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Proof that All Dissipation Rates are only Functions of Time for Transported Joint-Normal Distributions

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It has previously been proven that the conditional dissipation rate to transport a Gaussian distribution is equal to the mean dissipation rate throughout the variables' space and that only a Gaussian distribution can have a conditional dissipation rate that is only a function of time. This article extends both proofs to a joint-normal distribution for any number of dimensions.

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The Mapping Closure (MC)[1,2] and subsequently Multiple Mapping Conditioning (MMC)[3] rely on the fact that the conditional dissipation rate for a Gaussian probability density function (pdf) is equal to the mean dissipation rate and is not a function of the variable comprising the pdf. It was initially proven that if the conditional dissipation rate is modelled to be a constant, then a normal probability density function (pdf) preserves its shape and is always a normal pdf[4]. It was subsequently proven that if the pdf is Gaussian, then the conditional dissipation rate must be a function of time[5–7] and that only a Gaussian pdf can have a constant dissipation rate[5,6]. It has been assumed that the same behaviour can be extended to joint-normal joint-pdfs (jpdfs), which underpin general applications of MMC. A physical basis for this assumption is the argument[8] that the dissipation rate affects the small scales of turbulence, while the jpdf affects the large scales of turbulence, therefore these are uncorrelated. The benefit of this property is to simplify the modelling of the unknown conditional dissipation of the mapping variable in MC and MMC, making MMC an appealing approach. A Gaussian probability density function (pdf) and a joint-normal joint-pdf (jpdf) can be used to describe the marginal pdf and jpdf for the velocity components and scalar field in homogeneous shear flow with a uniform mean scalar gradient[9], while the velocity and scalar fields in the core of a mixing layer resemble a Gaussian pdf[10]. However, it is rare in practical applications for a field to resemble a joint-normal jpdf. Numerous models for the conditional dissipation have been devised for the Flamelet Model[11] and Conditional Moment Closure[12] to account for the relevant jpdf not being joint-normal. Because the conditioning ("reference") variable in MMC does not have to be a physical variable, it is possible to choose its distribution to be Gaussian. While most modern implementations of MMC only use a single conditioning variable[13–18]—for which the property of the pdf is proven—there are some implementations that use a multi-dimensional reference variable space[19,20]. In this article, the transport equation for a joint-normal jpdf is solved, thereby proving that the behaviour occurs for any number of dimensions.

Since the focus is on the modelling of the term involving the conditional dissipation rate, the passive variable ξ is considered. An important definition is the decay rate of the (co-)variance in homogeneous flow:

$$\frac{\partial \sigma^2}{\partial t} = -2\left\langle D\nabla\xi\cdot\nabla\xi\right\rangle \equiv -2\langle B\rangle \tag{1}$$

$$\frac{\partial \sigma_{ij}^2}{\partial t} = -2\sum_k \left\langle D_{ij}\frac{\partial \xi_i}{\partial x_k}\frac{\partial \xi_j}{\partial x_k}\right\rangle \equiv -2\langle B_{ij}\rangle\,, \tag{2}$$

where D_{ij} is the molecular diffusivity of variable ξ_i in variable ξ_j and $\langle\phi\rangle = \int \phi(\vec{\xi})P(\vec{\xi})d\vec{\xi}$ with $P(\vec{\xi})$ the jpdf of the variable space $\vec{\xi}$. The variable $\langle B_{ij}\rangle$ is commonly called the mean dissipation rate, and requires modelling in turbulent flows, with models developed from experimental measurements.

