A spectral collocation technique based on integrated Chebyshev polynomials for biharmonic problems in irregular domains

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Submitted to *Applied Mathematical Modelling*, 28-Feb-2007; Revised, 2-Nov-2007

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Abstract In this paper, an integral collocation approach based on Chebyshev polynomials for numerically solving biharmonic equations [Mai-Duy and Tanner, *Journal of Computational and Applied Mathematics*, **201**(1) (2007) 30–47] is further developed for the case of irregularly shaped domains. The problem domain is embedded in a domain of regular shape, which facilitates the use of tensor product grids. Two relevant important issues, namely the description of the boundary of the domain on a tensor product grid and the imposition of double boundary conditions, are handled effectively by means of integration constants. Several schemes of the integral collocation formulation are proposed, and their performances are numerically investigated through the interpolation of a function and the solution of 1D and 2D biharmonic problems. Results obtained show that they yield spectral accuracy.

Keywords: Integral collocation formulation; biharmonic problems; complex geometries; fictitious domains; Chebyshev polynomials

1 Introduction

Many engineering problems, such as the deformation of a thin plate and the motion of a fluid, are governed by the biharmonic equations – fourth-order partial differential equations (PDEs). Generally, problems involving high-order PDEs and complex geometries are more difficult to solve than those with second-order PDEs and regular geometries, respectively.

Spectral collocation/pseudo-spectral methods (cf. [2],[3],[4]) are known to have the capability to provide an exponential rate of convergence as the grid is refined or the degree of the interpolation polynomial is increased. The drawback of these techniques is that they require a computational domain be square $-1 \leq x, y \leq 1$. For solving problems with complex geometries, domain decompositions and coordinate transformations have usually been employed to convert irregular domains into regular ones (e.g. [5]). Another way, which is a subject of the present study, is based on the use of fictitious domain. It is noted that fictitious-domain/domain-embedding techniques can be traced back to the early 1950s. The basic idea behind these techniques is to extend domains of complicated shapes to those of simpler shapes from which the generation of meshes is simple and well-established efficient numerical solvers can be applied. Fictitious domains have been widely used in the context of finite elements, where the boundary conditions are implemented by means of Lagrange multipliers (e.g. [6] and references therein).

In the present work, the problem domain of irregular shape is embedded in the reference square. This new domain is then discretized using a tensor product grid. Clearly, the grid points do not generally lie on the boundary of the actual domain. It is thus difficult to impose boundary conditions here with conventional differential approaches.

In our earlier work [1] which deals with biharmonic problems defined in rectangular domains, it has been shown that the use of integration to construct the Chebyshev approximations provides an effective implementation of double boundary conditions. Unlike conventional differential formulations, the integral collocation formulation has the capability to generate extra expansion coefficients (integration constants). These additional unknown values can be utilized to incorporate normal derivative boundary conditions into the Chebyshev approximations.

This paper is concerned with the development of the integral collocation formulation for the case of irregularly shaped domains. Three schemes of the integral collocation formulation are presented, and their performances are numerically investigated by considering several 1D and 2D test problems.

The remainder of the paper is organized as follows. A brief review of differential and integral collocation formulations is given in Section 2. In this section, the integral collocation formulation is analyzed and several practical schemes of the formulation are presented. The proposed numerical procedure based on integral collocation schemes and fictitious domains is then described and verified through the interpolation of a function and the solution of 1D and 2D biharmonic equations in Sections 3, 4 and 5, respectively. Section 6 gives some concluding remarks.

2 Collocation formulations

2.1 Differential formulation

Consider a univariate function $f(x)$ defined in $[-1, 1]$. This function can be represented by the Chebyshev interpolant of degree N as follows

$$f(x) = \sum_{k=0}^{N} a_k T_k(x), \tag{1}$$

where $\{a_k\}_{k=0}^{N}$ are the coefficients of expansion and $\{T_k(x)\}_{k=0}^{N}$ are the Chebyshev polynomials of the first kind defined as $T_k(x) = \cos(k \arccos(x))$. Expressions of derivatives of f are then obtained through differentiation.

At the Gauss-Lobatto (G-L) points,

$$\{x_i\}_{i=0}^{N} = \left\{\cos\left(\frac{\pi i}{N}\right)\right\}_{i=0}^{N}, \tag{2}$$

the values of derivatives of f are simply computed by

$$\widehat{\frac{df}{dx}} = \widehat{\mathbf{D}}^{(1)}\widehat{f} = \widehat{\mathbf{D}}\widehat{f}, \tag{3}$$

$$\widehat{\frac{d^2 f}{dx^2}} = \widehat{\mathbf{D}}^{(2)}\widehat{f} = \widehat{\mathbf{D}}^2\widehat{f}, \tag{4}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\widehat{\frac{d^p f}{dx^p}} = \widehat{\mathbf{D}}^{(p)}\widehat{f} = \widehat{\mathbf{D}}^p\widehat{f}, \tag{5}$$

where the symbol $\widehat{.}$ is used to denote a vector/matrix that is associated with a grid line, $\widehat{f} = (f_0, f_1, \cdots, f_N)^T$, $\widehat{\frac{d^k f}{dx^k}} = \left(\frac{d^k f_0}{dx^k}, \frac{d^k f_1}{dx^k}, \cdots, \frac{d^k f_N}{dx^k}\right)^T$ with $k = \{1, 2, \cdots, p\}$, and $\widehat{\mathbf{D}}^{(.)}$ are the differentiation matrices. The entries of $\widehat{\mathbf{D}}$ ($\widehat{\mathbf{D}}^{(1)}$) are given by

$$\widehat{D}_{ij} = \frac{\bar{c}_i}{\bar{c}_j}\frac{(-1)^{i+j}}{x_i - x_j}, \qquad 0 \leq i, j \leq N, \quad i \neq j, \tag{6}$$

$$\widehat{D}_{ii} = -\frac{x_i}{2(1 - x_i^2)}, \qquad 1 \leq i \leq N-1, \tag{7}$$

$$\widehat{D}_{00} = -\widehat{D}_{NN} = \frac{2N^2+1}{6}, \tag{8}$$

where $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_i = 1$ for $i = \{1, 2, \cdots, N-1\}$. It is noted that the diagonal entries of $\widehat{\mathbf{D}}$ can also be obtained in the way that represents exactly the derivative of a constant

$$\widehat{D}_{ii} = -\sum_{j=0, j\neq i}^{N} \widehat{D}_{ij}. \tag{9}$$

For the case of smooth functions, the Chebyshev approximation scheme ((1)-(5)) is known to be very accurate (exponential accuracy) with the error being $O(N^{-\alpha})$ in which α depends on the regularity of a function. It should be emphasized that there is a reduction in accuracy for the approximation of derivatives and this reduction is an increasing function of derivative order (cf. [4]).

