

The inertial dynamics of thin film flow of non-Newtonian fluids

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Abstract

Consider the flow of a thin layer of non-Newtonian fluid over a solid surface. I model the case where the viscosity depends nonlinearly on the shear-rate; power law fluids are an important example, but the analysis here is for general nonlinear dependence. The modelling allows for large changes in film thickness provided the changes occur over a relatively large enough lateral length scale. Modifying the surface boundary condition for tangential stress forms an accessible foundation for the analysis where flow with constant shear is a neutral critical mode, in addition to a mode representing conservation of fluid. Perturbatively removing the modification then constructs a model for the coupled dynamics of the fluid depth and the lateral momentum. For example, the results model the dynamics of gravity currents of non-Newtonian fluids when the flow is not creeping.

Keywords: thin fluid flow; non-Newtonian fluid; inertia; power law rheology

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1 Introduction

Consider the two dimensional flow of a thin layer of fluid over a flat substrate. The fluid of thickness $\eta(x, t)$ spreads with mean lateral velocity $\bar{u}(x, t)$. Suppose the fluid has the non-Newtonian, power law, stress-strain relation that the stress \propto (strain-rate)^s for some fixed exponent s : the exponent $s = 1$ for a Newtonian fluid; $s < 1$ is shear thinning; and $s > 1$ is shear thickening. Such a power law is sometimes called Ostwald's or Norton's constitutive relation [5]. Then the systematic analysis developed in this article supports the nondimensional model of the flow

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} [\eta \bar{u}] = 0, \quad (1)$$

$$\begin{aligned} \text{Re} \left[\frac{\partial \bar{u}}{\partial t} + \frac{167 - 25/s}{96} \bar{u} \frac{\partial \bar{u}}{\partial x} + \frac{25 - 13/s}{96\eta} \bar{u}^2 \frac{\partial \eta}{\partial x} \right] \\ \approx - \frac{5(25 - 1/s)c_s}{48\sqrt{2}\eta} \left(\frac{\sqrt{2}\bar{u}}{\eta} \right)^s \\ + \frac{19 + 1/s}{24} \left(g_1 - g_2 \frac{\partial \eta}{\partial x} \right), \quad (2) \end{aligned}$$

where Re is the nondimensional Reynolds number, c_s is the coefficient of proportionality in the nonlinear stress-strain relation, and where g_1 and g_2 are the nondimensional components of gravity along and normal to the flat substrate, respectively. This model generalises the model of Newtonian fluids [17]. Fluid is conserved through (1). The momentum equation (2) incorporates effects of inertia, \bar{u}_t , self-advection, $\bar{u}\bar{u}_x$ and $\bar{u}^2\eta_x$, bed drag, $(\bar{u}/\eta)^s$, and gravitational forcing, $(g_1 - g_2\eta_x)$; the dependence of the coefficients upon s models the subtle effects of the power law rheology. For example, for flow down an inclined flat plate with lateral gravity g_1 , the nonlinear bed friction may balance gravitational forcing whence the above model predicts the equilibrium flow to have mean velocity

$$\bar{u} = \frac{\eta^{1+1/s}}{\sqrt{2}} \left[\frac{2\sqrt{2}(19 + 1/s)g_1}{5(25 - 1/s)c_s} \right]^{1/s}. \quad (3)$$

Just as for the special case of Newtonian fluids [17, §6.1], the model (1)–(2) also resolves instabilities from the equilibrium flow (3) and the emergence and interaction, or otherwise, of solitary waves on the falling fluid. Similarly, modulating gravity g_2 in time allows the above model to simulate Faraday waves as previously displayed for Newtonian fluids [17, §6.1]. Further, substituting the equilibrium mean velocity (3) into the fluid conservation equation (1), modelling the very slow dynamics at small Re , leads to an accurate lubrication model for nonlinear fluids, one previously approximated by others [10, 1, e.g.], and reducing to the classic lubrication model for Newtonian fluids when exponent $s = 1$.

The model (1)–(2) not only applies to the flow of simple liquids, it applies to: gravity currents of suspensions with medium to high volume fractions as these are non-Newtonian [18]; ice flow as power law rheologies are often used in models [9, 19, e.g.]—at even a few metres per year the Reynolds number is significant for a thick glacier; and a modified model would apply to turbulent flow as the Smagorinsky large eddy closure of turbulence corresponds to the shear thickening case of exponent $s = 2$ [8, Eqn. (6), e.g.]. This article puts models such as (1)–(2) within the sound support of modern dynamical systems theory, Section 3, to empower us to systematically control error, assess domains of validity, and to systematically account for further physical effects. For example, this analysis in the special case of Newtonian fluids is valid for free surface steepnesses η_x up to about one [17, Eqn. (62)].