Initially, a single dimension for ξ is considered—to follow the proof for a Gaussian pdf[5–7]—using the homogeneous transport equation for its pdf[21]:

$$\frac{\partial P(\xi;\mu,\sigma)}{\partial t} = -\frac{\partial^2 B(\xi)P(\xi;\mu,\sigma)}{\partial \xi^2}\,. \tag{3}$$

If the pdf is modelled to have a Gaussian distribution

$$P(\xi;\mu,\sigma) \equiv \frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right)\,, \tag{4}$$

where ξ has a single dimension, then the following derivatives are useful for solving Eq. (3):

$$\frac{\partial P}{\partial t} = \frac{\partial P}{\partial \sigma^2}\frac{\partial \sigma^2}{\partial t} \tag{5}$$

$$\begin{aligned}\frac{\partial P}{\partial \sigma^2} &= -\frac{\pi}{(2\pi\sigma^2)^{(1/2)+1}}\exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right) + \frac{1}{(2\pi\sigma^2)^{1/2}}\frac{(\xi-\mu)^2}{2\sigma^4}\exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right)\\ &= \frac{(\xi-\mu)^2-\sigma^2}{2\sigma^4}P\end{aligned} \tag{6}$$

$$\frac{\partial P}{\partial \xi} = -\frac{\xi-\mu}{\sigma^2}\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left(-\frac{(\xi-\mu)^2}{2\sigma^2}\right) = -\frac{\xi-\mu}{\sigma^2}P \tag{7}$$

$$\frac{\partial^2 P}{\partial \xi^2} = -\frac{1}{\sigma^2}P + \frac{(\xi-\mu)^2}{\sigma^4}P = \frac{(\xi-\mu)^2-\sigma^2}{\sigma^4}P \tag{8}$$

$$\frac{\partial^2 BP}{\partial \xi^2} = \frac{\partial^2 B}{\partial \xi^2}P + 2\frac{\partial B}{\partial \xi}\frac{\partial P}{\partial \xi} + B\frac{\partial^2 P}{\partial \xi^2} = \left[\frac{\partial^2 B}{\partial \xi^2} - 2\frac{\xi-\mu}{\sigma^2}\frac{\partial B}{\partial \xi} + \frac{(\xi-\mu)^2-\sigma^2}{\sigma^4}B\right]P\,. \tag{9}$$

Substituting Eqs. (5)–(6) and (9) into Eq. (3) and defining $\xi' = \xi-\mu$ yields

$$\frac{\xi'^2-\sigma^2}{2\sigma^4}P\frac{\partial \sigma^2}{\partial t} = -\left[\frac{\partial^2 B}{\partial \xi^2} - 2\frac{\xi'}{\sigma^2}\frac{\partial B}{\partial \xi} + \frac{\xi'^2-\sigma^2}{\sigma^4}B\right]P$$

$$\frac{\partial^2 B}{\partial \xi^2} - 2\frac{\xi'}{\sigma^2}\frac{\partial B}{\partial \xi} + \left[\frac{\xi'^2}{\sigma^4} - \frac{1}{\sigma^2}\right]B = -\frac{1}{2}\left[\frac{\xi'^2}{\sigma^4} - \frac{1}{\sigma^2}\right]\frac{\partial \sigma^2}{\partial t}\,. \tag{10}$$

Equation (10) is a linear nonhomogeneous ordinary differential equation for B. The result $B(\xi) = \langle B \rangle$ is the particular solution. The homogeneous solution, following[5], is:

$$B(\xi) = (C_1 + C_2\xi)\exp\left(\tfrac{1}{2}\xi^2/\sigma^2\right) . \tag{11}$$

Applying the symmetry condition $\partial B/\partial\xi|_{\xi'=0} = 0$, it follows that $C_2 = 0$; to comply with $\int BPd\xi = \langle B \rangle$, it is necessary that $C_1 = 0$.

Therefore, it is proven that the only mathematically viable form of the conditional dissipation for a Gaussian distribution is the constant value of the mean dissipation.

The general solution for multiple passive scalars is now derived by considering the homogeneous transport equation for the joint probability density function (jpdf):

$$\frac{\partial P\left(\vec{\xi};\vec{\mu},\mathbf{\Sigma}\right)}{\partial t} = -\sum_i\sum_j \frac{\partial^2 B_{ij}\left(\vec{\xi}\right) P\left(\vec{\xi};\vec{\mu},\mathbf{\Sigma}\right)}{\partial\xi_i\partial\xi_j} , \tag{12}$$

where $\vec{\xi}$ is the vector containing the n dimensions of ξ_k, $\vec{\mu}$ is the vector of means and $\mathbf{\Sigma}$ is the covariance matrix with elements $\Sigma_{ij} = \sigma_{ij}^2$.

