

Estimation of univariate normal mean using p -value

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Abstract

This paper estimates the mean of a normal distribution in presence of non-sample prior information regarding the value of the mean. We define the unrestricted estimator (UE), restricted estimator (RE), p -value based restricted estimator (PRE) and p -value based preliminary test estimator (PPTE) based on the sample, prior information, p -value of an appropriate test and combination of all them respectively. The relative performance of these estimators is studied on the basis of bias and mean square error (MSE) criteria. Both analytical and graphical comparisons are investigated. If the p -value is reasonably small, the PRE almost uniformly over performs the UE and PPTE, regardless of the level of significance.

1 Introduction

Estimation of parameters on the basis of sample responses is a common practice in statistics. The classical estimators of unknown parameters are based completely on the sample data, and ignore any other kind of non-sample prior information. However, it is a natural expectation that the quality of the estimators may improve if non sample prior information is incorporated in the estimation of the parameters. Any such estimator that combine both sample and non-sample prior information is likely to perform better than the exclusive sample based estimator under specific statistical criterion. A number of estimators have been introduced in the literature that, under particular situation, over performs the classical exclusive sample based estimators when judged by criteria such as the mean square error and square error loss function (Khan and Saleh, 2001).

There have been many studies in the area of improved estimators following the seminal work of Bancroft (1944) and later Han and Bancroft (1968). They developed the preliminary test estimator that uses uncertain non-sample prior information in addition to sample information. Stein (1956) introduced the Stein rule (shrinkage) estimator for multivariate normal population that dominates the usual maximum likelihood estimator (mle) under quadratic loss function. In a series of papers Saleh and Sen (1978, 1985) explored the preliminary test approach to Stein-rule estimation under nonparametric set up. Many authors have contributed to this area, notably Sclove et.al (1972,) Judge and Bock (1978), Stein (1981), Matta and Casella (1990), Khan (1998) and Saleh (2006). Ahmed and Saleh (1989) provided comparison of several improved estimators for two multivariate normal populations with a common covariance matrix. Later Khan and Saleh (1995, 1997) investigated the problem for a family of Student's t populations. Recently Khan et al. (2001, 2003) studied the performance of several improved estimators using the coefficient of distrust, d ($0 \leq d \leq 1$) on the value of the non sample prior information. The coefficient of distrust is a value decided subjectively by the researcher. Following Khan et al. (2001, 2003) our aim is to use the p -value of the test for testing the non-sample prior information, instead of the coefficient of distrust to prepare a new improved estimator. One of the main advantages of our proposed estimator is that p -value for testing

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prior information will come from observed sample data. Here we deal with the improved estimation of the population mean (μ) when σ is unknown, where sample and non-sample prior information about the value of the mean are available. The performances of the estimator are investigated on the basis of the unbiasedness and mean square error (MSE) criteria.

Let X_1, X_2, \dots, X_n be a random sample of size n from a normal population with unknown mean μ and unknown variance σ^2 . In the usual notation we write $X \sim N(\mu, \sigma^2)$. Assume that uncertain non-sample prior information on the value of μ is available, either from previous studies or practical experiences of the researchers or experts which may or may not be true. We can denote the uncertain non-sample prior information in the form of a null hypothesis,

$$H_0 : \mu = \mu_0 \quad (1.1)$$

against a two-sided alternative $H_A : \mu \neq \mu_0$. To remove the uncertainty in the null hypothesis we perform an appropriate test. Let θ be the p -value for testing the null hypothesis in (??). First we define the unrestricted maximum likelihood estimator (mle) of the unknown mean μ and the common variance σ^2 . Then define the RE of μ as $\hat{\mu} = \mu_0$ and that of σ^2 as $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu_0)^2$. Based on the unrestricted and restricted mle of σ^2 , we define the likelihood ratio test for testing $H_0 : \mu = \mu_0$ against $H_A : \mu \neq \mu_0$. Then use the test statistic as well as the sample and non sample information to define the preliminary test estimator of the unknown population mean.

It is well known that the mle of population mean is unbiased. However, there are estimators of the mean that are biased but may well have some superior statistical properties in terms of some other more popular statistical criterion. In view of this, we define two p -value based biased estimators: the restricted estimator (PRE) and the preliminary test estimator (PTE), as a linear combination of the mle and PRE. The later we call p -value based preliminary test estimator (PPTE). We investigate the bias and mean square error functions of the UE, RE, PRE and PPTE, both analytically and graphically to study the relative performances of the estimators. The analysis reveals the fact that if the p -value for testing the null hypothesis is reasonably small, then the proposed PRE is uniformly superior to other three estimators under the MSE criterion.