The analysis here encompasses not only power law fluids but also a general nonlinear rheology with a general de-

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pendence of the stress upon the strain-rate as described in Section 2. In contrast, almost all previous thin fluid film modelling use only a power law dependence. Some industrial plastics have a complicated non-monotonic dependence that cannot be represented by a simple power law [3]. Similarly, dense suspensions often have non-monotonic dependence [18]. The model derived in Section 4 applies to such complicated industrial plastics and dense suspensions.

The lubrication approximation of very slow flow, negligible Reynolds number, underpins previous theoretical models for non-Newtonian thin fluid films: Perazzo & Gratton [10] and Betelu & Fontelos [1] examined flow with surface tension; this followed experiments comparing travelling waves and similarity solutions by Gratton, Minotti & Mahajan [5]. Gratton et al. comment “the differences between Newtonian and non-Newtonian currents are significant and can clearly be observed in experiments”. But the lubrication approximation, that creates models expressed only in terms of the fluid thickness $\eta(x, t)$, does not model inertial dynamics and so cannot resolve any wave-like dynamics. To model faster flows, potentially with wave effects, we must resolve the dynamics of both the fluid thickness and a measure of horizontal momentum [12, 17], we used η and $\bar{\mathbf{u}}$ in (1)–(2). For example, Harris et al. [6] modelled particle driven gravity currents using shallow water equations that resolve the dynamics of both the fluid thickness and the mean lateral velocity. However, such modelling of essentially dissipative flows, albeit dissipative via turbulence, by the laminar inviscid foundation of shallow water equations appears a contradiction that demands resolution. This article shows how models of non-Newtonian rheology fluid flow may be put on a sound mathematical basis to empower accurate physical forecasts.

2 Differential equations to model non-Newtonian flow

Let the incompressible fluid have thickness $\eta(x, t)$, constant density ρ , a nonlinear rheology, and let the fluid flow with some varying velocity field $\mathbf{u} = (\mathbf{u}, \mathbf{v}) = (\mathbf{u}_1, \mathbf{u}_2)$ and pressure field p . In this letter we restrict attention to two dimensional fluid flow.

Nonlinear constitutive relation Define the strain-rate tensor [5, 18]¹

$$\dot{\epsilon}_{ij} = \frac{1}{2} \left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right), \quad (4)$$

where $x_1 = x$ and $x_2 = y$ are distances along and normal to the solid substrate, respectively. Then the stress tensor for the fluid is $\sigma_{ij} = -p\delta_{ij} + 2\rho\nu\dot{\epsilon}_{ij}$: the kinematic viscosity ν is constant for a Newtonian fluid; but when the kinematic viscosity varies with strain-rate then we model shear thickening or shear thinning non-Newtonian fluids.

The important class of non-Newtonian fluids that we address has viscosity which depends only upon the magnitude $\dot{\epsilon}$

of the second invariant of the strain-rate tensor [1]:

$$\dot{\epsilon}^2 = \sum_{i,j} \dot{\epsilon}_{ij}^2. \quad (5)$$

For example, Bird et al. [2, see [1]] report that a solution of 0.5% Hydroxyethylcellulose is shear thinning: at 20 °C the solution has viscosity $\mu = m\dot{\epsilon}^{s-1}$ for exponent $s = 1/1.96$ and coefficient $m = 0.84 \text{ N s}^s / \text{m}^2$.

Partial differential equations Make variables nondimensional with respect to some velocity scale, a typical fluid thickness, and the fluid density. The nondimensional PDEs for the incompressible, two dimensional, fluid flow are firstly the continuity equation

$$\nabla \cdot \mathbf{u} = \frac{\partial \mathbf{u}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} = 0, \quad (6)$$

and secondly the momentum equation

$$\text{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \mathbf{g}, \quad (7)$$

where Re is the appropriate Reynolds number, $\boldsymbol{\tau}$ is the nondimensional deviatoric stress tensor, and $\mathbf{g} = (g_1, g_2)$ is the nondimensional forcing of gravity. For a fluid with a nonlinear stress-strain relation, the nondimensional deviatoric stress tensor

$$\tau_{ij} = 2\nu(\dot{\epsilon})\dot{\epsilon}_{ij} = \nu(\dot{\epsilon}) \left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i} \right). \quad (8)$$

