that h_2 is only used for the pretest on β_2 (i.e., PT), as such its value remains the same when testing β_1.

3. SOME PRELIMINARIES

To formally define the tests let us consider the following expressions, partitions and results. For the full rank design matrix X we write

$$X'X = \begin{pmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{pmatrix}, \quad (X'X)^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{6}$$

where

$$A_{11}^{-1} = X_1'X_1 - X_1'X_2\left(X_2'X_2\right)^{-1}X_2'X_1 \text{ and } A_{22}^{-1} = X_2'X_2 - X_2'X_1\left(X_1'X_1\right)^{-1}X_1'X_2.$$

Then the unrestricted least squares estimator of the regression parameters is given by

$$\tilde{\beta} = (X'X)^{-1}X'Y = \begin{pmatrix} \tilde{\delta}_1 \\ \tilde{\delta}_2 \end{pmatrix}, \tag{7}$$

so that the UE of the two subvectors are

$$\tilde{\delta}_1 = A_{11}X_1'Y + A_{12}X_2'Y \text{ and } \tilde{\delta}_2 = A_{22}X_2'Y + A_{21}X_1'Y. \tag{8}$$

Then the sum of square errors for the *full regression model* with k regressors is given by

$$SSE_F = \left(Y - X\tilde{\beta}\right)'\left(Y - X\tilde{\beta}\right), \tag{9}$$

so an unbiased estimator of σ^2 is $MSE_F = SSE_F/\left(n-p\right)$.

Let δ_2 be specified to be δ_{20}, so the RE of β becomes,

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \hat{\delta}_1 \\ \hat{\delta}_2 \end{pmatrix} = \tilde{\beta} - C^{-1}A'\left(AC^{-1}A'\right)^{-1}\left(A\tilde{\beta} - h\right), \tag{10}$$

where $C = X'X$. Since $\tilde{\beta} \sim N_p\left(\beta, \sigma^2C^{-1}\right)$, we get

$$\begin{aligned} \tilde{\delta}_1 &\sim N_r\left(\delta_1, \sigma^2A_{11}^{-1}\right) \\ \tilde{\delta}_2 &\sim N_s\left(\delta_2, \sigma^2A_{22}^{-1}\right). \end{aligned}$$

Similarly, as $\hat{\beta} \sim N_p\left(\beta, \sigma^2D^{-1}\right)$, where $D = \left[C^{-1} - C^{-1}A'\left(AC^{-1}A'\right)^{-1}AC^{-1}\right]^{-1}$, we get

$$\begin{aligned} \hat{\delta}_1 &\sim N_r\left(\delta_1, \sigma^2D_{11}^{-1}\right) \\ \hat{\delta}_2 &\sim N_s\left(\delta_2, \sigma^2D_{22}^{-1}\right), \end{aligned}$$

in which

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.$$

Since $A\widetilde{\beta}$ is linear combination of normal variables $A\widetilde{\beta} \sim N_q\left(A\beta, \sigma^2\left[AC^{-1}A'\right]^{-1}\right)$ and $A\hat{\beta} \sim N_q\left(A\beta, \sigma^2\left[AD^{-1}A'\right]^{-1}\right)$.

Furthermore, the test statistic for testing H_0: $A_1\delta_1 = h_1$ is given by

$$F_* = \frac{1}{qs_e^2}\left\{\left(A_1\tilde{\delta}_1 - h_1\right)'\left[A_1\left(X_1'X_1\right)^{-1}A_1'\right]^{-1}\left(A_1\tilde{\delta}_1 - h_1\right)\right\}, \tag{11}$$

where $s_e^2 = \frac{1}{n-p}\left(Y - X\widetilde{\beta}\right)'\left(Y - X\widetilde{\beta}\right)$ is an unrestricted unbiased estimator of σ^2.

It is clear that $\frac{1}{\sigma^2}\left[\left(A_1\tilde{\delta}_1 - h_1\right)'\left(A_1C_1^{-1}A_1'\right)^{-1}\left(A_1\tilde{\delta}_1 - h_1\right)\right]$, where $C_1 = X_1'X_1$, follows a noncentral chi-squared distribution with q_1 degrees of freedom (df) and noncentrality parameter $\Delta_1^2/2$, where

$$\Delta_1^2 = \frac{\left(A_1\delta_1 - h_1\right)'\left[A_1C_1^{-1}A_1'\right]^{-1}\left(A_1\delta_1 - h_1\right)}{\sigma^2}. \tag{12}$$

Under H_a, the F_* statistic follows a noncentral F distribution with $\left(q_1, n-p\right)$ df and noncentrality parameter $\Delta_1^2/2$, and under H_0, F_* follows a central F distribution with $\left(q_1, n-p\right)$ df. Ohtani and Toyoda [29] and Gurland and McCullough [30] also used the above F test for testing linear hypotheses.