If the pdf is modelled to have a joint-normal distribution

$$P\left(\vec{\xi};\vec{\mu},\mathbf{\Sigma}\right) \equiv \frac{1}{(2\pi)^{n/2}|\mathbf{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\left[\vec{\xi}-\vec{\mu}\right]^T \mathbf{\Sigma}^{-1}\left[\vec{\xi}-\vec{\mu}\right]\right) , \tag{13}$$

where $|\mathbf{\Sigma}|$ is the determinant of $\mathbf{\Sigma}$, then the form of Eq. (12) is:

$$\sum_i\sum_j \frac{\partial P}{\partial\sigma_{ij}^2}\frac{\partial\sigma_{ij}^2}{\partial t} = -\sum_i\sum_j \frac{\partial^2 B_{ij}P}{\partial\xi_i\partial\xi_j} . \tag{14}$$

By definition[8], the conditional dissipation rate B_{ij} only directly affects the evolution of the covariance σ_{ij}^2, so, for the purposes of determining the form of B_{ij}, Eq. (14) can be solved without considering the summations. A useful definition is the fluctuations of each variable:

$$\xi_i' = \xi_i - \mu_i \to d\xi_i = d\xi_i' . \tag{15}$$

To solve Eq. (14) using Eq. (13) for any dimension, the general form of $\mathbf{\Sigma}^{-1}$ is required:

$$\mathbf{\Sigma}^{-1} \equiv \frac{1}{|\mathbf{\Sigma}|}\tilde{\mathbf{\Sigma}} , \tag{16}$$

where $\tilde{\mathbf{\Sigma}}$ is the adjugate matrix for $\mathbf{\Sigma}$. Let $\mathbf{M}_{ij}$ be a "minor" matrix of $\mathbf{\Sigma}$, with $\mathbf{M}_{ij}$ constructed by removing row i and column j from $\mathbf{\Sigma}$. The elements of the cofactor matrix $\mathbf{C}$ are therefore:

$$C_{ij} \equiv (-1)^{i+j}|\mathbf{M}_{ij}| \tag{17}$$

and the adjugate matrix is:

$$\tilde{\mathbf{\Sigma}} = \mathbf{C}^T \tag{18}$$

$$\tilde{\Sigma}_{ij} = (-1)^{i+j}|\mathbf{M}_{ji}|\,. \tag{19}$$

It follows that:

$$\left[\vec{\xi}-\vec{\mu}\right]^T \mathbf{\Sigma}^{-1}\left[\vec{\xi}-\vec{\mu}\right] = \frac{1}{|\mathbf{\Sigma}|}\sum_k\sum_l \tilde{\Sigma}_{kl}\xi'_k\xi'_l \tag{20}$$

$$\frac{\partial|\mathbf{\Sigma}|}{\partial\sigma_{ij}^2} = (-1)^{i+j}|\mathbf{M}_{ij}| \tag{21}$$

$$\frac{\partial|\mathbf{M}_{kl}|}{\partial\sigma_{ij}^2} = (-1)^{i+j+q_{ik,jl}}|\mathbf{M}_{ik,jl}| \tag{22}$$

$$|\mathbf{M}_{kl}| = \sum_m (-1)^{r+m+q_{kr,lm}}\Sigma_{rm}|\mathbf{M}_{kr,lm}| \tag{23}$$

$$q_{kr,lm} = \begin{cases} 0\,, & \mathrm{sign}(r-k) = \mathrm{sign}(m-l) \\ 1\,, & \mathrm{sign}(r-k) \neq \mathrm{sign}(m-l) \end{cases} \tag{24}$$

$$|\mathbf{M}_{ij,kl}| \equiv |\mathbf{M}_{ij,lk}| \equiv |\mathbf{M}_{ji,lk}| \equiv |\mathbf{M}_{ji,kl}|\,, \tag{25}$$