Boundary conditions Solve these PDEs with nondimensional boundary conditions:

- on the bed of no-slip,²

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad y = 0; \quad (9)$$

- the kinematic condition on the free-surface of

$$\frac{\partial \eta}{\partial t} + \mathbf{u} \frac{\partial \eta}{\partial x} = \mathbf{v} \quad \text{on} \quad y = \eta; \quad (10)$$

- the stress normal to the free surface comes from constant environmental pressure and surface tension, that is,

$$\begin{aligned} -p + \frac{1}{1 + \eta_x^2} (\tau_{22} - 2\eta_x \tau_{12} + \eta_x^2 \tau_{11}) \\ = \frac{\text{We} \eta_{xx}}{(1 + \eta_x^2)^{3/2}} \quad \text{on} \quad y = \eta, \end{aligned} \quad (11)$$

where We is a nondimensional Weber number characterising the importance of surface tension;

²If modelling turbulent flows by a large eddy closure, we may justifiably replace this no-slip bed condition by a mixed boundary condition on the lateral velocity: $\mathbf{u} \propto \frac{\partial \mathbf{u}}{\partial y}$.

¹Some, such as Betelu & Fontelos [1], use double this tensor.

- and there must be no tangential stress at the free surface,

$$(1 - \eta_x^2)\tau_{12} + \eta_x(\tau_{22} - \tau_{11}) = 0 \quad \text{on } y = \eta. \quad (12)$$

This boundary condition of zero tangential stress implicitly is effectively one of zero shear at the surface; this zero shear would not be appropriate for material with a finite yield stress. Here we assume the fluid yields for arbitrarily small stress.

3 Centre manifold theory supports the modelling

This section describes one approach to placing models such as (1)–(2) on a sound theoretical base. Artificially modify the zero tangential stress free surface condition (12) to have an artificial forcing proportional to the local velocity, a forcing which we later remove by evaluating at parameter $\gamma = 1$:

$$\begin{aligned} & (1 - \frac{1}{6}\gamma) [(1 - \eta_x^2)\tau_{12} + \eta_x(\tau_{22} - \tau_{11})] \\ & = (1 - \gamma) \frac{\nu(E)}{\eta} \mathbf{u} \quad \text{on } y = \eta. \end{aligned} \quad (13)$$

Evaluated at $\gamma = 1$ this artificial right-hand side becomes zero so the boundary condition (13) reduces to the physical boundary condition of zero tangential stress (12). However, when the parameter $\gamma = 0$ and the lateral gravity and lateral derivatives negligible, $g_1 = \partial_x = 0$, a neutral mode of the dynamics is the lateral shear flow $\mathbf{u} = \sqrt{2}E\mathbf{y}$ where I define E to be proportional to the mean lateral strain-rate:

$$E = \frac{1}{\sqrt{2}\eta} \int_0^\eta \frac{\partial \mathbf{u}}{\partial y} dy = \frac{1}{\sqrt{2}\eta} \mathbf{u}|_{y=\eta}.$$

This neutral lateral shear mode arises because in pure shear flow $\tau_{12} = \nu \mathbf{u}_y$ and hence the artificial free surface condition (13) reduces to $\nu \mathbf{u}_y = \nu \mathbf{u}/\eta$ on $y = \eta$. Conservation of fluid provides a second neutral mode in the dynamics. That is, when $\gamma = g_1 = \partial_x = 0$ then a two parameter family of equilibria exists corresponding to some uniform lateral shear flow, $\mathbf{u} = E\mathbf{y}$, on a fluid of any constant thickness η . For large enough lateral length scales, these equilibria occur independently at each location x [11, 13, e.g.] and hence the space of equilibria are in effect parametrised by $E(x)$ and $\eta(x)$. Provided we can treat lateral derivatives ∂_x as a modifying influence, that is provided solutions vary slowly enough in x , centre manifold theorems [4, 7, 14, e.g.] assure us three vitally important properties:

1. this space of equilibria is perturbed to a slow manifold, on which the evolution is slow, that *exists for a finite range* of gradients ∂_x , and parameters γ and g_1 , and which may be parametrised by the mean lateral shear $E(x, t)$ and the local thickness of the fluid $\eta(x, t)$;
2. the slow manifold *attracts solutions from all nearby initial conditions*; and that

3. a formal power series in the parameters γ , g_1 and gradients ∂_x *approximates* the slow manifold to the same order of error as the order of the residuals of the governing differential equations.

That is, the theorems support the existence, accurate relevance and construction of slow manifold models such as (1)–(2).