4. THE THREE TESTS

For testing δ_1 when NSPI is available on δ_2, define the tests as

i. For the UT, let ϕ^{UT} be the test function and T^{UT} be the test statistic for testing H_0: $A_1\delta_1 = h_1$, a vector of order q_1, against H_a: $A_1\delta_1 \neq h_1$ when δ_2 is unspecified,
ii. For the RT, let ϕ^{RT} be the test function and T^{RT} be the test statistic for testing H_0: $A_1\delta_1 = h_1$ against H_a: $A_1\delta_1 \neq h_1$ when $\delta_2 = \delta_2^0$ (specified) and
iii. For the PTT, let ϕ^{PTT} be the test function and T^{PTT} be the test statistic for testing H_0: $A_1\delta_1 = h_1$ against H_a: $A_1\delta_1 \neq h_1$ when δ_2 is suspected to be δ_2^0 following a pretest (PT) on δ_2. For the PT, let ϕ^{PT} be the test function for testing H_0^*: $A_2\delta_2 = h_2$ (a suspected vector of order q_2) against H_a^*: $A_2\delta_2 \neq h_2$. If H_0^* is rejected in the PT, then the UT is used to test on δ_1, otherwise the RT is used to test H_0.

Then the proposed three test statistics are defined as follows:

i. The UT for testing H_0: $A_1\beta_1 = h_1$ is given by

$$L^{UT} = \frac{\left(A_1\widetilde{\beta}_1 - h_1\right)'\left[A_1\left(X_1'X_1\right)^{-1}A_1'\right]^{-1}\left(A_1\widetilde{\beta}_1 - h_1\right)}{q\,s_e^2}, \tag{13}$$

where s_e^2 is the unbiased estimator of σ^2. Under H_0, L^{UT} follows an F distribution with q_1 and $\left(n-p\right)$ df whereas under H_a the L^{UT} follows a noncentral F distribution with $\left(q_1, n-p\right)$ df and noncentrality parameter $\Delta_1^2/2$.

ii. The RT is given by

$$L^{RT} = \frac{\left(A_1\hat{\delta}_1 - h_1\right)'\left[\left(A_1\left(D_{11}\right)^{-1}A_1'\right]^{-1}\left(A_1\hat{\delta}_1 - h_1\right)}{q_1\,s_e^2}. \tag{14}$$

Under H_a, L^{RT} follows a noncentral F distribution with $\left(q_1, n-p\right)$ df and noncentrality parameter $\Delta_2^2/2$, where

$$\Delta_2^2 = \frac{\left(A_1\delta_1 - h_1\right)'\left[A_1D_{11}^{-1}A_1'\right]^{-1}\left(A_1\delta_1 - h_1\right)}{\sigma^2}. \tag{15}$$

iii. For the preliminary test on δ_2, we test H_0^*: $A_2\delta_2 = h_2$ against H_a^*: $A_2\delta_2 \neq h_2$ using the statistic

$$L^{PT} = \frac{\left(A_2\widetilde{\delta}_2 - h_2\right)' \left[A_2 A_{22}^{-1} A_2'\right]^{-1} \left(A_2\widetilde{\delta}_2 - h_2\right)}{q_1\, s_e^2}, \tag{16}$$

where s_e^2 is an unbiased estimator of σ^2. Under H_a, L^{PT} follows a noncentral F distribution with $\left(q_2, n-p\right)$ df and noncentrality parameter $\Delta_3^2/2$, where

$$\Delta_3^2 = \frac{(A_2\delta_2 - h_2)' \left[A_2 D_{22}^{-1} A_2'\right]^{-1} (A_2\delta_2 - h_2)}{\sigma^2}. \tag{17}$$

Let α_j $\left(0 < \alpha_j < 1, \text{ for j} = 1, 2, 3\right)$ be a positive number. Then set F_{ν_1,ν_2,α_j}, in which ν_1 and ν_2 are the numerator and denominator d.f., respectively, such that

$$P\left(L^{UT} > F_{q_1,n-p,\alpha_1} | A_1\delta_1 = h_1\right) = \alpha_1, \tag{18}$$

$$P\left(L^{RT} > F_{q_1,n-p,\alpha_2} | A_1\delta_1 = h_1\right) = \alpha_2, \tag{19}$$

$$P\left(L^{PT} > F_{q_2,n-p,\alpha_3} | A_2\delta_2 = h_2\right) = \alpha_3. \tag{20}$$

To test H_0: $A_1\beta_1 = h_1$ against H_a: $A_1\beta_1 \neq h_1$, after pretesting on δ_2, the test function is

$$\Phi = \begin{cases} 1, & \text{if} \left(\mathrm{L}^{\mathrm{PT}} \leq \mathrm{F_c}, \mathrm{L}^{\mathrm{RT}} > \mathrm{F_b}\right) \text{ or } \left(\mathrm{L}^{\mathrm{PT}} > \mathrm{F_c}, \mathrm{L}^{\mathrm{UT}} > \mathrm{F_a}\right); \\ 0, & \text{otherwise,} \end{cases} \tag{21}$$

where $F_a = F_{\alpha_1,q_1,n-p}$, $F_b = F_{\alpha_2,q_1,n-p}$ and $F_c = F_{\alpha_3,q_2,n-p}$.