2.2 Integral formulation

The integral collocation formulation uses a truncated Chebyshev series of degree N to represent a derivative of an unknown function f, e.g. a derivative of order p,

$$\frac{d^p f(x)}{dx^p} = \sum_{k=0}^{N} a_k T_k(x) = \sum_{k=0}^{N} a_k I_k^{(p)}(x). \tag{10}$$

Expressions for lower-order derivatives and the function itself are then obtained through integration as

$$\frac{d^{p-1} f(x)}{dx^{p-1}} = \sum_{k=0}^{N} a_k I_k^{(p-1)}(x) + c_1, \tag{11}$$

$$\frac{d^{p-2} f(x)}{dx^{p-2}} = \sum_{k=0}^{N} a_k I_k^{(p-2)}(x) + c_1 x + c_2, \tag{12}$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{df(x)}{dx} = \sum_{k=0}^{N} a_k I_k^{(1)}(x) + c_1 \frac{x^{p-2}}{(p-2)!} + c_2 \frac{x^{p-3}}{(p-3)!} + \cdots + c_{p-2} x + c_{p-1}, \tag{13}$$

$$f(x) = \sum_{k=0}^{N} a_k I_k^{(0)}(x) + c_1 \frac{x^{p-1}}{(p-1)!} + c_2 \frac{x^{p-2}}{(p-2)!} + \cdots + c_{p-1} x + c_p, \tag{14}$$

where $I_k^{(p-1)}(x) = \int I_k^{(p)}(x)dx,\ I_k^{(p-2)}(x) = \int I_k^{(p-1)}(x)dx, \cdots, I_k^{(0)}(x) = \int I_k^{(1)}(x)dx$, and $\{c_i\}_{i=1}^{p}$ are the constants of integration.

2.3 An analysis of the integral formulation

Since a truncated Chebyshev series expansion representing $d^p f/dx^p$ is the interpolant of degree N, the integration process defined by (10)-(14) leads to an approximate expression for f that is the interpolation polynomial of degree $(N+p)$ with $(N+p)$ coefficients. Based on this observation, in addition to (14), we also consider the following expansion

$$f(x) = \sum_{k=0}^{N} a_k T_k(x) + c_1 g_1(x) + c_2 g_2(x) + \cdots + c_p g_p(x), \tag{15}$$

where $\{g_i(x)\}_{i=1}^{p}$ are chosen in such a way that

- they are polynomials,
- all basis functions in (15) are linearly independent, and
- the resultant expansion (15) has the same degree of the polynomial and the same number of expansion coefficients as (14).

Possible choices for such basis functions include

$$\{T_{N+1}(x), T_{N+2}(x), \cdots, T_{N+p}(x)\} \quad \text{and} \tag{16}$$

$$\left\{x^{N+1}, x^{N+2}, \cdots, x^{N+p}\right\}. \tag{17}$$

From here on, ICS^I, ICS^{II} and ICS^{III} are used to represent three schemes of the integral collocation formulation, (14), (15)&(16) and (15)&(17), respectively. The value of p in (10) is regarded as the order of the integral collocation scheme, denoted by ICS_p. A differential collocation scheme can be considered as a special case of ICS by letting p be zero (ICS_0).

To make notations simple, we also use $\left\{I_k^{(i)}(x)\right\}_{k=0,i=0}^{k=N,i=p}$ to denote the basis functions associated with $\{a_k\}_{k=0}^{N}$ in (15) and its derivatives, and $\{g_i(x)\}_{i=1}^{p}$ to represent the basis functions associated with $\{c_i\}_{i=1}^{p}$ in (14). Solution procedures for the three schemes are exactly the same.

The evaluation of f and its derivatives at the G-L points leads to

$$\widehat{\frac{d^p f}{dx^p}} = \widehat{\mathcal{I}}_{[p]}^{(p)} \widehat{s}, \tag{18}$$

$$\widehat{\frac{d^{p-1} f}{dx^{p-1}}} = \widehat{\mathcal{I}}_{[p]}^{(p-1)} \widehat{s}, \tag{19}$$

$$\cdots\cdots\cdots$$

$$\widehat{\frac{df}{dx}} = \widehat{\mathcal{I}}_{[p]}^{(1)} \widehat{s}, \tag{20}$$

$$\widehat{f} = \widehat{\mathcal{I}}_{[p]}^{(0)} \widehat{s}, \tag{21}$$

where subscript [.] and superscript (.) are used to indicate the orders of ICS and derivative function, respectively; $\widehat{s} = (\widehat{a}, \widehat{c})^T$ in which $\widehat{a} = (a_0, a_1, \cdots, a_N)^T$ and $\widehat{c} = (c_1, c_2, \cdots, c_p)^T$;

$$\widehat{\mathcal{I}}_{[p]}^{(p)} = \begin{bmatrix} I_0^{(p)}(x_0), & I_1^{(p)}(x_0), & \cdots, & I_N^{(p)}(x_0), & \frac{d^p g_1}{dx^p}(x_0), & \frac{d^p g_2}{dx^p}(x_0), & \cdots, & \frac{d^p g_p}{dx^p}(x_0) \\ I_0^{(p)}(x_1), & I_1^{(p)}(x_1), & \cdots, & I_N^{(p)}(x_1), & \frac{d^p g_1}{dx^p}(x_1), & \frac{d^p g_2}{dx^p}(x_1), & \cdots, & \frac{d^p g_p}{dx^p}(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_0^{(p)}(x_N), & I_1^{(p)}(x_N), & \cdots, & I_N^{(p)}(x_N), & \frac{d^p g_1}{dx^p}(x_N), & \frac{d^p g_2}{dx^p}(x_N), & \cdots, & \frac{d^p g_p}{dx^p}(x_N) \end{bmatrix};$$

$$\widehat{\mathcal{I}}_{[p]}^{(p-1)} = \begin{bmatrix} I_0^{(p-1)}(x_0), & \cdots, & I_N^{(p-1)}(x_0), & \frac{d^{p-1} g_1}{dx^{p-1}}(x_0), & \frac{d^{p-1} g_2}{dx^{p-1}}(x_0), & \cdots, & \frac{d^{p-1} g_p}{dx^{p-1}}(x_0) \\ I_0^{(p-1)}(x_1), & \cdots, & I_N^{(p-1)}(x_1), & \frac{d^{p-1} g_1}{dx^{p-1}}(x_1), & \frac{d^{p-1} g_2}{dx^{p-1}}(x_1), & \cdots, & \frac{d^{p-1} g_p}{dx^{p-1}}(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_0^{(p-1)}(x_N), & \cdots, & I_N^{(p-1)}(x_N), & \frac{d^{p-1} g_1}{dx^{p-1}}(x_N), & \frac{d^{p-1} g_2}{dx^{p-1}}(x_N), & \cdots, & \frac{d^{p-1} g_p}{dx^{p-1}}(x_N) \end{bmatrix};$$

$\cdots\cdots$; and

$$\widehat{\mathcal{I}}_{[p]}^{(0)} = \begin{bmatrix} I_0^{(0)}(x_0), & I_1^{(0)}(x_0), & \cdots, & I_N^{(0)}(x_0), & g_1(x_0), & g_2(x_0), & \cdots, & g_p(x_0) \\ I_0^{(0)}(x_1), & I_1^{(0)}(x_1), & \cdots, & I_N^{(0)}(x_1), & g_1(x_1), & g_2(x_1), & \cdots, & g_p(x_1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_0^{(0)}(x_N), & I_1^{(0)}(x_N), & \cdots, & I_N^{(0)}(x_N), & g_1(x_N), & g_2(x_N), & \cdots, & g_p(x_N) \end{bmatrix}.$$

3 Function interpolation

Consider a function $f(x)$ defined in [-1,1]. The domain of interest is discretized using the G-L points. Here, we concern the case, where given information consists of the values of the function at the grid points and some "extra" values. The latter can be the values of f and its derivatives at some points that do not coincide with the grid nodes. Let x_{bi} and f_{bi} $(d^k f_{bi}/dx^k)$ with $i = \{1, 2, \cdots\}$ denote the extra points and the extra information values, respectively. Unlike conventional differential formulations, the integral collocation formulation can easily incorporate extra information into the Chebyshev approximations. Two approaches are proposed below.