The next Section deals with the specification of the model and definition of the unrestricted estimators of μ , and σ^2 as well as the definition of the likelihood ratio test statistic. The alternative 'improved' estimators are defined in Section ???. The expressions and comparison of bias and MSE functions of the estimators are obtained in Section ???. Some concluding remarks are included in Section ??.

2 The Model and Some Preliminaries

Let us express the n sample responses in the following form

$$\mathbf{X}_n = \mu \mathbf{l}_n + \mathbf{e} \quad (2.1)$$

where $\mathbf{X}_n = (x_1, x_2, \dots, x_n)'$ is an $n \times 1$ vector of observations, $\mathbf{l}_n = (1, 1, \dots, 1)'$ is a vector of n -tuple of one's, μ is a scalar unknown parameter (mean) and $\mathbf{e} = (e_1, e_2, \dots, e_n)'$ is a vector of errors with independent components. Assume that \mathbf{e} is distributed as $N_n(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ where $E(\mathbf{e}) = \mathbf{0}$, and $E(\mathbf{e}\mathbf{e}') = \sigma^2 \mathbf{I}_n$. Here, σ^2 stands for the common variance of each of the error components in \mathbf{e} and \mathbf{I}_n is the identity matrix of order n . Then $\mathbf{X} \sim N_n(\mu \mathbf{l}_n, \sigma^2 \mathbf{I}_n)$. The exclusive sample information based UE of μ is given by

$$\tilde{\mu} = \frac{\sum_{i=1}^n X_i}{n} = \bar{X} \quad (2.2)$$

where \bar{X} is the sample mean. It is known that $\tilde{\mu}$ is the mle of μ and the sampling distribution is normal with mean $E(\tilde{\mu}) = \mu$ and variance, $\text{Var}(\tilde{\mu}) = \frac{\sigma^2}{n}$. Infact, $\tilde{\mu}$ is uniformly minimum variance unbiased estimator (UMVUE) for μ . The bias and MSE of $\tilde{\mu}$ is given by

$$B_1(\tilde{\mu}) = 0 \text{ and } M_1(\tilde{\mu}) = \frac{\sigma^2}{n} \text{ respectively.} \quad (2.3)$$

The RE of μ is biased. The bias and MSE of $\hat{\mu}$ are given by

$$B_2(\hat{\mu}) = (\mu_0 - \mu) \text{ and } M_2(\hat{\mu}) = (\mu_0 - \mu)^2 = \frac{\sigma^2}{n} \Delta^2 \quad (2.4)$$

where $\Delta = \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}$. It follows from (??) that the MSE of RE increases with the increase of Δ and zero when $\Delta = 0$. It is well known that the mle of σ^2 is $S_*^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n}$. This estimator is biased. However, an unbiased estimator of σ^2 is given by

$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}. \quad (2.5)$$

The unbiased estimator of σ^2 has a scaled χ^2 distribution with shape parameter $v = (n-1)$. Also it is well-known that $\tilde{\mu}$ and S^2 are independently distributed. For testing the null hypothesis $H_0 : \mu = \mu_0$ against the alternative hypothesis $H_A : \mu \neq \mu_0$, an appropriate test statistic is the likelihood ratio test (LRT) given by

$$\zeta_v = \frac{\sqrt{n}(\tilde{\mu} - \mu_0)}{S}. \quad (2.6)$$

Under alternative hypothesis, ζ_v follows a non-central Student-t distribution with $v = (n-1)$ degrees of freedom (d.f.), and noncentrality parameter $\frac{1}{2}\Delta$, where

$$\Delta = \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}. \quad (2.7)$$

Under H_A , ζ_v^2 follows a non-central F-distribution with $(1, v)$ d.f. and noncentrality parameter $\frac{1}{2}\Delta^2$. Under the null hypothesis ζ_v and ζ_v^2 follow central Student's t and F distributions respectively with appropriate degrees of freedom.