5. SAMPLING DISTRIBUTION OF TEST STATISTICS

The sampling distribution of the test statistics are discussed in this section. For the power function of the PTT the joint distribution of $\left(L^{UT}, L^{PT}\right)$ and $\left(L^{RT}, L^{PT}\right)$ are essential. Following Khan and Pratikno [31], let $\{M_n\}$ be a sequence of alternative hypotheses defined as

$$M_n\text{: } (A_1\beta_1 - h_1, A_2\beta_2 - h_2) = \left(\frac{\lambda_1}{\sqrt{n}}, \frac{\lambda_2}{\sqrt{n}}\right) = \lambda, \tag{22}$$

where $\lambda_{(q\times 2)}$ is a vector of fixed real numbers. Under M_n both $(A_1\beta_1 - h_1)$ and $(A_2\beta_2 - h_2)$ are nonsingular matrices and under H_0 they are singular matrices.

From Yunus and Khan [28] and (13), define the test statistic of the UT when δ_2 is unspecified, under M_n, as

$$L_1^{UT} = L^{UT} - \frac{n\sigma}{q_1 s_e^2}\left(A_1\widetilde{\beta}_1 - h_1\right)' \left[A_1\left(X_1'X_1\right)^{-1} A_1'\right]^{-1} \left(A_1\widetilde{\beta}_1 - h_1\right). \tag{23}$$

The statistic L_1^{UT} follows a noncentral F distribution with $\left(q_1, n-p\right)$ df and a noncentrality parameter which is a function of $(A_1\beta_1 - h_1)$.

From (14), under M_n, $(A_1\beta_1 - h_1)$ the test statistic of the RT becomes

$$L_2^{RT} = L^{RT} - \frac{n\sigma}{q_1 s_e^2}\left(A_1\hat{\delta}_1 - h_1\right)' \left[A_1 D_{11}^{-1} A_1'\right]^{-1} \left(A_1\hat{\delta}_1 - h_1\right). \tag{24}$$

The statistic L_2^{RT} also follows a noncentral F distribution with $\left(q_1, n-p\right)$ df and a noncentrality parameter which is a function of $(A_1\beta_1 - h_1)$ under M_n. From (16) the test statistic of the PT is given by

$$L_3^{PT} = L^{PT} - \frac{n\sigma}{q_2 s_e^2}\left(A_2\tilde{\delta}_2 - h_2\right)' \left[A_2\left(A_{22}\right)^{-1} A_2'\right]^{-1} \left(A_2\tilde{\delta}_2 - h_2\right). \tag{25}$$

Under H_a, the L_3^{PT} follows a noncentral F distribution with $\left(q_2, n-s\right)$ df and a noncentrality parameter which is a function of $(A_2\beta_2 - h_2)$.

From (13), (14), and (16) we observe that the L^{UT} and L^{PT} are correlated, and that L^{RT} and L^{PT} are uncorrelated. The joint distribution of the L^{UT} and L^{PT} is a correlated bivariate F distribution with $\left(q_1, n-p\right)$ and $\left(q_2, n-p\right)$ df. The details on the bivariate central F distribution is found in Krishnaiah [32], Amos and Bulgren [33], and El-Bassiouny and Jones [34]. Khan *et al.* [35] provided the probability density function and some properties of correlated noncentral bivariate F distribution. The covariance and correlation of the correlated bivariate F distribution for the $L^{UT} \sim F_{1(q_1,n-p)}$ and $L^{PT} \sim F_{2(q_2,n-p)}$ are then given, respectively, as

$$\begin{aligned} \operatorname{Cov}\left(L^{UT}, L^{PT}\right) &= \frac{2\left(n-p\right)\left(n-p\right)}{\left(n-p-2\right)\left(n-p-2\right)\left(n-p-4\right)} \text{ and} \\ \rho_{L^{UT},L^{PT}} &= \left\{\frac{q_1 q_2\left(n-p-4\right)}{\left(n-p+q_1-2\right)\left(n-p+q_2-2\right)\left(n-p-4\right)}\right\}^{1/2}. \end{aligned} \tag{26}$$

6. POWER FUNCTION AND SIZE OF TESTS

The power function and size of the three tests are derived in this section.