3.1 Approach 1

For the sake of simplicity, assume that there are $p/2$ extra points (p—an even number) and each point is associated with two given values, f and df/dx. One thus has p extra values

$$\widehat{f}_{\text{extra}} = \left(f_{b1}, df_{b1}/dx, \cdots f_{b\frac{p}{2}}, df_{b\frac{p}{2}}/dx \right)^T. \tag{22}$$

The expansion coefficients can be determined using the ICS scheme of order p (ICS_p)

$$\begin{pmatrix} \widehat{f} \\ \widehat{f}_{\text{extra}} \end{pmatrix} = \begin{bmatrix} \widehat{\mathcal{I}}^{(0)}_{[p]} \\ \widehat{\mathcal{B}} \end{bmatrix} \widehat{s} = \widehat{\mathcal{C}}\,\widehat{s}, \tag{23}$$

where

$$\widehat{\mathcal{B}} = \begin{bmatrix} I_0^{(0)}(x_{b1}), & I_1^{(0)}(x_{b1}), & \cdots, & I_N^{(0)}(x_{b1}), & g_1(x_{b1}), & g_2(x_{b1}), & \cdots, & g_p(x_{b1}) \\ I_0^{(1)}(x_{b1}), & I_1^{(1)}(x_{b1}), & \cdots, & I_N^{(1)}(x_{b1}), & \frac{dg_1}{dx}(x_{b1}), & \frac{dg_2}{dx}(x_{b1}), & \cdots, & \frac{dg_p}{dx}(x_{b1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ I_0^{(0)}(x_{b\frac{p}{2}}), & I_1^{(0)}(x_{b\frac{p}{2}}), & \cdots, & I_N^{(0)}(x_{b\frac{p}{2}}), & g_1(x_{b\frac{p}{2}}), & g_2(x_{b\frac{p}{2}}), & \cdots, & g_p(x_{b\frac{p}{2}}) \\ I_0^{(1)}(x_{b\frac{p}{2}}), & I_1^{(1)}(x_{b\frac{p}{2}}), & \cdots, & I_N^{(1)}(x_{b\frac{p}{2}}), & \frac{dg_1}{dx}(x_{b\frac{p}{2}}), & \frac{dg_2}{dx}(x_{b\frac{p}{2}}), & \cdots, & \frac{dg_p}{dx}(x_{b\frac{p}{2}}) \end{bmatrix},$$

$\widehat{\mathcal{C}}$ is the system matrix of dimension $(N+1+p) \times (N+1+p)$ and other notations are defined as before. The above expression indicates that the integral formulation takes into account the extra information values. After solving (23) for $\widehat{s}$, one can easily calculate the values of derivatives of f at the grid points using (18)-(20).

3.2 Approach 2

As mentioned earlier, the constants of integration are generated for the purpose of dealing with extra information. It can be seen that every grid point is associated with the same set of integration constants. The relationship between $\widehat{c}$ and $\widehat{f}_{\text{extra}}$ is as follows

$$\widehat{f}_{\text{extra}} = \widehat{\mathcal{B}} \begin{pmatrix} \widehat{a} \\ \widehat{c} \end{pmatrix} = \widehat{\mathcal{B}}_1 \widehat{a} + \widehat{\mathcal{B}}_2 \widehat{c}, \tag{24}$$

or

$$\widehat{c} = -\widehat{\mathcal{B}}_2^{-1} \widehat{\mathcal{B}}_1 \widehat{a} + \widehat{\mathcal{B}}_2^{-1} \widehat{f}_{\text{extra}}, \tag{25}$$

where $\widehat{\mathcal{B}}_1$ and $\widehat{\mathcal{B}}_2$ are the first $(N+1)$ and the last p columns of $\widehat{\mathcal{B}}$, respectively. It is noted that one can solve (24) for $\widehat{c}$ in an analytical manner.

Substitution of (25) into (21) yields

$$\widehat{f} = \widehat{\mathcal{C}}\,\widehat{a} + \widehat{k}, \tag{26}$$

where

$$\widehat{\mathcal{C}} = \left(\left(\widehat{\mathcal{I}}^{(0)}_{[p]} \right)_1 - \left(\widehat{\mathcal{I}}^{(0)}_{[p]} \right)_2 \widehat{\mathcal{B}}_2^{-1} \widehat{\mathcal{B}}_1 \right) \quad \text{and}$$
$$\widehat{k} = \left(\widehat{\mathcal{I}}^{(0)}_{[p]} \right)_2 \widehat{\mathcal{B}}_2^{-1} \widehat{f}_{\text{extra}},$$

in which $\left(\widehat{\mathcal{I}}^{(0)}_{[p]} \right)_1$ and $\left(\widehat{\mathcal{I}}^{(0)}_{[p]} \right)_2$ are the first $(N+1)$ and the last p columns of $\widehat{\mathcal{I}}^{(0)}_{[p]}$, respectively.

The expansion coefficients are then obtained through (26) for $\widehat{a}$ and (25) for $\widehat{c}$.

Approach 1 and Approach 2 are equivalent from the mathematical point of view. However, Approach 1 involves solving one set of equations, while Approach 2 involves solving two smaller sets of equations.

Consider a function $f = \sin(\pi x)$ with $-1 \leq x \leq 1$. In addition to the grid function values, there are 4 extra values given (f and df/dx at $x = -1/3$ and $x = 1/3$). The three integral collocation schemes are employed to evaluate the values of derivatives of f at the grid points. Since there are 4 extra information values, one can employ ICSs of order 4. Each scheme is implemented in conjunction with Approach 1 and Approach 2. Results obtained are given in Tables 1, 2 and 3. They indicate that the three schemes of the integral formulation yield similar degrees of accuracy on grids where their system matrices are well-conditioned.

It should be pointed out that the system matrix of each integral collocation scheme has an entirely different range of the condition number. In each scheme, Approach 1 and Approach 2 also strongly affect the matrix condition number. Approach 2 is seen to be much better than Approach 1, except for ICS^{III}. The ICS^{II} scheme appears to be the best one as its condition number is very low, ranged from 10^1 to 10^2 (Table 2). The reason for that is probably due to the fact that the system matrix of ICS^{II} is composed largely of Chebyshev polynomials. The approximation scheme based on ICS^{II} and Approach 2 is recommended for use in the interpolation of a function and its derivatives.

4 One-dimensional biharmonic problems

Consider the following 1D biharmonic equation

$$\frac{d^4u}{dx^4} + \frac{d^2u}{dx^2} = b(x), \quad x_{b1} \leq x \leq x_{b2}, \quad |x_{bi}| \leq 1, \tag{27}$$

where $b(x)$ is a known driving function, subject to Dirichlet boundary conditions at both ends

$$\begin{aligned} u(x_{b1}) &= \bar{u}_1, \quad \frac{du}{dx}(x_{b1}) = \frac{d\bar{u}_1}{dx}, \\ u(x_{b2}) &= \bar{u}_2, \quad \frac{du}{dx}(x_{b2}) = \frac{d\bar{u}_2}{dx}. \end{aligned}$$

The problem domain is embedded in [-1,1] and the extended domain is discretized using the G-L points.