3 Proposed Estimator of the Mean

In this Section we incorporate the uncertain non-sample prior information in estimating mean. First we combine the exclusive sample based estimator UE with the non-sample based restricted estimator in the following way

$$\hat{\mu}^{PRE} = \theta\hat{\mu} + (1-\theta)\tilde{\mu} = \tilde{\mu} - \theta(\tilde{\mu} - \hat{\mu}) \quad (3.1)$$

where θ is the p -value for testing the null hypothesis $H_0 : \mu = \mu_0$. Given the sample observations, θ is obtainable and always $0 \leq \theta \leq 1$. This estimator of μ is called p -value based restricted estimator (PRE). Here, $\theta = 0$ implies that $H_0 : \mu = \mu_0$, has no support from the sample and then we get $\hat{\mu}^{PRE} = \tilde{\mu}$, the UE. On the other hand $\theta = 1$ suggests that $H_0 : \mu = \mu_0$ is certainly true and then we get $\hat{\mu}^{PRE} = \hat{\mu}$, the RE. If $0 < \theta < 1$ (that is, credibility of the null hypothesis varies) then PRE of μ may take any value between $\tilde{\mu}$ and $\hat{\mu}$. The p -value based restricted estimator, as defined above, is normally distributed with mean and variance given by

$$\begin{aligned} E(\hat{\mu}^{PRE}) &= \theta\mu_0 + (1-\theta)\mu = \mu - \theta(\mu - \mu_0) \\ \text{and } Var(\hat{\mu}^{PRE}) &= \frac{\sigma^2}{n}(1-\theta)^2. \end{aligned} \quad (3.2)$$

Following Khan and Saleh (2001), the p -value based preliminary test estimator (PPTE) of the population mean μ is defined as

$$\begin{aligned} \hat{\mu}^{PPTe} &= \tilde{\mu}I(|t| \geq t_{\alpha/2}) + \hat{\mu}^{PRE}I(|t| < t_{\alpha/2}) \\ &= \tilde{\mu} - \theta(\tilde{\mu} - \hat{\mu})I(|t| < t_{\alpha/2}) \end{aligned} \quad (3.3)$$

where $I(A)$ is an indicator function of the set A and $t_{\alpha/2}$ is the critical value chosen for two-sided α -level test based on the Student's t distribution with $v = (n-1)$ degrees of freedom. For $\theta = 1$ the above PPTE becomes

$$\hat{\mu}^{PTE} = \tilde{\mu}I(|t| \geq t_{\alpha/2}) + \mu_0I(|t| < t_{\alpha/2}), \quad (3.4)$$

the ordinary preliminary test estimator (PTE) of μ . For the convenience of the derivation of the bias and mean square error functions of the PPTE this estimator is written as

$$\hat{\mu}^{PPTe} = \tilde{\mu} - \theta(\tilde{\mu} - \mu_0)I(F < F_\alpha) \quad (3.5)$$

where F_α is the $(1-\alpha)^{th}$ upper quantile of a central F-distribution with $(1, v)$ degrees of freedom. As a special case, when $\theta = 1$,

$$\hat{\mu}^{PPTe}(\theta = 1) = \hat{\mu}^{PTE} = \tilde{\mu} - (\tilde{\mu} - \mu_0)I(F < F_\alpha). \quad (3.6)$$