6.1. The Power of the Tests

From (13) and (23), (14) and (24), and (16), (21), and (25), the power function of the UT, RT, and PTT are given below.

i. The power of the UT,

$$\begin{aligned} \pi^{UT}(\lambda) &= P\left(L^{UT} > F_{\alpha_1,q,n-r}|M_n\right) = 1 - P\left(L_1^{UT} \leq F_{\alpha_1,q_1,n-p} - \Omega_{ut}\right) \\ &= 1 - P\left(L_1^{UT} \leq F_{\alpha_1,q_1,n-p} - k_{ut}\zeta_1\right), \end{aligned} \tag{27}$$

where $\Omega_{ut} = \frac{\sigma}{q_1 s_e^2}(\lambda_1)'[\gamma_1]^{-1}(\lambda_1)$, $\gamma_1 = A_1\left(X_1'X_1\right)^{-1}A_1'$, $\zeta_1 = (\lambda_1)'[\gamma_1]^{-1}(\lambda_1)$ and $k_{ut} = \frac{\sigma}{q_1 s_e^2}$.

ii. The power of the RT,

$$\begin{aligned} \pi^{RT}(\lambda) &= P\left(L^{RT} > F_{\alpha_1,q_1,n-p}|M_n\right) = P\left(L_2^{RT} > F_{\alpha_2,q_1,n-p} - \Omega_{rt}\right) \\ &= 1 - P\left(L_2^{RT} \leq F_{\alpha_2,q_1,n-p} - \Omega_{rt}\right) = 1 - P\left(L_2^{RT} \leq F_{\alpha_2,q_1,n-p} - k_{rt}\zeta_1\right) \end{aligned} \tag{28}$$

where $\Omega_{rt} = \frac{\sigma}{q_1 s_e^2}(\lambda_1)'[\gamma_1]^{-1}(\lambda_1)$, $\gamma_1 = A_1\left(X_1'X_1\right)^{-1}A_1'$, $\zeta_1 = (\lambda_1)'[\gamma_1]^{-1}(\lambda_1)$ and $k_{rt} = \frac{\sigma}{q s_{rt}^2}$.
The power function of the PT is

$$\begin{aligned} \pi^{PT}(\lambda) &= P\left(T^{PT} > F_{\alpha_3,q_2,n-p}|M_n\right) \\ &= 1 - P\left(L_3^{PT} \leq F_{\alpha_3,q_2,n-p} - k_{pt}\zeta_2\right), \end{aligned} \tag{29}$$

where $k_{pt} = \frac{\sigma}{q_2 s_e^2}$ and $\zeta_2 = (\lambda_2)'[\gamma_2]^{-1}(\lambda_2)$ with $\gamma_2 = A_2\left(X_2'X_2\right)^{-1}A_2'$.

iii. Then the power of the PTT becomes,

$$\begin{aligned} \pi^{PTT}(\lambda) &= P\left(L^{PT} < F_{\alpha_3,q_2,n-p}, L^{RT} > F_{\alpha_2,q_2,n-p}|M_n\right) \\ &\quad + P\left(L^{PT} \geq F_{\alpha_3,q_2,n-p}, L^{UT} > F_{\alpha_1,q_2,n-p}|M_n\right) \\ &= P\left[L^{PT} < F_{\alpha_3,q,n-s}\right]P\left[L^{RT} > F_{\alpha_2,q,n-p}\right] + d_{1r}(a,b) \\ &= \left[1 - P\left(L^{PT} > F_{\alpha_3,q_2,n-p}\right)\right]P\left(L^{RT} > F_{\alpha_2,q_2,n-p}\right) + d_{1r}(a,b), \end{aligned} \tag{30}$$

where $a = F_{\alpha_3,q_1,n-p} - \frac{\sigma}{q s_e^2}(\lambda_2)'\left[\gamma_{pt}\right]^{-1}(\lambda_2) = F_{\alpha_3,q_2,n-p} - k_{pt}\zeta_2$, and $d_{1r}(a,b)$ is bivariate F probability integral. The value of ζ_1 and ζ_2 depend on λ_1 and λ_2, respectively, and

$$\begin{aligned} d_{1r}(a,b) &= \int_a^\infty \int_b^\infty f(F^{PT}, F^{UT})dF^{PT}dF^{UT} \\ &= 1 - \int_0^b \int_0^a f(F^{PT}, F^{UT})dF^{PT}dF^{UT}, \end{aligned} \tag{31}$$

with $b = F_{\alpha_1,q_1,n-p} - \Omega_{ut}$. The integral $\int_0^b \int_0^a f\left(F^{PT}, F^{UT}\right) dF^{PT} dF^{UT}$ is the cdf of the correlated bivariate noncentral F (BNCF) distribution of the UT and PT. Following Yunus and Khan [36], we define the pdf and cdf of the BNCF distribution as

$$f(y_1, y_2) = \left(\frac{m}{n}\right)^m \left[\frac{\left(1-\rho^2\right)^{\frac{m+n}{2}}}{\Gamma(n/2)}\right] \sum_{j=0}^{\infty} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \left[\rho^{2j} \left(\frac{m}{n}\right)^{2j} \Gamma(m/2+j)\right]$$
$$\times \left[\left(\frac{e^{-\theta_1/2} (\theta_1/2)^{r_1}}{r_1!}\right) \left(\frac{\left(\frac{m}{n}\right)^{r_1}}{\Gamma\left(m/2+j+r_1\right)}\right) \left(y_1^{m/2+j+r_1-1}\right)\right]$$
$$\times \left[\left(\frac{e^{-\theta_2/2} (\theta_2/2)^{r_2}}{r_2!}\right) \left(\frac{\left(\frac{m}{n}\right)^{r_2}}{\Gamma\left(m/2+j+r_2\right)}\right) \left(y_2^{m/2+j+r_2-1}\right)\right]$$
$$\times \Gamma\left(q_{rj}\right) \left[\left(1-\rho^2\right) + \frac{m}{n} y_1 + \frac{m}{n} y_2\right]^{-\left(q_{rj}\right)}, \text{ and} \tag{32}$$