Making use of (14)/(15) and its relevant derivatives with $p = 4$, the governing equation (27) and the boundary conditions can be transformed into

$$\sum_{k=0}^{N} a_k I_k^{(4)}(x) + \sum_{k=0}^{N} a_k I_k^{(2)}(x) + c_1 x + c_2 = b(x), \tag{28}$$

$$\sum_{k=0}^{N} a_k I_k^{(0)}(x_{b1}) + c_1 \frac{x_{b1}^3}{6} + c_2 \frac{x_{b1}^2}{2} + c_3 x_{b1} + c_4 = \bar{u}_1, \tag{29}$$

$$\sum_{k=0}^{N} a_k I_k^{(1)}(x_{b1}) + c_1 \frac{x_{b1}^2}{2} + c_2 x_{b1} + c_3 = \frac{d\bar{u}_1}{dx}, \tag{30}$$

$$\sum_{k=0}^{N} a_k I_k^{(0)}(x_{b2}) + c_1 \frac{x_{b2}^3}{6} + c_2 \frac{x_{b2}^2}{2} + c_3 x_{b2} + c_4 = \bar{u}_2, \tag{31}$$

$$\sum_{k=0}^{N} a_k I_k^{(1)}(x_{b2}) + c_1 \frac{x_{b2}^2}{2} + c_2 x_{b2} + c_3 = \frac{d\bar{u}_2}{dx}. \tag{32}$$

The evaluation of (28) at the whole set of the G-L points $\{x_i\}_{i=0}^N$ plus the boundary conditions (29)-(32) leads to a determinate system of equations

$$\begin{bmatrix} \widehat{\mathcal{I}}_{[4]}^{(4)} + \widehat{\mathcal{I}}_{[4]}^{(2)} \\ \widehat{\mathcal{B}} \end{bmatrix} \widehat{s} = \widehat{t}, \tag{33}$$

where $\widehat{s} = (a_0, a_1, \cdots, a_N, c_1, c_2, c_3, c_4)^T$, $\widehat{t} = (b_0, b_1, \cdots, b_N, \bar{u}_1, d\bar{u}_1/dx, \bar{u}_2, d\bar{u}_2/dx)^T$, and

$$\widehat{\mathcal{B}} = \begin{bmatrix} I_0^{(0)}(x_{b1}), & I_1^{(0)}(x_{b1}), & \cdots, & I_N^{(0)}(x_{b1}), & x_{b1}^3/6, & x_{b1}^2/2, & x_{b1}, & 1 \\ I_0^{(1)}(x_{b1}), & I_1^{(1)}(x_{b1}), & \cdots, & I_N^{(1)}(x_{b1}), & x_{b1}^2/2, & x_{b1}, & 1, & 0 \\ I_0^{(0)}(x_{b2}), & I_1^{(0)}(x_{b2}), & \cdots, & I_N^{(0)}(x_{b2}), & x_{b2}^3/6, & x_{b2}^2/2, & x_{b2}, & 1 \\ I_0^{(1)}(x_{b2}), & I_1^{(1)}(x_{b2}), & \cdots, & I_N^{(1)}(x_{b2}), & x_{b2}^2/2, & x_{b2}, & 1, & 0 \end{bmatrix}.$$

The resultant system (33) can be solved in a direct manner (like Approach 1 in the case of function interpolation) or by splitting it into 2 smaller sets of equations (like Approach 2).

The three schemes are numerically verified using the following data

$$x_{b1} = -2/3, \quad x_{b2} = +2/3,$$

$$b = -\pi^2 \sin(\pi x) + \pi^4 \sin(\pi x),$$

$$\bar{u}_1 = -\sqrt{3}/2, \quad d\bar{u}_1/dx = -\pi/2,$$

$$\bar{u}_2 = +\sqrt{3}/2, \quad d\bar{u}_2/dx = -\pi/2.$$

The exact solution of this problem is given by

$$u_e = \sin(\pi x).$$

Results obtained are presented in Tables 4, 5 and 6. Relative L_2 errors of u ($N_e(u)$) are computed at the grid points. Unlike the case of function interpolation, the construction of the system matrix here is mainly based on the approximation of derivative functions (the differential equation) rather than based on the original function. It can be seen that the first scheme involves more Chebyshev polynomials $T_k(x)$ than the others. Numerical results show that ICS^I yields a system matrix with the condition number much lower than those associated with ICS^{II} and ICS^{III}. Its values are considerably small, especially for Approach 2 (Table 4). It is recommended that the numerical scheme based on ICS^I and Approach 2 be considered for solving 1D biharmonic equations.

5 Two-dimensional biharmonic problems

Consider a 2D Dirichlet biharmonic problem. The governing equation takes the form

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = b(x,y), \tag{34}$$

where $b(x,y)$ is a known driving function, subject to double boundary conditions (u and $\partial u/\partial n$, $n-$the normal direction) along the boundary. The proposed numerical procedure is presented in detail through a domain of irregular shape Ω depicted in Figure 1. This irregular domain is embedded in the reference square, which allows the use of tensor product grids $(N_x+1)\times(N_y+1)$. The present method divides the prescribed boundary conditions into two groups. The first group is made of the given values of the solution at the regular boundary points (grid points which lie on the actual boundary), while the second group is formed from the remaining boundary conditions. The latter consists of normal derivative boundary conditions at the regular boundary points, and the boundary conditions at the irregular boundary points (the intersections of grid lines and irregular boundaries). The construction of approximate expressions for $\partial^2 u/\partial x^2$ and $\partial^4 u/\partial x^4$ is similar to that for $\partial^2 u/\partial y^2$ and $\partial^4 u/\partial y^4$. Only derivatives of u with respect to y are considered here. Unlike the case of 1D biharmonic problems, the Chebyshev approximations will be expressed in terms of nodal variable values (physical space) to avoid the problem of increasing the system matrix size. Some typical cases are as follows.

Case 1 - Line aa':

Along this line, one needs to impose the values of u only. The ICS_0 scheme can be employed here. The values of $\partial^2 u/\partial y^2$ and $\partial^4 u/\partial y^4$ at the grid points are computed using (3)-(5).

Case 2 - Line bb':

This line intersects the actual boundary at two points y_{b1} and y_{b2} ($y_{b1} < y_{b2}$). The first boundary point y_{b1} is a grid node (regular boundary point). There is one extra value associated with this node, namely $\partial \bar{u}_1/\partial n$.

If the second boundary point y_{b2} is also a grid node, the treatment for y_{b2} will be the same as that for y_{b1}. There are thus two extra values in total along this line. To impose them, one can use the ICS_2 scheme. The transformation of the spectral space into the physical space is based on the following system

$$\begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \frac{\partial \bar{u}_2}{\partial y} \end{pmatrix} = \begin{bmatrix} \widehat{\mathcal{I}}^{(0)}_{[2]} \\ \widehat{\mathcal{B}} \end{bmatrix} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \end{pmatrix} = \widehat{\mathcal{C}} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \end{pmatrix}, \tag{35}$$

where $\widehat{\mathcal{C}}$ is the conversion matrix of dimension $(N_y+3)\times(N_y+3)$, $\widehat{a} = \left(a_0, a_1, \cdots, a_{N_y}\right)^T$, $\widehat{u} = \left(u_0, u_1, \cdots, u_{N_y}\right)^T$, and

$$\widehat{\mathcal{B}} = \begin{bmatrix} I_0^{(1)}(y_{b1}), & I_1^{(1)}(y_{b1}), & \cdots, & I_{N_y}^{(1)}(y_{b1}), & \frac{dg_1}{dy}(y_{b1}), & \frac{dg_2}{dy}(y_{b1}) \\ I_0^{(1)}(y_{b2}), & I_1^{(1)}(y_{b2}), & \cdots, & I_{N_y}^{(1)}(y_{b2}), & \frac{dg_1}{dy}(y_{b2}), & \frac{dg_2}{dy}(y_{b2}) \end{bmatrix}_{[2]}.$$