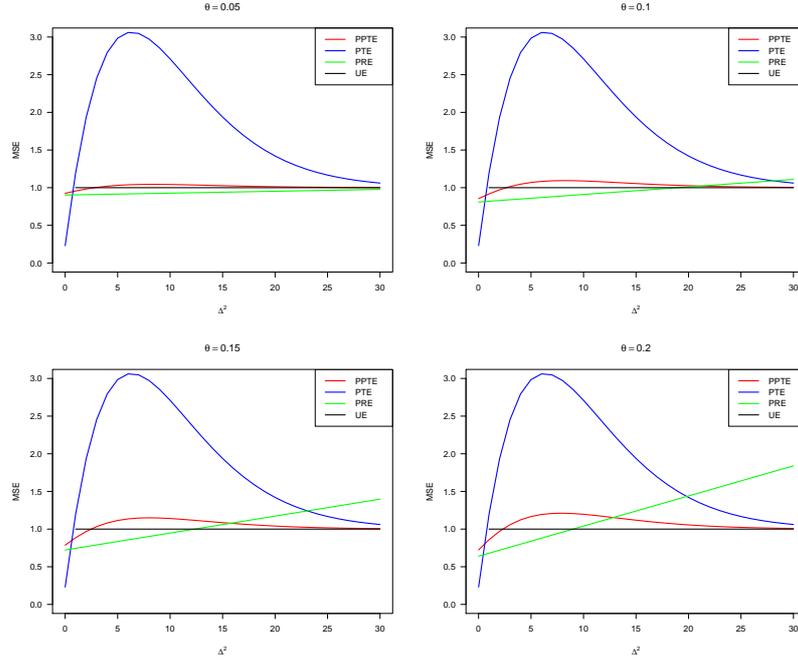


Figure 1: Graph of MSE as a function of Δ^2 , for $\alpha = 0.05$ and varying θ .

4 Some Statistical Properties

In this Section, the bias and mean square error of PRE and PPTE of the population mean μ are derived. We also discuss some important properties of these functions. Some comparisons of these functions are also discussed.

4.1 Properties of PRE

By definition, the bias function of the PRE is

$$B_3[\hat{\mu}^{PRE}] = E[\hat{\mu}^{PRE} - \mu] = -\theta \frac{\sigma}{\sqrt{n}} \Delta \quad (4.1)$$

where $\Delta = \frac{\sqrt{n}(\mu - \mu_0)}{\sigma}$ and $\frac{\Delta^2}{2}$ is the non-centrality parameter of the non-central F distribution with 1 and v d.f. Similarly, the MSE function of the PRE is

$$\begin{aligned} M_3[\hat{\mu}^{PRE}] &= Var[\hat{\mu}^{PRE}] + B_3^2[\hat{\mu}^{PRE}] \\ &= \frac{\sigma^2}{n} [(1 - \theta)^2 + \theta^2 \Delta^2]. \end{aligned} \quad (4.2)$$

Δ^2 is also called the departure parameter that measure the distance between the true value of the population mean and the value under H_0 . The value of the departure parameter is 0 when the null hypothesis is true; otherwise it is always positive. The statistical properties of the PRE and PPTE depend on the value of the above departure parameter. The performance of the estimators varies with the change in the value of Δ^2 .

4.2 Comparison of MSE

The MSE functions of the estimators are compared in this subsection.

- The MSE function of the PRE is $M_3[\hat{\mu}^{PRE}] = \frac{\sigma^2}{n} [(1 - \theta)^2 + \theta^2 \Delta^2]$. This is an equation of a straight line in Δ^2 . That means, the MSE of PRE changes with the change in the value of Δ^2 and the rate of change is $\frac{\sigma^2}{n} \theta^2$.

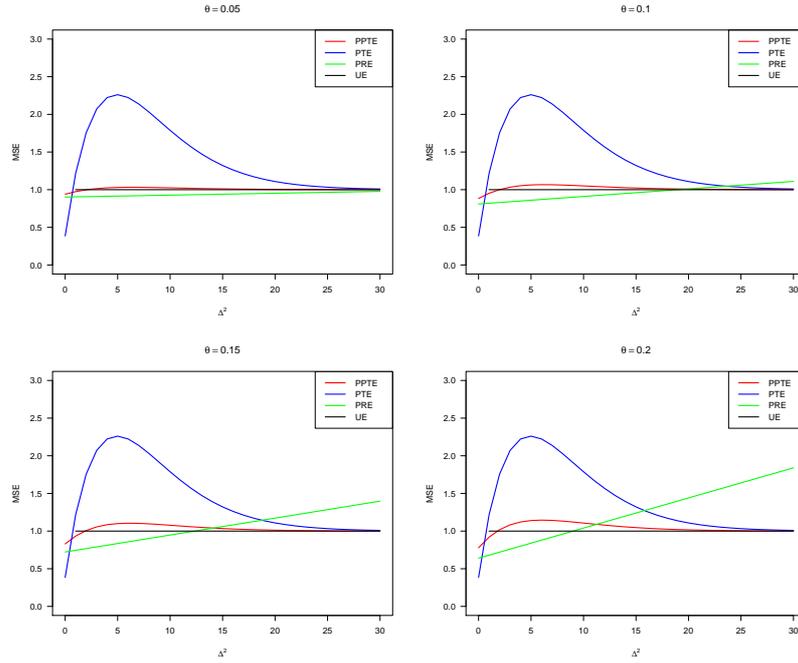


Figure 2: Graph of MSE as a function of Δ^2 , for $\alpha = 0.10$ and varying θ .