$$F_{Y_1,Y_2}(a, b) = P(Y_1 < a, Y_2 < b) = \int_0^a \int_0^b f\left(y_1, y_2\right) dy_1 dy_2. \tag{33}$$

By setting $a = b = d$, Schuurmann *et al.* [37] presented the critical values of d for the probability table of multivariate F distribution.

From (30), it is clear that the cdf of the BNCF distribution involved in the expression of the power function of the PTT.

6.2. The Size of the Tests

The size of a test is the value of its power under the null hypothesis, H_0. The size of the UT, RT, and PTT are given by

i. The size of the UT

$$\begin{aligned} \alpha^{UT} &= P\left(L^{UT} > F_{\alpha_1,q_1,n-p} | H_0: A_1\beta_1 = h_1\right) \\ &= 1 - P\left(L_1^{UT} \leq F_{\alpha_1,q_1,n-p} | H_0: A_1\beta_1 = h_1\right) \\ &= 1 - P\left(L_1^{UT} \leq F_{\alpha_1,q_1,n-p}\right), \end{aligned} \tag{34}$$

ii. The size of the RT

$$\begin{aligned} \alpha^{RT} &= P\left(L^{RT} > F_{\alpha_2,q_1,n-p} | H_0: A_1\beta_1 = h_1\right) \\ &= 1 - P\left(L_2^{RT} \leq F_{\alpha_2,q_1,n-p} | H_0: A_1\beta_1 = h_1\right) \\ &= 1 - P\left(L_2^{RT} \leq F_{\alpha_2,q_1,n-p} - k_2\zeta_2\right), \end{aligned} \tag{35}$$

where the value of $\zeta_1 = 0$ but $\zeta_2 \neq 0$. The size of the PT is given by

$$\begin{aligned} \alpha^{PT}(\lambda) &= P\left(T^{PT} > F_{\alpha_3,q_2,n-p} | H_0: A_2\beta_2 = h_2\right) \\ &= 1 - P\left(L_3^{PT} \leq F_{\alpha_3,q_2,n-p}\right) \text{ and then} \end{aligned} \tag{36}$$

iii. The size of the PTT

$$\begin{aligned} \alpha^{PTT} &= P\left(L^{PT} \leq a, L^{RT} > d | H_0\right) + P\left(L^{PT} > a, L^{UT} > h | H_0\right) \\ &= P\left(L^{PT} \leq a\right) P\left(L^{RT} > d\right) + d_{1r}(a, h) \\ &= \left[1 - P\left(L^{PT} \leq a\right)\right] P\left(L^{RT} > d\right) + d_{1r}(a, h), \end{aligned} \tag{37}$$

where $h = F_{\alpha_1,q_1,n-p}$, $d = F_{\alpha_2,q_1,n-p}$, and under H_0 the value of a is $F_{\alpha_1,q_1,n-p}$.