Solving (35), in a direct manner (Approach 1), yields

$$\begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \end{pmatrix} = \widehat{\mathcal{C}}^{-1} \begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \frac{\partial \bar{u}_2}{\partial y} \end{pmatrix}. \tag{36}$$

The values of $\partial^2 u/\partial y^2$ and $\partial^4 u/\partial y^4$ at the grid points are then computed by

$$\widehat{\frac{\partial^2 u}{\partial y^2}} = \widehat{\mathcal{I}}^{(2)}_{[2]} \widehat{\mathcal{C}}^{-1} \begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \frac{\partial \bar{u}_2}{\partial y} \end{pmatrix}, \tag{37}$$

$$\widehat{\frac{\partial^4 u}{\partial y^4}} = \widehat{\mathcal{I}}^{(4)}_{[2]} \widehat{\mathcal{C}}^{-1} \begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \frac{\partial \bar{u}_2}{\partial y} \end{pmatrix}, \tag{38}$$

where

$$
\widehat{\mathcal{I}}^{(4)}_{[2]} = \begin{bmatrix} \frac{d^2T_0}{dy^2}(y_0), & \frac{d^2T_1}{dy^2}(y_0), & \cdots, & \frac{d^2T_{N_y}}{dy^2}(y_0), & 0, & 0 \\ \frac{d^2T_0}{dy^2}(y_1), & \frac{d^2T_1}{dy^2}(y_1), & \cdots, & \frac{d^2T_{N_y}}{dy^2}(y_1), & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \frac{d^2T_0}{dy^2}(y_{N_y}), & \frac{d^2T_1}{dy^2}(y_{N_y}), & \cdots, & \frac{d^2T_{N_y}}{dy^2}(y_{N_y}), & 0, & 0 \end{bmatrix}.
$$

If the second boundary point y_{b2} does not coincide with any grid points, there are two extra values, namely $\bar{u}_2$ and $\partial\bar{u}_2/\partial n$, at y_{b2}, leading to a total of three extra values along line bb'. They can be imposed through the ICS_3 scheme. The transformation system is given by

$$
\begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \bar{u}_2 \\ \frac{\partial \bar{u}_2}{\partial y} \end{pmatrix} = \begin{bmatrix} \widehat{\mathcal{I}}^{(0)}_{[3]} \\ \widehat{\mathcal{B}} \end{bmatrix} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \widehat{\mathcal{C}} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{39}
$$

where $\widehat{\mathcal{C}}$ is the conversion matrix of dimension $(N_y + 4) \times (N_y + 4)$ and

$$
\widehat{\mathcal{B}} = \begin{bmatrix} I^{(1)}_0(y_{b1}), & I^{(1)}_1(y_{b1}), & \cdots, & I^{(1)}_{N_y}(y_{b1}), & \frac{dg_1}{dy}(y_{b1}), & \frac{dg_2}{dy}(y_{b1}), & \frac{dg_3}{dy}(y_{b1}) \\ I^{(0)}_0(y_{b2}), & I^{(0)}_1(y_{b2}), & \cdots, & I^{(0)}_{N_y}(y_{b2}), & g_1(y_{b2}), & g_2(y_{b2}), & g_3(y_{b2}) \\ I^{(1)}_0(y_{b2}), & I^{(1)}_1(y_{b2}), & \cdots, & I^{(1)}_{N_y}(y_{b2}), & \frac{dg_1}{dy}(y_{b2}), & \frac{dg_2}{dy}(y_{b2}), & \frac{dg_3}{dy}(y_{b2}) \end{bmatrix}_{[3]}.
$$

The remaining steps for obtaining expressions of $\partial^2 u/\partial y^2$ and $\partial^4 u/\partial y^4$ are similar to the previous case and therefore omitted here for brevity.

Case 3 - Line cc':

This case involves 4 intersection points: y_{b1}, y_{b2}, y_{b3} and y_{b4}. The first and last points are regular boundary points. Assume that y_{b2} and y_{b3} are not grid points. There are 6 extra values to be imposed. The process of transforming the expansion coefficients into the nodal variable values is based on the following system

$$\begin{pmatrix} \widehat{u} \\ \frac{\partial \bar{u}_1}{\partial y} \\ \bar{u}_2 \\ \frac{\partial \bar{u}_2}{\partial y} \\ \bar{u}_3 \\ \frac{\partial \bar{u}_3}{\partial y} \\ \frac{\partial \bar{u}_4}{\partial y} \end{pmatrix} = \begin{bmatrix} \widehat{\mathcal{I}}^{(0)}_{[6]} \\ \widehat{\mathcal{B}} \end{bmatrix} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix} = \widehat{\mathcal{C}} \begin{pmatrix} \widehat{a} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}, \tag{40}$$

where $\widehat{\mathcal{C}}$ is the conversion matrix of dimension $(N_y + 7) \times (N_y + 7)$ and

$$\widehat{\mathcal{B}} = \begin{bmatrix} I_0^{(1)}(y_{b1}), & \cdots, & I_{N_y}^{(1)}(y_{b1}), & \frac{dg_1}{dy}(y_{b1}), & \cdots, & \frac{dg_6}{dy}(y_{b1}) \\ I_0^{(0)}(y_{b2}), & \cdots, & I_{N_y}^{(0)}(y_{b2}), & g_1(y_{b2}), & \cdots, & g_6(y_{b2}) \\ I_0^{(1)}(y_{b2}), & \cdots, & I_{N_y}^{(1)}(y_{b2}), & \frac{dg_1}{dy}(y_{b2}), & \cdots, & \frac{dg_6}{dy}(y_{b2}) \\ I_0^{(0)}(y_{b3}), & \cdots, & I_{N_y}^{(0)}(y_{b3}), & g_1(y_{b3}), & \cdots, & g_6(y_{b3}) \\ I_0^{(1)}(y_{b3}), & \cdots, & I_{N_y}^{(1)}(y_{b3}), & \frac{dg_1}{dy}(y_{b3}), & \cdots, & \frac{dg_6}{dy}(y_{b3}) \\ I_0^{(1)}(y_{b4}), & \cdots, & I_{N_y}^{(1)}(y_{b4}), & \frac{dg_1}{dy}(y_{b4}), & \cdots, & \frac{dg_6}{dy}(y_{b4}) \end{bmatrix}_{[6]}.$$

It can be seen that the Chebyshev approximations of derivatives at a grid point are now expressed in terms of the nodal values of u along the grid line that goes through that point. As with finite-difference and finite-element techniques, one will gather these approximations together to form global matrices for the discretization of the PDE. Their final forms can be written as

$$\widetilde{\frac{\partial^i u}{\partial x^i}} = \widetilde{\mathcal{D}}_{ix}\widetilde{u} + \widetilde{k}_{ix} \tag{41}$$

$$\widetilde{\frac{\partial^i u}{\partial y^i}} = \widetilde{\mathcal{D}}_{iy}\widetilde{u} + \widetilde{k}_{iy}, \tag{42}$$

where $\widetilde{.}$ is used to denote a vector/matrix that is associated with a 2D tensor product grid, $\widetilde{\mathcal{D}}_{ix}$ and $\widetilde{\mathcal{D}}_{iy}$ are known matrices of dimension $(N_x+1)(N_y+1)\times(N_x+1)(N_y+1)$, and $\widetilde{k}_{ix}$ and $\widetilde{k}_{iy}$ are known vectors of length $(N_x+1)(N_y+1)$.