- The MSE function of PRE coincides with the variance of $\tilde{\mu}$ if $\Delta^2 = \frac{1-(1-\theta)^2}{\theta^2}$. That means, if

$$\Delta^2 < \frac{1 - (1 - \theta)^2}{\theta^2}, \quad (4.3)$$

then MSE of PRE is always smaller than the variance of $\tilde{\mu}$. In other word, if p -value (θ) is too small then the PRE has smaller MSE than the UE.

- Under the null hypothesis $\Delta^2 = 0$, and hence the MSE of PRE becomes

$$\begin{aligned} M_3[\hat{\mu}^{PRE}] &= \frac{\sigma^2}{n}[(1 - \theta)^2] = 0, \quad \text{if } \theta = 1 \\ &= \frac{\sigma^2}{n} \quad \text{if } \theta = 0 \\ &< \frac{\sigma^2}{n} \quad \text{if } 0 < \theta < 1. \end{aligned} \quad (4.4)$$

For $\theta = 0$, the PRE has the same MSE as the UE. For all other values of θ , the PRE has a smaller MSE than the UE. So, under the MSE criterion, the biased estimator, PRE performs better than the unbiased estimator, UE.

4.3 Properties of PPTE

By definition, the bias function of the PPTE is

$$\begin{aligned} B_4[\hat{\mu}^{PPTE}] &= E[\hat{\mu}^{PPTE} - \mu] \\ &= -\theta E[(\tilde{\mu} - \mu_0)I(F < F_\alpha)] \\ &= -\frac{\sigma}{\sqrt{n}}\theta E\left[ZI\left(\frac{vZ^2}{\chi_v^2} < F_\alpha\right)\right]. \end{aligned} \quad (4.5)$$

Note $Z = \frac{(\tilde{\mu} - \mu_0)\sqrt{n}}{\sigma}$ is distributed as $N(\Delta, 1)$, and $v\sigma^{-2}s^2$ is distributed as a central chi-square variable with $v = (n - 1)$ degrees of freedom. Applying Theorem 4, Saleh(2006, p.32), the bias

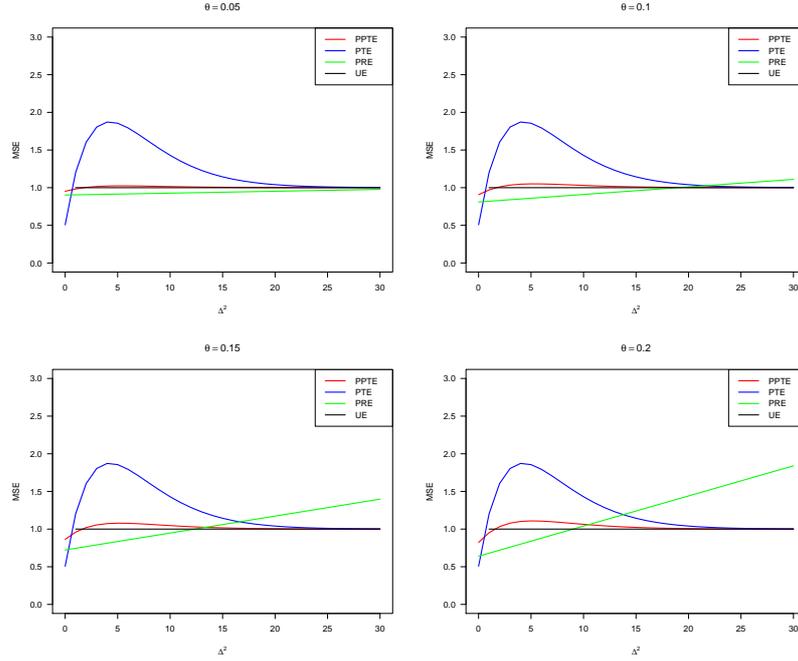


Figure 3: Graph of MSE as a function of Δ^2 , for $\alpha = 0.15$ and varying θ .