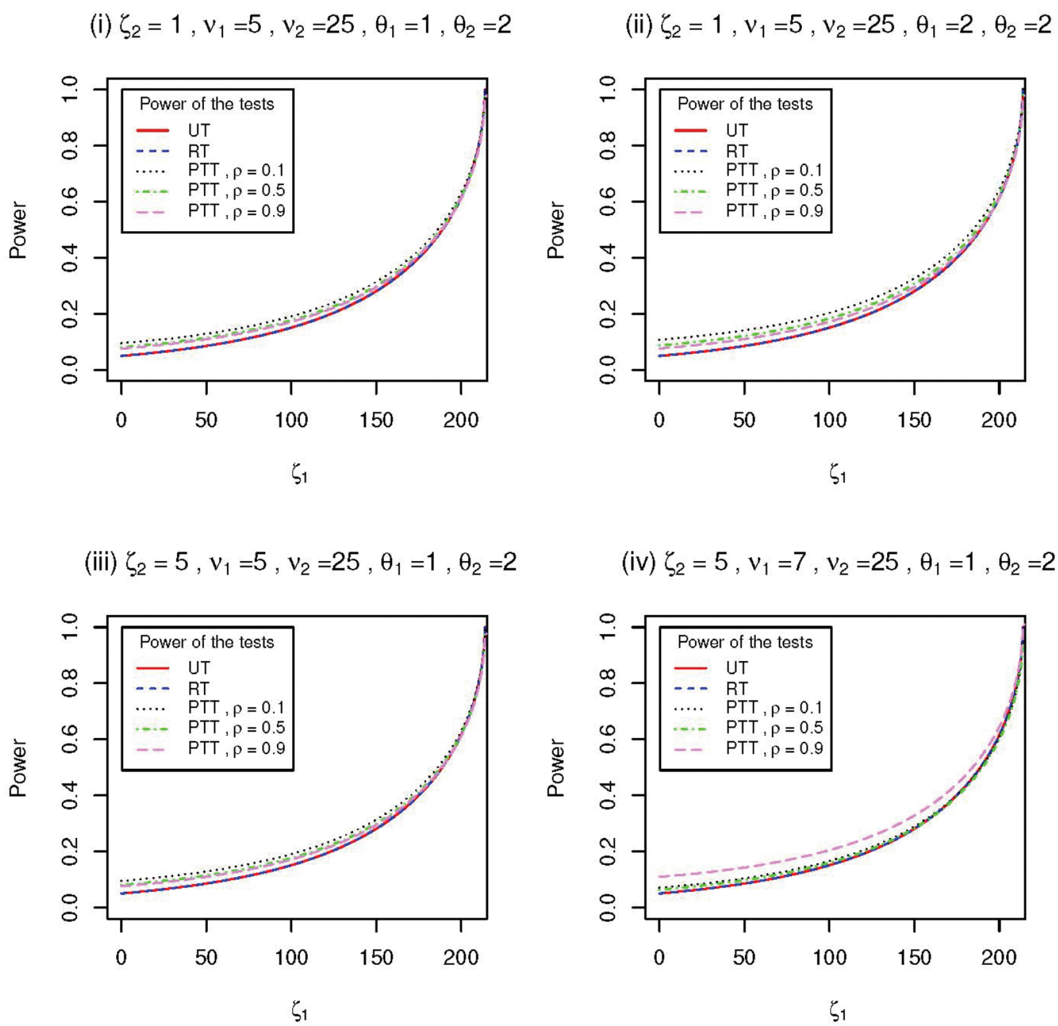

Figure 1 Comparing power of three tests against ζ_1 with selected values of ρ, ζ_2, df, and noncentrality parameters.

7. ILLUSTRATIVE EXAMPLE

To compare the tests, the properties of the three tests are studied using simulated data. The R statistical package was used to generate data on Y and X. Using $k = 3$, three covariates $\left(x_j, j = 1, 2, 3\right)$ were generated from the $U(0, 1)$ distribution. The error vector (e) was generated from the $N\left(\mu = 0, \Sigma = \sigma^2 I_3\right)$ distribution. For $n = 100$ random variates the dependent variable $\left(y\right)$ was determined by $y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + e_i$ for $i = 1, 2, \cdots, n$.

The power functions of the tests are computed for $k = 3$, $p = 4$, $r = 2$, and $s = 2$ so that $\delta_1 = (\beta_0, \beta_1)$, $\delta_2 = (\beta_2, \beta_3)$, and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.05$. Thus to compute the power of the tests, we fix the size to be 0.05 for all the tests. The power functions of the tests are calculated using the formulas in (27), (28), and (30). Whereas the graphs for the size of the three tests are produced using formulas in (34), (35), and (37). The power and size curves of the tests are shown in Figs. 1 and 2.

8. POWER AND SIZE COMPARISON

Figure 1 shows that the power of the UT does not depend on ζ_2 and ρ, but it slowly increases as the value of ζ_1 increases. The form of the power curve of the UT is concave. For very small values of ζ_1, near 0, the power curve of the UT slowly increases as ζ_1 becomes larger. The power of the UT reaches its minimum, around 0.05, for $\zeta_1 = 0$ and for any value of ζ_2.

Like the power of the UT, the power of the RT increases as the values of ζ_1 increases and reaches 1 for large values of ζ_1 (see Fig. 1). The power of the RT is greater, or equal to, than that of the UT for all values of ζ_1 and/or ζ_2. The RT achieves its minimum power, around 0.05, for $\zeta_1 = 0$ and all values of ζ_2 (see Fig. 1). The maximum power of the RT is 1 for reasonably larger values of ζ_1.

The power of the PTT depends on the values of ζ_1, ζ_2, and ρ. The power of the RT and PTT increases as the values of ζ_1 and ζ_2 increase for $\rho = 0.9$. For $\zeta_2 = 5$ and $\rho = 0.9$, the power of the PTT increases as the value of ν_1 increases (see Fig. 1(d)), but not for $\rho = 0.1$ and $\rho = 0.5$.