The mixed partial fourth-order derivatives can be computed using the following relation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \frac{1}{2}\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 u}{\partial x^2}\right)\right]. \tag{43}$$

In the calculation of the RHS of (43), approximate expressions (41) and (42) are used to evaluate the values of $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ at the grid points, while second-order differential operators are simply replaced by

$$\frac{\partial^2}{\partial x}() = \left(\widehat{\mathbf{D}}_x^2 \otimes \mathbf{I}_y\right)(), \tag{44}$$

$$\frac{\partial^2}{\partial y}() = \left(\mathbf{I}_x \otimes \widehat{\mathbf{D}}_y^2\right)(), \tag{45}$$

where $\otimes$ denotes the Kronecker tensor product; $\widehat{\mathbf{D}}_x$ and $\widehat{\mathbf{D}}_y$ are the differentiation matrices in the $x-$ and $y-$ directions, respectively ($\widehat{D}_{ij}$ are defined by (6)-(8)); and $\mathbf{I}_y$ and $\mathbf{I}_x$ are identity matrices of dimension $(N_y+1)\times(N_y+1)$ and $(N_x+1)\times(N_x+1)$, respectively. In (44) and (45), the grid points are numbered from bottom to top and from left to right.

It is worth mentioning that approximate expressions for derivatives of u already contain information about the boundary of Ω (location and value).

By collocating the governing equation at the grid points and then deleting rows corresponding to points that lie on the boundary, a determinate system of algebraic equations is obtained, which is solved for the approximate solution.

The proposed procedure is tested using the following functions

$$b = 4\cos(\pi x)\cos(\pi y) + \cos(\pi x) + \cos(\pi y), \quad (46)$$

$$u_e = \frac{1}{\pi^4}\left[1 + \cos(\pi x)\right]\left[1 + \cos(\pi y)\right]. \quad (47)$$

Two cases, namely single domain and multi-domains, are studied.

5.1 Single domain

A unit circular domain is considered (Figure 2). This domain is embedded in the reference square. Results obtained by the ICS^I and ICS^{II} schemes are presented in Table 7. Unlike the cases of function approximations and 1D biharmonic equations, a numerical solution here is solved directly in the physical space. It can be seen that, in the physical space, the ICS^I and ICS^{II} schemes essentially yield the same results with respect to the condition number and the relative L_2 error. An exponential rate of convergence with grid refinement is achieved.

5.2 Multi-domains

An irregular domain, which is displayed in Figure 3, is divided into three subdomains. Sudomain 1 is a simply-connected domain, while subdomains 2 and 3 are multiply-connected domains. Points A, B, C, D, E, F and G are located at (0,0), (0,-1), (-1,-1), (1,1), (-7/12,1), (-1,7/12) and (-1,0), respectively. The circular hole is of radius 1/3 and centered at (1/2,-1/2), while the square hole is taken as $[1/6, 1/6] \times [5/6, 5/6]$. Along the subdomain interfaces, the unknowns are chosen to be u and $\partial u/\partial n$. These unknown values are determined using the continuity of $\partial^2 u/\partial n^2$ and $\partial^3 u/\partial n^3$ across the interfaces. Table 8 presents relative L_2 errors of the solution u at the interior points of the three subdomains and of the whole domain. It can be seen that the proposed technique yields spectral accuracy.

6 Concluding remarks

This paper reports a Chebyshev integral collocation approach for solving biharmonic equations in irregular domains. The problem domain is embedded in the reference square, and this extended domain is handled using integral collocation schemes. Boundary conditions are simply divided into two groups. The first group is made of the given values of the solution at the regular boundary points, while the second group is formed from the remaining boundary conditions. The latter consists of normal derivative boundary conditions at the regular boundary points, and the boundary conditions at the irregular boundary points. All boundary conditions in the second group can be implemented in a similar fashion, making the present numerical procedure very attractive in terms of simplicity. Very accurate results are achieved using coarse grids.

ACKNOWLEDGEMENTS

This research is supported by the Australian Research Council. The authors are grateful to Prof. R.I. Tanner and Prof. X.-J. Fan for fruitful discussions throughout this study. We would like to thank the referees for their helpful comments.

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Table 1: $f = \sin(\pi x)$, $-1 \le x \le 1$: Relative L_2 errors, denoted by N_e, of derivatives of f at the grid points and the condition numbers of the system matrix A, denoted by cond(A), obtained by the ICS_4^I scheme. In addition to the grid function values, there are 4 extra values imposed (f and df/dx at $x = -1/3$ and $x = 1/3$).

Grid	cond(A)	$N_e(df/dx)$	$N_e(d^2f/dx^2)$	$N_e(d^3f/dx^3)$	$N_e(d^4f/dx^4)$
$(N+1)$			Approach 1		
5	3.4e+5	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	2.0e+6	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	1.0e+8	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	1.6e+9	1.0e-07	3.1e-06	2.9e-05	3.4e-04
13	4.8e+8	1.1e-09	4.4e-08	5.7e-07	8.8e-06
15	5.4e+8	9.1e-12	4.7e-10	7.9e-09	1.5e-07
17	3.9e+9	4.7e-14	2.9e-12	6.3e-11	1.5e-09
$(N+1)$			Approach 2		
5	7.1e+3	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	4.8e+4	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	1.8e+6	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	2.7e+7	1.0e-07	3.1e-06	2.9e-05	3.4e-04
13	8.9e+6	1.1e-09	4.4e-08	5.7e-07	8.8e-06
15	1.2e+7	9.1e-12	4.7e-10	7.9e-09	1.5e-07
17	7.2e+7	6.1e-14	3.9e-12	8.5e-11	2.1e-09

Table 2: $f = \sin(\pi x)$, $-1 \le x \le 1$: Relative L_2 errors of derivatives of f at the grid points and the condition numbers of the system matrix A by the ICS_4^{II} scheme. In addition to the grid function values, there are 4 extra values imposed (f and df/dx at $x = -1/3$ and $x = 1/3$).

Grid	cond(A)	$N_e(df/dx)$	$N_e(d^2f/dx^2)$	$N_e(d^3f/dx^3)$	$N_e(d^4f/dx^4)$
$(N+1)$			Approach 1		
5	3.7e+1	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	5.6e+1	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	7.1e+2	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	3.6e+3	1.0e-07	3.1e-06	2.9e-05	3.4e-04
13	3.9e+2	1.1e-09	4.4e-08	5.7e-07	8.8e-06
15	1.7e+2	9.1e-12	4.7e-10	7.9e-09	1.5e-07
17	6.9e+2	8.6e-14	5.3e-12	1.1e-10	2.7e-09
$(N+1)$			Approach 2		
5	1.8e+1	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	1.6e+1	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	1.6e+2	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	6.8e+2	1.0e-07	3.1e-06	2.9e-05	3.4e-04
13	8.6e+1	1.1e-09	4.4e-08	5.7e-07	8.8e-06
15	4.1e+1	9.2e-12	4.7e-10	7.9e-09	1.5e-07
17	1.3e+2	3.6e-14	1.2e-12	3.0e-11	5.5e-10

Table 3: $f = \sin(\pi x)$, $-1 \leq x \leq 1$: Relative L_2 errors of derivatives of f at the grid points and the condition numbers of the system matrix A by the ICS_4^{III} scheme. In addition to the grid function values, there are 4 extra values imposed (f and df/dx at $x = -1/3$ and $x = 1/3$).