An alternative and powerful view of these theorems is that they follow from a nonlinear, normal form, coordinate transform that decouples the slow and fast modes in the fluid dynamics [15, e.g.]. That is, the models we discuss are essentially just a reparametrisation of the state space, restricted to the slow dynamics.

4 Low order models of the dynamics

The detailed and complicated algebra deriving a model is of little interest to users of the model. Computer algebra readily constructs slow manifold models [16, §3]. Those interested should check the code and verify that the algorithm solves the governing differential equations and boundary conditions as specified [16, p.17–23]. The solution is valid for in small lateral derivatives, small lateral forcing and small perturbation of the free surface condition [Property 3]. Here we focus on the resulting model and its interpretation.

4.1 Power law fluids

For simplicity, suppose the rheology is a nondimensional power law for the kinematic viscosity, $\nu = c_s \dot{\epsilon}^{s-1}$.

Computer algebra [16, §3] derives that for such a power law fluid, the evolution of the fluid thickness η and the stress parameter E is

$$\begin{aligned} \frac{\partial \eta}{\partial t} &= -\frac{\partial}{\partial x} \left[\left(1 + \frac{5\sqrt{2}}{48s}\gamma \right) \frac{1}{2}\eta^2 E \right] \\ &\quad + \mathcal{O}(\partial_x^2 + g_1^2 + \gamma^3), \quad (14) \\ \text{Re } \frac{\partial E}{\partial t} &= -\frac{5}{2} \left(\gamma + \frac{1-1/s}{4}\gamma^2 \right) c_s \frac{E^s}{\eta^2} \\ &\quad - \text{Re } \sqrt{2} \left[\left(\frac{3}{8} + \frac{1-8/s}{96}\gamma \right) \eta E \frac{\partial E}{\partial x} - \frac{1}{6s}\gamma E^2 \frac{\partial \eta}{\partial x} \right] \\ &\quad + \sqrt{2} \left[\frac{3}{4} - \frac{1+1/s}{16}\gamma \right] \eta^{-1} \left(g_1 - g_2 \frac{\partial \eta}{\partial x} \right) \\ &\quad + \mathcal{O}(\partial_x^2 + g_1^2 + \gamma^3). \quad (15) \end{aligned}$$

The nonlinear rheology primarily appears in the first line of (15) as a nonlinear drag on the bed. However, the different power laws also change the vertical profiles of velocity and pressure; these changes affect the coefficients of the model (14)–(15) through their dependence upon exponent s .

In modelling the flow of thin fluid layers, we generally prefer to use the mean lateral velocity or the lateral fluid flux instead of the shear parameter E . Using the velocity fields computed at the same time as the evolution (14)–(15),

the computer algebra [16, §3] also derives the mean lateral velocity

$$\bar{\mathbf{u}} = \frac{1}{\eta} \int_0^\eta \mathbf{u} \, dy = \frac{1}{\sqrt{2}} \left(1 + \frac{5}{24s} \gamma + \frac{5(4-1/s)}{288s} \gamma^2 \right) \eta E + \frac{\eta^2 E^{1-s}}{sc_s} \left[\frac{Re}{160} E \frac{\partial E}{\partial x} + \frac{1}{48} \left(g_1 - g_2 \frac{\partial \eta}{\partial x} \right) \right] + \dots$$

Reverting this series to express E in terms of $\bar{\mathbf{u}}$, and substituting into the model (14)–(15) leads to a model for the coupled evolution of fluid thickness $\eta(x, t)$ and mean lateral velocity $\bar{\mathbf{u}}(x, t)$. Evaluating at the physically relevant $\gamma = 1$, to remove the artifice in the surface boundary condition (13), then gives the model (1)–(2) discussed in the Introduction of this article.

Computer algebra experiments [16, §1] suggest that the convergence of the asymptotic series in γ is markedly improved by the factor $(1 - \frac{1}{6}\gamma)$ on the left-hand side of the tangential stress boundary condition (13). This factor is equivalent to an Euler transformation of the asymptotic series. As shown in other similar applications [12, 17, e.g.], evaluation at $\gamma = 1$ is physically valid.

Computer algebra [16, §3] may construct terms in the formal power series solutions to higher order in the notionally small parameters γ , g_1 and ∂_x . Various truncations of the multivariate power series generate many valid approximations of varying orders of accuracy. For example, to resolve any effects of surface tension we need to compute terms in ∂_x^2 that are neglected in (2) and (15). With the support of centre manifold theory, researchers may choose an approximate model that suits the parameter regime of their application.