function of the PPTE of μ can be written as

$$B_4[\hat{\mu}^{PPTE}] = -\theta \frac{\sigma}{\sqrt{n}} \Delta G_{3,v} \left(\frac{1}{3} F_\alpha; \Delta^2 \right) \quad (4.6)$$

where, $G_{m,n}(\cdot; \Delta^2)$ is the c.d.f of a non-central F-distribution with (m, n) degrees of freedom and non-centrality parameter $\frac{\Delta^2}{2}$. Thus the bias function of the PPTE depends on the p -value for testing $H_0 : \mu = \mu_0$ and the departure parameter. For $\Delta = 0$, the PPTE is unbiased, otherwise it is biased. From the definition, the MSE of the PPTE is

$$\begin{aligned} M_4[\hat{\mu}^{PPTE}] &= E[\hat{\mu}^{PPTE} - \mu]^2 \\ &= E[(\tilde{\mu} - \mu) - \theta(\tilde{\mu} - \mu_0)I(F < F_\alpha)]^2 \\ &= \frac{\sigma^2}{n} + \theta^2 E[(\tilde{\mu} - \mu_0)^2 I(F < F_\alpha)] \\ &\quad - 2\theta E[(\tilde{\mu} - \mu)(\tilde{\mu} - \mu_0)I(F < F_\alpha)]. \end{aligned} \quad (4.7)$$

The second term of the right hand side of the above equation is

$$\begin{aligned} \theta^2 E[(\tilde{\mu} - \mu_0)^2 I(F < F_\alpha)] &= \theta^2 \frac{\sigma^2}{n} E \left[\left(\frac{(\tilde{\mu} - \mu_0)\sqrt{n}}{\sigma} \right)^2 I \left(\frac{n(\tilde{\mu} - \mu_0)}{s^2} < F_\alpha \right) \right] \\ &= \theta^2 \frac{\sigma^2}{n} E \left[Z^2 I \left(\frac{Z^2}{s^2/\sigma^2} < F_\alpha \right) \right]. \end{aligned} \quad (4.8)$$

Again applying Theorem 5, Saleh (2006, p.32), we get $\theta^2 E[(\tilde{\mu} - \mu_0)^2 I(F < F_\alpha)] = \theta^2 \frac{\sigma^2}{n} G_{3,v}(\frac{1}{3} F_\alpha; \Delta^2) + \theta^2 \frac{\sigma^2}{n} \Delta^2 G_{5,v}(\frac{1}{5} F_\alpha; \Delta^2)$. Similarly we can write $E[(\tilde{\mu} - \mu)(\tilde{\mu} - \mu_0)I(F < F_\alpha)] = \frac{\sigma^2}{n} [\Delta^2 G_{5,v}(\frac{1}{5} F_\alpha; \Delta^2) + (1 - \Delta^2) G_{3,v}(\frac{1}{3} F_\alpha; \Delta^2)]$. Collecting all terms of (4.7), the MSE function of the PPTE is expressed as

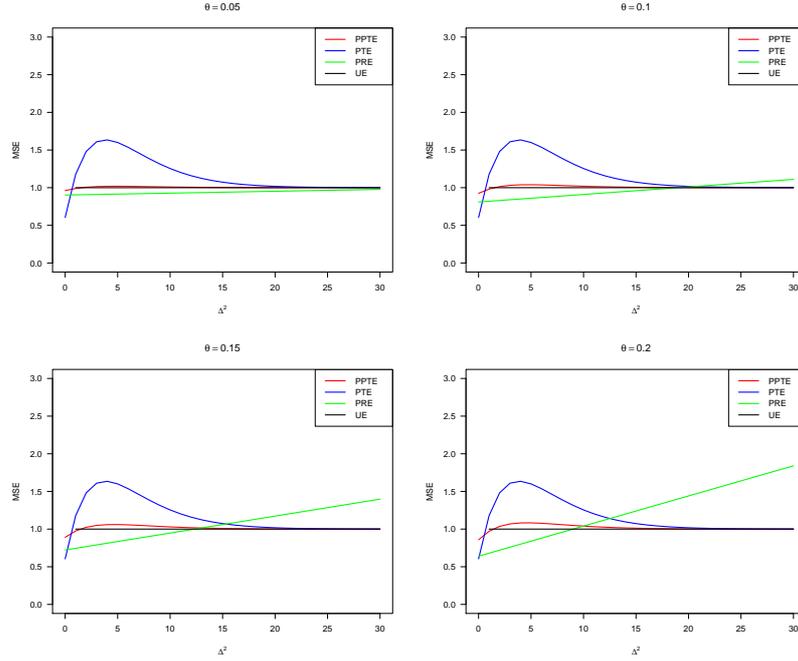


Figure 4: Graph of MSE as a function of Δ^2 , for $\alpha = 0.20$ and varying θ .