Grid	cond(A)	$N_e(df/dx)$	$N_e(d^2f/dx^2)$	$N_e(d^3f/dx^3)$	$N_e(d^4f/dx^4)$
$(N+1)$			Approach 1		
5	2.5e+3	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	1.5e+4	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	8.3e+5	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	2.2e+7	1.0e-07	3.1e-06	2.9e-05	3.4e-04
13	1.3e+7	1.1e-09	4.4e-08	5.7e-07	8.8e-06
15	3.4e+7	9.1e-12	4.7e-10	7.9e-09	1.5e-07
17	5.0e+8	8.6e-14	5.3e-12	1.1e-10	2.7e-09
$(N+1)$			Approach 2		
5	2.4e+06	1.1e-02	1.2e-01	2.7e-01	1.2e+00
7	1.5e+08	3.8e-04	5.5e-03	2.3e-02	1.4e-01
9	8.7e+10	7.6e-06	1.6e-04	1.0e-03	9.0e-03
11	2.4e+13	1.1e-07	3.3e-06	3.0e-05	3.5e-04
13	1.5e+14	2.0e-07	3.5e-06	2.4e-05	2.3e-04
15	3.8e+15	1.0e-06	1.7e-05	1.7e-04	1.9e-03
17	4.1e+17	2.9e-05	7.9e-04	8.8e-03	1.3e-01

Table 4: 1D biharmonic problem, Dirichlet boundary conditions, $-2/3 \leq x \leq 2/3$: Relative L_2 errors of the solution u and the condition numbers of the system matrix A by the ICS_4^I scheme.

Grid	Approach 1		Approach 2	
$(N+1)$	cond(A)	$N_e(u)$	cond(A)	$N_e(u)$
5	4.6e+1	1.0e-01	2.2e+0	1.0e-01
7	5.3e+1	2.3e-03	2.3e+0	2.3e-03
9	6.0e+1	1.2e-05	2.2e+0	1.2e-05
11	6.6e+1	2.4e-07	2.2e+0	2.4e-07
13	7.2e+1	4.9e-10	2.1e+0	4.9e-10
15	7.7e+1	1.4e-11	2.1e+0	1.4e-11
17	8.2e+1	1.5e-14	2.1e+0	1.5e-14
19	8.6e+1	2.2e-15	2.1e+0	1.0e-15
21	9.1e+1	6.8e-16	2.1e+0	4.4e-16

Table 5: 1D biharmonic problem, Dirichlet boundary conditions, $-2/3 \le x \le 2/3$: Relative L_2 errors of the solution u and the condition numbers of the system matrix A by the ICS_4^{II} scheme.

Grid	Approach 1		Approach 2	
$(N+1)$	cond(A)	$N_e(u)$	cond(A)	$N_e(u)$
5	1.4e+5	1.0e-01	1.5e+3	1.0e-01
7	9.4e+5	2.3e-03	4.6e+3	2.3e-03
9	4.3e+6	1.2e-05	4.1e+4	1.2e-05
11	1.5e+7	2.4e-07	8.9e+4	2.4e-07
13	4.8e+7	4.9e-10	4.2e+5	4.9e-10
15	1.2e+8	1.4e-11	8.2e+5	1.4e-11
17	3.1e+8	2.3e-14	2.5e+6	6.2e-14
19	6.8e+8	4.9e-15	4.9e+6	3.9e-14
21	1.4e+9	2.0e-15	1.0e+7	2.2e-13

Table 6: 1D biharmonic problem, Dirichlet boundary conditions, $-2/3 \leq x \leq 2/3$: Relative L_2 errors of the solution u and the condition numbers of the system matrix A by the ICS_4^{III} scheme.

Grid	Approach 1		Approach 2	
$(N+1)$	cond(A)	$N_e(u)$	cond(A)	$N_e(u)$
5	1.9e+3	1.0e-01	2.3e+03	1.0e-01
7	1.6e+4	2.3e-03	2.2e+04	2.3e-03
9	1.6e+5	1.2e-05	1.4e+05	1.2e-05
11	1.0e+6	2.4e-07	1.5e+06	2.4e-07
13	4.9e+6	4.9e-10	1.5e+07	4.9e-10
15	1.7e+7	1.4e-11	1.8e+08	1.5e-11
17	5.3e+7	1.5e-14	1.7e+09	1.7e-12
19	2.3e+8	1.5e-14	2.0e+10	5.0e-12
21	1.5e+9	2.7e-15	2.0e+11	4.5e-12

Table 7: Biharmonic equation, single domain, Approach 2: Relative L_2 norms of the solution u at the interior points of the actual domain by the ICS^I and ICS^{II} schemes. Results concerning the matrix condition number are also included.

Grid	ICS^I		ICS^{II}	
	cond(A)	$N_e(u)$	cond(A)	$N_e(u)$
4×4	2.1e+2	3.0e-01	2.1e+2	3.0e-01
6×6	7.7e+3	1.4e-02	7.7e+3	1.4e-02
8×8	8.6e+4	1.5e-04	8.6e+4	1.5e-04
10×10	6.5e+5	6.0e-07	6.5e+5	6.0e-07
12×12	4.4e+6	8.7e-09	4.4e+6	8.7e-09
14×14	3.4e+7	7.5e-11	3.4e+7	7.5e-11

Table 8: Biharmonic equation, three subdomains: Relative L_2 norms of the solution u at the interior points of the three subdomains and of the whole domain by the ICS scheme.

Grid	$N_e^1(u)$	$N_e^2(u)$	$N_e^3(u)$	$N_e(u)$
5×5	1.4e-03	4.5e-03	1.4e-03	2.8e-03
7×7	6.3e-05	4.9e-05	1.5e-05	4.8e-05
9×9	2.1e-07	3.6e-07	1.3e-07	2.5e-07
11×11	9.6e-10	2.0e-09	2.5e-09	1.9e-09
13×13	4.5e-12	1.1e-11	9.4e-12	8.7e-12