4.2 More general non-Newtonian fluids

We now return to the more general rheology where the viscosity ν of the fluid depends arbitrarily upon the magnitude of the shear-rate $\dot{\epsilon}$, instead of being a simple power law. In this more general rheology the expressions for the modelling are much more complicated. For conciseness define

$$\bar{E} = \frac{\sqrt{2}\bar{\mathbf{u}}}{\eta}, \quad \bar{\nu} = \nu(\bar{E}) \quad \text{and} \quad R_{\bar{\nu}} = \frac{1}{\bar{\nu} + \bar{E}\bar{\nu}'}, \quad (16)$$

where primes on $\bar{\nu}$ denote the derivatives d/dE of the viscosity $\nu(E)$ and evaluated at $E = \sqrt{2}\bar{\mathbf{u}}/\eta$.

Theory [§3] supports a model obtained through solving asymptotically the governing differential equations. The procedure is as for the power law rheology: computer algebra [16, §3] constructs the slow manifold and evolution thereon to some order of error; then revert the asymptotic series to find stress parameter E as a function of mean velocity $\bar{\mathbf{u}}$; and substitute to express the model in terms of η and $\bar{\mathbf{u}}$. Conservation of fluid again derives (1) (to any order of error). The momentum dynamics leads to

$$\begin{aligned} Re \frac{\partial \bar{\mathbf{u}}}{\partial t} &= - \left[\frac{5\gamma}{2} + \frac{5\gamma^2}{48} \bar{E}\bar{\nu}R_{\bar{\nu}}^2(2\bar{\nu}' + \bar{E}\bar{\nu}'') \right] \frac{\bar{\nu}\bar{\mathbf{u}}}{\eta^2} \\ &- Re \left[\frac{7}{4} - \frac{13\gamma}{48} + \frac{\gamma}{96} \bar{E}R_{\bar{\nu}}^2(38\bar{\nu}\bar{\nu}' + 12\bar{E}\bar{\nu}'^2 + 13\bar{E}\bar{\nu}\bar{\nu}'') \right] \bar{\mathbf{u}} \frac{\partial \bar{\mathbf{u}}}{\partial x} \\ &- Re \sqrt{2} \left[\frac{1}{8} - \frac{\gamma}{16} + \frac{13\gamma}{192} \bar{E}^2 R_{\bar{\nu}}^2(2\bar{\nu}'^2 - \bar{\nu}\bar{\nu}'') \right] \bar{E}\bar{\mathbf{u}} \frac{\partial \eta}{\partial x} \end{aligned}$$

$$\begin{aligned} &+ \left[\frac{3}{4} + \frac{\gamma}{12} - \frac{\gamma}{24} \bar{E}\bar{\nu}R_{\bar{\nu}}^2(2\bar{\nu}' + \bar{E}\bar{\nu}'') \right] \left(g_1 - g_2 \frac{\partial h}{\partial x} \right) \\ &+ \mathcal{O}(\partial_x^2 + g_1^2 + \gamma^3). \end{aligned} \quad (17)$$

As before, the terms on the right-hand side represent, respectively, bed drag through the nonlinear rheology, self advection of momentum, and forcing due to gravity and hydrostatic pressure. Evaluate this equation at $\gamma = 1$ to recover a physically relevant model of the dynamics of lateral momentum.

The power law model (2) is just one specific subclass of the general model (17): obtain (2) by the specific choice of a power law viscosity, $\nu(\dot{\epsilon}) = c_s \dot{\epsilon}^{s-1}$.

5 Conclusion

Following similar modelling for Newtonian thin films [12, 17], this innovation of modifying the free surface condition to (13) places the modelling of a physically important class of non-Newtonian fluids upon the powerful and sound basis of centre manifold theory [4, 7, 14, e.g.]. This modern dynamical system foundation empowers us to systematically derive the novel and accurate models (2), (15) and (17) for the lateral momentum of fluids with nonlinear rheology.

These models of thin fluid flow can be directly applied to flows as diverse as those of industrial plastics [3, e.g.], ice [9, 19, e.g.], and medium to dense suspensions [18, e.g.]. The models replace lubrication theory when inertia becomes important in the flow. When you desire more accuracy than that presented here, computer algebra readily computes higher order approximations [16, §3]. Modifying the no-slip boundary condition on the bed, (9), will empower the modelling of turbulent layers of flow over a substrate via large eddy closures. There are enormous applications for this approach to modelling the dynamics of relatively thin layers of fluids flowing over substrates.

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