$$\begin{aligned}
M_4[\hat{\mu}^{PPTE}] &= \theta^2 \frac{\sigma^2}{n} G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) + \theta^2 \frac{\sigma^2}{n} \Delta^2 G_{5,v}(\frac{1}{5}F_\alpha, \Delta^2) \\
&- 2\theta \frac{\sigma^2}{n} [\Delta^2 G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2) + (1 - \Delta^2) G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2)] \\
&= \frac{\sigma^2}{n} [(1 - \theta(2 - \theta) G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2)) + \theta \Delta^2 (2G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) \\
&- (2 - \theta) G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2))]. \tag{4.9}
\end{aligned}$$

As we know, for $\theta = 1$, $\hat{\mu}^{PPTE} = \hat{\mu}^{PTE}$. Therefore, the bias function of $\hat{\mu}^{PTE}$ is

$$B_5[\hat{\mu}^{PTE}] = -\frac{\sigma}{\sqrt{n}} \Delta G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) \tag{4.10}$$

and the MSE function of $\hat{\mu}^{PTE}$ is given by

$$\begin{aligned}
M_5[\hat{\mu}^{PTE}] &= \frac{\sigma^2}{n} \left[\left(1 - G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) \right) + \Delta^2 \right. \\
&\quad \left. \times \left\{ 2G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) - G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2) \right\} \right] \tag{4.11}
\end{aligned}$$

Figures ??- ??, display the MSE curves of the UE, PRE, PTE and PPTE for different values of θ and the level of significance (α).

4.4 Comparison of MSE

- Under the null hypothesis, $\Delta^2 = 0$ and hence the MSE of $\hat{\mu}^{PPTE}$ is

$$\begin{aligned}
\frac{\sigma^2}{n} [1 - \theta(2 - \theta) G_{3,v}(\frac{1}{3}F_\alpha; 0)] &< \frac{\sigma^2}{n} \quad \text{if } 0 < \theta \leq 1 \\
&= \frac{\sigma^2}{n} \quad \text{if } \theta = 0. \tag{4.12}
\end{aligned}$$

Thus, when $\Delta^2 = 0$, the PPTE of μ performs better than $\tilde{\mu}$, the UE whatever may be the level of significance.

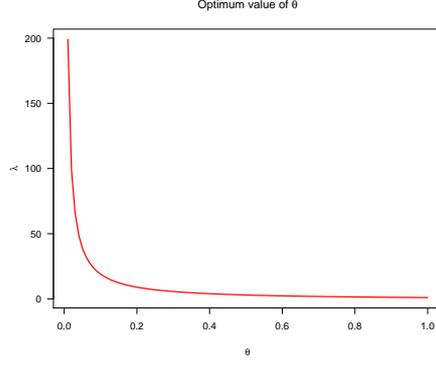


Figure 5: Graph of $\lambda = \frac{1-(1-\theta)^2}{\theta^2}$ as a function of p -value (θ).

- As $\alpha \rightarrow 0$, $G_{3,v}(\frac{1}{3}F_\alpha; 0) \rightarrow 1$ and hence

$$\begin{aligned} \frac{\sigma^2}{n}[1 - \theta(2 - \theta)G_{3,v}(\frac{1}{3}F_\alpha; 0)] &\rightarrow \frac{\sigma^2}{n}[1 - \theta(2 - \theta)] \\ &< \frac{\sigma^2}{n} \quad \text{if } 0 < \theta < 1 \\ &= \frac{\sigma^2}{n} \quad \text{if } \theta = 0 \\ &= 0 \quad \text{if } \theta = 1. \end{aligned} \quad (4.13)$$

Thus for any value of θ the MSE of $\hat{\mu}^{PTE}$ is always less than the MSE of $\tilde{\mu}$ when $\alpha \rightarrow 0$.

- As $\alpha \rightarrow 1$, $F_\alpha \rightarrow 0$ then $G_{3,v}(\frac{1}{3}F_\alpha; 0) \rightarrow 0$ and hence

$$\frac{\sigma^2}{n}[1 - \theta(2 - \theta)G_{3,v}(\frac{1}{3}F_\alpha; 0)] \rightarrow \frac{\sigma^2}{n} \quad (4.14)$$

which is the MSE of $\tilde{\mu}$.