where the sub-minor matrix $\mathbf{M}_{ik,jl}$ is practically missing both rows i and k from $\mathbf{\Sigma}$ as well as both columns j and l, but by definition is missing row i and column j from $|\mathbf{M}_{kl}|$. If $k = i$ and/or $l = j$, then $|\mathbf{M}_{ik,jl}| = 0$. If $n = 2$, $k \neq i$, and $l \neq j$; then $|\mathbf{M}_{ik,jl}| = (-1)^{k+l}$; this property can be determined by direct substitution into Eq. (14) of the well-known formula for $\mathbf{\Sigma}^{-1}$ for rank 2 matrices—which takes the form of Eq. (16).

The general formulae for the necessary derivatives are:

$$\begin{aligned}\frac{\partial P}{\partial \sigma_{ij}^2} &= -\frac{1}{2|\mathbf{\Sigma}|}\frac{\partial|\mathbf{\Sigma}|}{\partial \sigma_{ij}^2}P - \frac{1}{2|\mathbf{\Sigma}|^2}\left[|\mathbf{\Sigma}|\sum_k\sum_l \frac{\partial\tilde{\Sigma}_{kl}}{\partial\sigma_{ij}^2}\xi_k'\xi_l' - \frac{\partial|\mathbf{\Sigma}|}{\partial\sigma_{ij}^2}\sum_k\sum_l \tilde{\Sigma}_{kl}\xi_k'\xi_l'\right]P \\ &= -\frac{1}{2|\mathbf{\Sigma}|^2}\left[|\mathbf{\Sigma}|\sum_k\sum_l(-1)^{k+l}\frac{\partial|\mathbf{M}_{lk}|}{\partial\sigma_{ij}^2}\xi_k'\xi_l' + \frac{\partial|\mathbf{\Sigma}|}{\partial\sigma_{ij}^2}\left(|\mathbf{\Sigma}| - \sum_k\sum_l\tilde{\Sigma}_{kl}\xi_k'\xi_l'\right)\right]P \\ &= -\frac{(-1)^{i+j}}{2|\mathbf{\Sigma}|^2}\left[|\mathbf{\Sigma}|\sum_k\sum_l(-1)^{k+l+q_{il,jk}}|\mathbf{M}_{il,jk}|\xi_k'\xi_l' + |\mathbf{M}_{ij}|\left(|\mathbf{\Sigma}| - \sum_k\sum_l\tilde{\Sigma}_{kl}\xi_k'\xi_l'\right)\right]P \\ &= -\frac{(-1)^{i+j}}{2|\mathbf{\Sigma}|^2}\left\{|\mathbf{\Sigma}||\mathbf{M}_{ij}| + \sum_k\sum_l(-1)^{k+l}\left[(-1)^{q_{il,jk}}|\mathbf{\Sigma}||\mathbf{M}_{il,jk}| - |\mathbf{M}_{ij}||\mathbf{M}_{lk}|\right]\xi_k'\xi_l'\right\}P\end{aligned} \tag{26}$$

$$\frac{\partial P}{\partial \xi_i} = -\frac{1}{2|\mathbf{\Sigma}|}\left[\sum_l\tilde{\Sigma}_{il}\xi_l' + \sum_k\tilde{\Sigma}_{ki}\xi_k'\right]P \tag{27}$$

$$\begin{aligned}\frac{\partial^2 P}{\partial\xi_i\partial\xi_j} &= -\frac{\tilde{\Sigma}_{ij}+\tilde{\Sigma}_{ji}}{2|\mathbf{\Sigma}|}P + \frac{1}{4|\mathbf{\Sigma}|^2}\left[\sum_l\tilde{\Sigma}_{il}\xi_l' + \sum_k\tilde{\Sigma}_{ki}\xi_k'\right]\left[\sum_l\tilde{\Sigma}_{jl}\xi_l' + \sum_k\tilde{\Sigma}_{kj}\xi_k'\right]P \\ &= \frac{1}{|\mathbf{\Sigma}|^2}\left\{\frac{1}{4}\left[\sum_l\tilde{\Sigma}_{il}\xi_l' + \sum_k\tilde{\Sigma}_{ki}\xi_k'\right]\left[\sum_l\tilde{\Sigma}_{jl}\xi_l' + \sum_k\tilde{\Sigma}_{kj}\xi_k'\right] - \frac{(-1)^{i+j}}{2}|\mathbf{\Sigma}|\left(|\mathbf{M}_{ji}| + |\mathbf{M}_{ij}|\right)\right\}P\end{aligned} \tag{28}$$