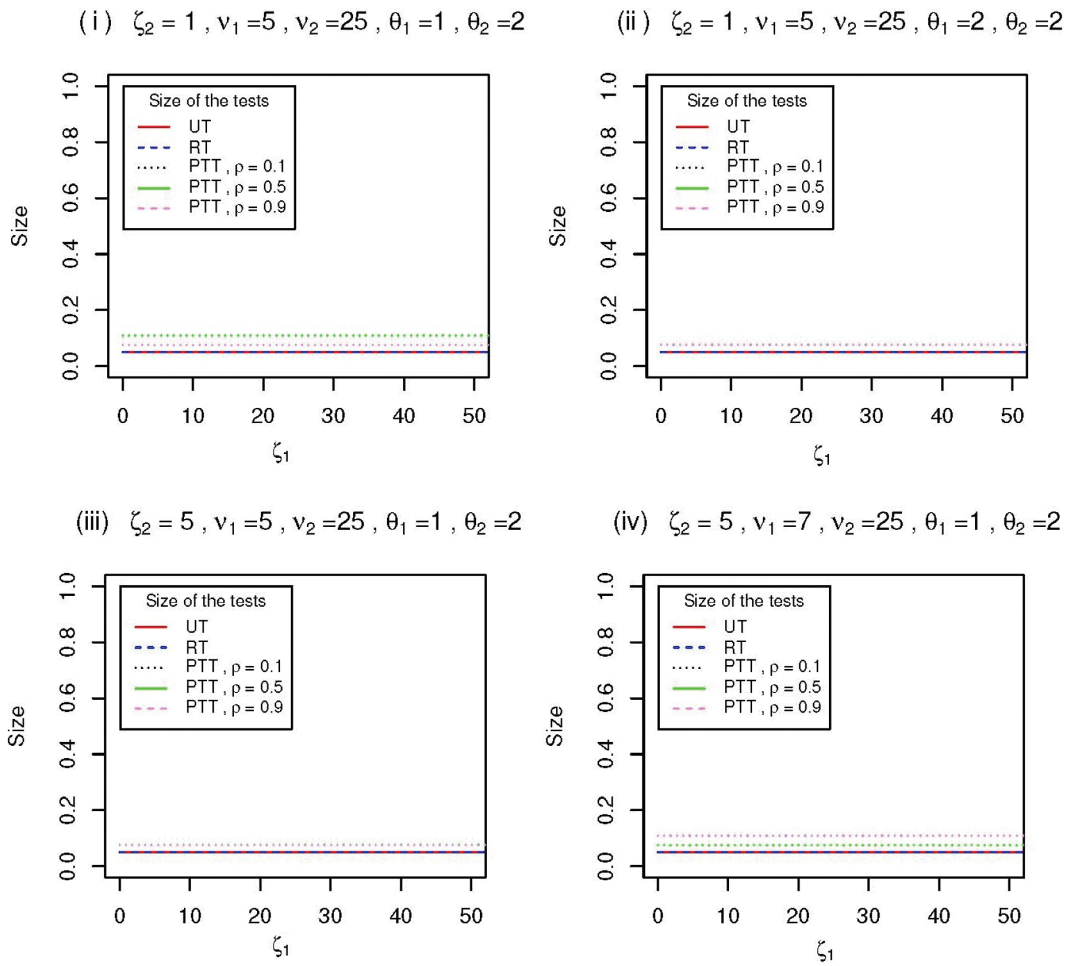

Figure 2 | Comparison of size of three tests against ζ_1 for selected values of ρ, ζ_2, df, and noncentrality parameters.

Moreover, the power of the PTT is always larger than that of the UT and tends to be the same as that of the RT for large values of ζ_1 (see Fig. 1(d)). The minimum power of the PTT is around 0.07 for $\zeta_1 = 0$, and $\rho = 0.1,\ 0.5$ (see Fig. 1(d)), and it decreases (close to RT) as the value of ζ_2 and ν_1 becomes larger.

From Fig. 2 or (34) it is evident that the size of the UT does not depend on ζ_2. It is constant for all values of ζ_1 and ζ_2. Like the size of the UT, the size of the RT is also constant for all values of ζ_1 and ζ_2. Moreover, the size of the RT is the same or larger than that of the UT for all values of ζ_2 and does not depend on ρ.

The size of the PTT increases as the values of ν_1 and ζ_2 increase for $\rho = 0.9$ (see Fig. 2(c) and 2(d)). But it decreases as the values of θ_1 increases (see Fig. 2(a) and 2(b)).

The size of the UT is $\alpha^{UT} = 0.05$ for all values of ζ_1 and ζ_2. The size of the RT, $\alpha^{RT} \geq \alpha^{UT}$ for all values of ζ_2. The size of the PTT, $\alpha^{PTT} \geq \alpha^{RT}$ for all values of ζ_1, a_2 and ρ.

9. CONCLUSION

The above analyses reveal that the UT has lower power than the RT. The power of the UT is also less than that of the PTT for all values of ζ_1 and ζ_2 and ρ. The size of the RT and PTT is larger or equal to that of the UT for all values of ζ_1 and ζ_2.

For smaller values of ζ_1, the UT and RT have lower power than the PTT. But for larger values of ζ_1 the RT has higher, or same, power than the PTT and UT. Thus when the NSPI is reasonably accurate (i.e., ζ_1 is small) the PTT over performs the UT and RT with higher power.