- As $\Delta^2 \rightarrow \infty$, $G_{m,v}(\frac{1}{m}F_\alpha; \Delta^2) \rightarrow 0$ and $M_4[\hat{\mu}^{PTE}]$ tends to $\frac{\sigma^2}{n}$, the MSE of $\tilde{\mu}$.
- Since $G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2)$ is always greater than $G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2)$ for any value of α , replacing $G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2)$ by $G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2)$ gives

$$\begin{aligned} M_4[\hat{\mu}^{PTE}] &\geq \frac{\sigma^2}{n} \left[1 - \theta(2 - \theta)G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2) \{ \theta^2 \Delta^2 - \theta(2 - \theta) \} \right] \\ &\geq \frac{\sigma^2}{n} \quad \text{whenever } \Delta^2 \geq \frac{2}{\theta} - 1. \end{aligned} \quad (4.15)$$

- On the other hand, $M_4[\hat{\mu}^{PTE}]$ may be rewritten as

$$\begin{aligned} M_4[\hat{\mu}^{PTE}] &= \frac{\sigma^2}{n} [1 + \theta G_{3,v}(\frac{1}{3}F_\alpha; \Delta^2)(2\Delta^2 - (2 - \theta)) \\ &\quad - \theta \Delta^2 (2 - \theta) G_{5,v}(\frac{1}{5}F_\alpha; \Delta^2)] \\ &\leq \frac{\sigma^2}{n} \quad \text{whenever } \Delta^2 < 1 - \frac{\theta}{2} \end{aligned} \quad (4.16)$$

This means that $M_4[\hat{\mu}^{PTE}]$ is a function of Δ^2 and crosses from below the constant line of $\frac{\sigma^2}{n}$ in the interval $(1 - \frac{\theta}{2}, \frac{2}{\theta} - 1)$.

- A general picture of the MSE function of the PPTE of μ can be described as follows: The MSE function begins with the smallest value $\frac{\sigma^2}{n}[1 - \theta(2 - \theta)G_{3,v}(\frac{1}{3}F_\alpha; 0)]$ at $\Delta^2 = 0$. As Δ^2 grows larger, the MSE function increases monotonically, crossing the constant line $\frac{\sigma^2}{n}$ in the interval $(1 - \frac{\theta}{2}, \frac{2}{\theta} - 1)$ from below and reaches its maximum in the interval $(\frac{2}{\theta} - 1, \infty)$. Finally, as $\Delta^2 \rightarrow \infty$, the MSE of the PPTE monotonically decreases and approaches $\frac{\sigma^2}{n}$, the MSE of UE from above.

It follows from Figure ?? that the value of Δ^2 decreases with the increase of θ in case of the MSE of PRE. It also shows that if $\theta = 0.01$ and $\Delta^2 < 199$, then MSE of PRE is smaller than the variance of UE. That means, if p -value for testing $H_0 : \mu = \mu_0$ is too small (≤ 0.05), proposed PRE is the best choice for estimating the population mean.

5 Conclusion

The UE is based on the sample data alone, and it is the only unbiased estimator among the five estimators considered in this paper. The introduction of non-sample prior information in the formation of the estimator causes the estimators to be biased. However, the biased estimators perform better than the unbiased estimator when they are judged based on the MSE criterion. The performance of the biased estimators depend on the value of Δ and the p -value for testing $H_0 : \mu = \mu_0$. If p -value is too small (may be less than or equal to .05), PRE uniformly over performs all biased and unbiased estimators for any value of α and Δ . However, there is a risk of using PRE if p -value is too large because in such cases the MSE of $\hat{\mu}^{PRE}$ increases at the rate of $\frac{\sigma^2}{n}\theta^2$. The MSE of the proposed PPTE is very close to the UE for any p -value and level of significance. The PPTE perform better than UE if the difference between the true value of μ and that specified by the prior information is small. However, the proposed PPTE performs better than the usual PTE except $\Delta = 0$. Therefore, if the p -value for testing $H_0 : \mu = \mu_0$ is too small or if $\Delta^2 < \frac{1-(1-\theta)^2}{\theta^2}$, then the proposed PRE of μ is the best choice as an improved estimator for estimating the population mean.

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