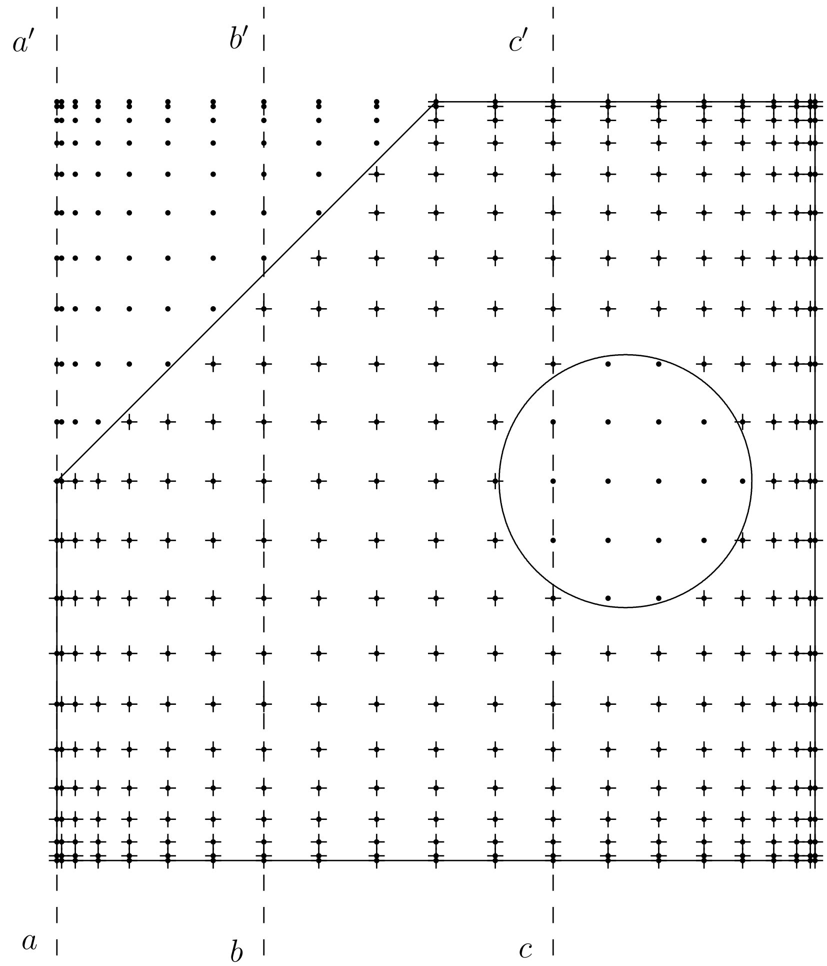

Figure 1: 2D Biharmonic equation: Irregular domain, extended domain and discretization. The mark + is used to denote the interior points of the actual domain Ω.

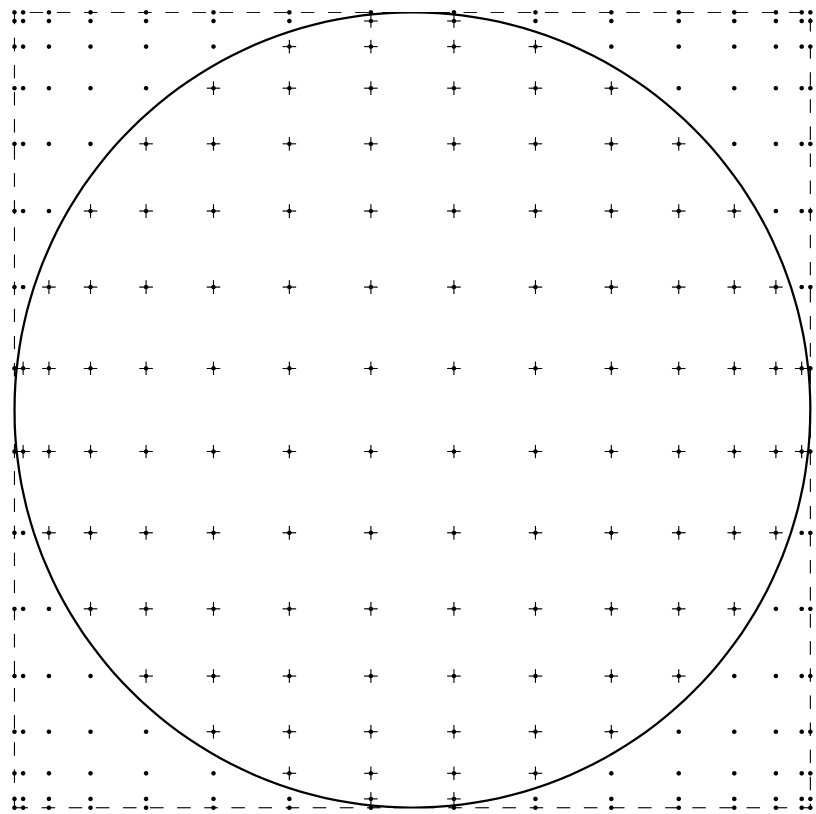

Figure 2: Biharmonic equation, single domain: Geometry and discretization. The mark + is used to denote the interior points of the actual domain Ω.

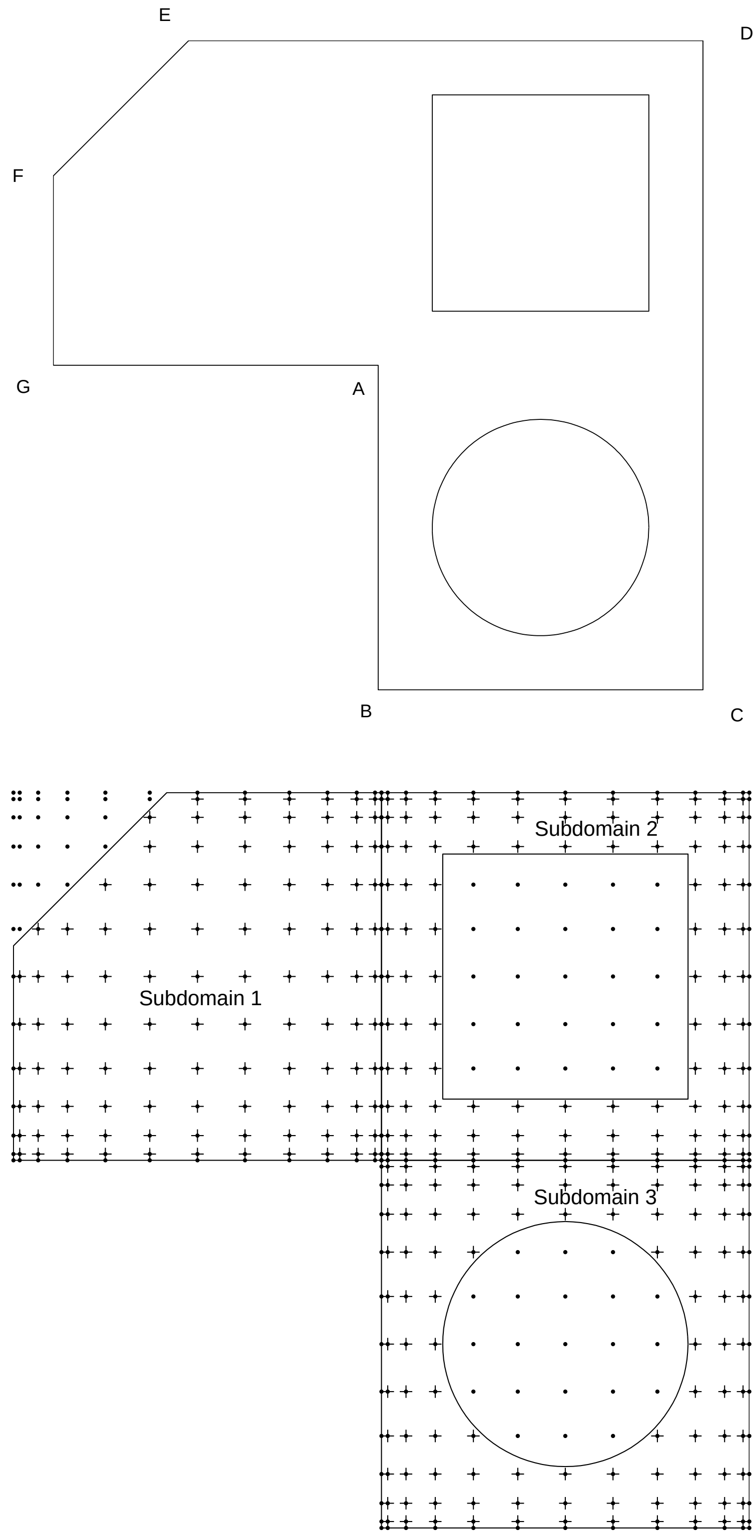

Figure 3: Biharmonic equation, three subdomains: Geometry, extended subdomains and discretization. The mark + is used to denote the interior points of the actual domain Ω.