The UT has the smallest size among the three tests. But it also has the lowest power. The RT has the highest power and highest size. The PTT achieves higher power than the UT and lower size than the RT. Thus in the face of uncertainty, if NSPI is reasonably close to the true value of the parameters than the PTT is a better choice compared to UT and RT.

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REFERENCES

1. R. Mead, The Design of Experiments, Cambridge University Press, Cambridge, 1988, pp. 368.
2. K. Hinkelmann, O. Kempthorne, Design and Analysis of Experiments: Introduction to Experimental Design, vol. 1, Wiley, India, 1994, pp. 350.
3. P. Kabaila, D. Tissera, Aust. NZ. J. Stat. 54 (2014), 371–383.
4. T.A. Bancroft, Ann. Math. Stat. 15 (1944), 190–204.
5. J.L. Hodges, E.L. Lehmann, Ann. Math. Stat. 23 (1952), 396–407.
6. P. Kabaila, G. Dharmarathne, Aust. NZ. J. Stat. 57 (2015), 99–118.
7. P.J. Kempthorne, 1983 Business and Economic Statistics Proceedings of the American Statistical Association, MIT, USA, 1983, pp. 568–573.
8. P.J. Kempthorne, Ann. Stat. 16 (1988), 1594–1608.
9. P.J. Bickel, Ann. Stat. 12 (1984), 864–879.
10. S. Khan, Biom. J. 44 (2003), 73–90.
11. S. Khan, J. Stat. Res. 39 (2005), 79–94.
12. S. Khan, Commun. Stat. Theory Methods. 37 (2008), 247–260.
13. S. Khan, A.K.Md.E. Saleh, Stat. Pap. 42 (2001), 451–473. http://www.springerlink.com/content/mbqc7a26qm6cf41c/fulltext.pdf
14. S. Khan, A.K.Md.E. Saleh, Stat. Pap. 46 (2005), 379–394.
15. S. Khan, A.K.Md.E. Saleh, Commun. Stat. Theory Methods. 37 (2008), pp. 2564–2581.
16. A.K.Md.E. Saleh, Theory of Preliminary Test and Stein-Type Estimation with Applications, John Wiley and Sons, Inc., New Jersey, 2006.
17. S. Khan Biometrical J. vol 45 (2003), 73–90.
18. R.M. Yunus, Increasing power of M-test through pretest, Unpublished PhD Thesis, University of Southern Queensland, Australia, 2010.
19. G.G. Judge, M.E. Bock, The Statistical Implications of Pre-test and Stein-pule Estimators in Econoetrics, North-Holland, New York, 1978.
20. T.A. Bancroft, Biometrics. 20 (1964), 427–442.
21. C.P. Han, T.A. Bancroft, J. Am. Stat. Assoc. 63 (1968), 1333–1342.
22. S. Khan, Z. Hoque, J. Stat. Res. 37 (2003), 43–55.
23. R. Tamura, Bull. Math. Stat. 11 (1965), 38–61.
24. A.K.Md.E. Saleh, P.K. Sen, Ann. Stat. 6 (1978), 154–168.
25. A.K.Md.E. Saleh, P.K. Sen, Procedings of the Fifth Pannonian Symposium on Mathematical Statistics, Visegrád, Hungary, 13-18 September, 1982, pp. 307–325.
26. R.M. Yunus, S. Khan, 9th Islamic Countries Conference on Statistical Sciences (ICCS-IX): Statistics in the Contemporary World - Theories, Methods and Applications, Kuala Lumpur, Malaysia, 2007.
27. R.M. Yunus, S. Khan, Commun. Stat. Theory Methods. 40 (2011), 581–597.
28. R.M. Yunus, S. Khan, J. Nonparamatr. Stat. 23 (2011), 201–218.
29. K. Ohtani, T. Toyoda, J. Econ. Lett. 17 (1985), 111–114.
30. J. Gurland, R.S. McCullough, J. Biometrika. 49 (1962), 403–417.
31. S. Khan, B. Pratikno, Stat. Pap. 54 (2013), 605–617.
32. P.R. Krishnaiah, Part of PhD thesis, University of Minnesota, 1964.
33. D.E. Amos, W.G. Bulgren, J. Math. Comput. 26 (1972), 255–264.
34. A.H. El-Bassiouny, M.C. Jones, Stat. Methods Appl. 18 (2009), 465–481.
35. S. Khan, B. Pratikno, A.I.N. Ibrahim, M.R. Yunus, Commun. Stat. Simul. Comput. 45 (2016), 3491–3507.
36. R.M. Yunus, S. Khan, Appl. Math. Comput. 217 (2011), 6237–6247.
37. F.J. Schuurmann, P.R. Krishnaiah, A.K. Chattopadhyay, Indian J. Stat. 37 (1975), 308–331.
38. S. Khan, Z. Hoque, A.K.Md.E. Saleh, J. Stat. Res. 31 (2002), 51–72.