$$\frac{\partial^2 B_{ij}P}{\partial\xi_i\partial\xi_j} = \frac{\partial^2 B_{ij}}{\partial\xi_i\partial\xi_j}P + \frac{\partial B_{ij}}{\partial\xi_i}\frac{\partial P}{\partial\xi_j} + \frac{\partial B_{ij}}{\partial\xi_j}\frac{\partial P}{\partial\xi_i} + B_{ij}\frac{\partial^2 P}{\partial\xi_i\partial\xi_j}\,. \tag{29}$$

Substituting into Eq. (14) yields

$$\begin{aligned}&\frac{1}{2|\mathbf{\Sigma}|^2}\left\{(-1)^{i+j}\left[|\mathbf{\Sigma}||\mathbf{M}_{ij}| + \sum_k\sum_l(-1)^{k+l}\left[(-1)^{q_{il,jk}}|\mathbf{\Sigma}||\mathbf{M}_{il,jk}| - |\mathbf{M}_{ij}||\mathbf{M}_{lk}|\right]\xi_k'\xi_l'\right]\right\}\frac{\partial\sigma_{ij}^2}{\partial t} \\ &= \frac{\partial^2 B_{ij}}{\partial\xi_i\partial\xi_j} - \frac{1}{2|\mathbf{\Sigma}|}\left[\sum_l\tilde{\Sigma}_{jl}\xi_l' + \sum_k\tilde{\Sigma}_{kj}\xi_k'\right]\frac{\partial B_{ij}}{\partial\xi_i} - \frac{1}{2|\mathbf{\Sigma}|}\left[\sum_l\tilde{\Sigma}_{il}\xi_l' + \sum_k\tilde{\Sigma}_{ki}\xi_k'\right]\frac{\partial B_{ij}}{\partial\xi_j} \\ &\quad - \frac{1}{|\mathbf{\Sigma}|^2}\left\{\frac{(-1)^{i+j}}{2}|\mathbf{\Sigma}|\left(|\mathbf{M}_{ji}| + |\mathbf{M}_{ij}|\right) - \frac{1}{4}\left[\sum_l\tilde{\Sigma}_{il}\xi_l' + \sum_k\tilde{\Sigma}_{ki}\xi_k'\right]\left[\sum_l\tilde{\Sigma}_{jl}\xi_l' + \sum_k\tilde{\Sigma}_{kj}\xi_k'\right]\right\}B_{ij}\,.\end{aligned} \tag{30}$$

Equation (30) has the same form as Eq. (10). To obtain the particular solution, it will now be proven that the portion of the coefficients within $\{\cdot\}$ for the first and final terms in Eq. (30) are identical. Due to the symmetry of $\mathbf{\Sigma}$, $|\mathbf{M}_{ij}| = |\mathbf{M}_{ji}|$, which means that the terms not involving ξ' are identical. For the remaining terms, after applying the symmetry of $|\mathbf{\Sigma}|$, it is required to prove the following:

$$\begin{aligned}\sum_k \sum_l (-1)^{i+j+k+l} \left[|\mathbf{M}_{ij}||\mathbf{M}_{kl}| - (-1)^{q_{il,jk}} |\mathbf{\Sigma}||\mathbf{M}_{il,jk}|\right] \xi'_k \xi'_l &= \left[\sum_k \tilde{\Sigma}_{ki}\xi'_k\right]\left[\sum_l \tilde{\Sigma}_{jl}\xi'_l\right] \\ &= \sum_k \sum_l \tilde{\Sigma}_{ki}\tilde{\Sigma}_{jl}\xi'_k\xi'_l \\ &= \sum_k \sum_l (-1)^{i+j+k+l} |\mathbf{M}_{ik}||\mathbf{M}_{jl}|\xi'_k\xi'_l\end{aligned} \tag{31}$$

Compiling terms, what remains is to prove:

$$\forall_{k,l} |\mathbf{M}_{ik}||\mathbf{M}_{jl}| - |\mathbf{M}_{ij}||\mathbf{M}_{kl}| + (-1)^{q_{il,jk}} |\mathbf{\Sigma}||\mathbf{M}_{il,jk}| = 0\,. \tag{32}$$

All the matrices are expanded to sub-minor matrices as a common basis (in a two-step process because of the rank of $\mathbf{\Sigma}$ and to compile terms into a double-summation):

$$\begin{aligned}&\left(\sum_r \Sigma_{lr}(-1)^{l+r+q_{il,kr}}|\mathbf{M}_{il,kr}|\right)\left(\sum_m \Sigma_{im}(-1)^{i+m+q_{il,jm}}|\mathbf{M}_{il,jm}|\right) \\ &\quad - \left(\sum_r \Sigma_{lr}(-1)^{l+r+q_{il,jr}}|\mathbf{M}_{il,jr}|\right)\left(\sum_m \Sigma_{im}(-1)^{i+m+q_{il,km}}|\mathbf{M}_{il,km}|\right) \\ &\quad + (-1)^{q_{il,jk}} \sum_m (-1)^{i+m}\Sigma_{im}|\mathbf{M}_{im}||\mathbf{M}_{il,jk}| \\ &= \sum_m \sum_r \Sigma_{im}\Sigma_{lr}(-1)^{i+l+m+r}\left[(-1)^{q_{il,jm}+q_{il,kr}}|\mathbf{M}_{il,jm}||\mathbf{M}_{il,kr}|\right. \\ &\quad \left. -(-1)^{q_{il,jr}+q_{il,km}}|\mathbf{M}_{il,jr}||\mathbf{M}_{il,km}| + (-1)^{q_{il,jk}+q_{il,mr}}|\mathbf{M}_{il,mr}||\mathbf{M}_{il,jk}|\right]\,.\end{aligned} \tag{33}$$

Because by definition:

$$(-1)^{q_{ij,kl}+q_{ij,mr}}|\mathbf{M}_{ij,kl}||\mathbf{M}_{ij,mr}| - (-1)^{q_{ij,lr}+q_{ij,mk}}|\mathbf{M}_{ij,lr}||\mathbf{M}_{ij,mk}| = (-1)^{q_{ij,kr}+q_{ij,lm}}|\mathbf{M}_{ij,kr}||\mathbf{M}_{ij,lm}|\,, \tag{34}$$

all the terms in Eq. (33) cancel irrespective of the number of dimensions n, the value of ξ, and the choice of indices, so it is proven that the particular solution is $B_{ij}(\vec{\xi}) = \langle B_{ij}\rangle$. The homogeneous solution to Eq. (30) is:

$$B_{ij} = \left(C_1 + C_2\xi'_i + C_3\xi'_j\right)\exp\left(\frac{\sum_k\sum_l \tilde{\Sigma}_{kl}\xi'_k\xi'_l}{2|\mathbf{\Sigma}|}\right)\,. \tag{35}$$

Just like Eq. (11), the homogeneous solution must be zero. Therefore, every conditional (cross-)dissipation rate must be the mean (cross-)dissipation rate for joint-normal jpdfs of any dimension. Furthermore, because Eq. (12) yields the solution that the Fourier transform of a joint-normal jpdf is the initial value of the joint-normal jpdf's Fourier transform multiplied by the exponential in Eq. (35), the proof that only a Gaussian pdf can have a constant dissipation rate[5] can be directly used to prove that only a joint-normal jpdf can have a constant (cross-)dissipation rate.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.