Modelling the confinement of spilled oil with floating booms

SongPing Zhu and Dmitry Strunin

School of Mathematics and Applied Statistics,

University of Wollongong, Wollongong, NSW 2522, Australia

An effective mechanical method of confining the oil spills in an open ocean is to use barriers such as floating booms. However, the confined oil may leak beneath a boom if either the towing speed of the boom or the amount of the oil is too large. In this paper a simple mathematical model based on the potential theory is presented for the two-layer (oil & water) flow near a vertical barrier. A set of nonlinear integral equations are formulated and solved numerically. For the indirect approach we adopted to solve the nonlinear integral equations, the water velocity at the water-oil upstream contact point becomes a determining parameter of the final results. It is shown that the oil leakage under the barrier is impossible if the contact point is a stagnation one. For other non-stagnation cases, we were able to compute flows up to critical Froude numbers beyond which the oil will leak underneath the barrier.

1. Introduction.

As our society becomes more and more dependent on petroleum and petroleum products (e.g., gasoline, diesel, etc.), the demand of using supertankers to ship crude oils is undoubtedly going to keep increasing and consequently disasters of oil spills off the surface of open oceans or bays are unavoidable. There have been many oil spills from supertankers in the past ten years already. Any disaster of oil spill, particularly those with tons of spilled oil involved, is devastating to our environment and ecological system (e.g., hundreds and thousands sea birds died as a result of these accidents [1]). Examples of these type of accidents range from the most recent one taking place in Tokyo Bay with about 10,000 bbl of crude oils spilled on July 2, 1997 [2] to the most notorious Exxon Valdez tanker spill off Alaska's Prince William Sound nine years ago on May 24, 1989 [3].

As demanded by the devastating nature of oil spills, they have been studied quite intensively in the past decade. Majority of these studies are dedicated to modelling the spread of spilled oil on the surface of an open ocean, understanding the basic physico-chemical mechanisms of the spreading and developing hindcast models for some specific accidents. In the earlier seventies, Hoult [4] undertook a detailed study of the oil spreading mechanisms and outlined three stages of the spreading - inertial, viscous and surface-tension-driven. For each stage, a relevant theory was proposed and a comparison with some available experimental data were presented. Later on, further work was done in this direction, for example, by Sebastiao and Soares [5] who took into account evaporation, emulsification and change in viscosity and density of the oil. Empirical relations were extensively used, which gave good correlation with field observations for the problem with such a high degree of complexity. However, models heavily depending on empirical formulae generally tend to be less versatile. Reed *et al.* [6] and Reed and Gundlach [7] developed a numerical oil spill model with a number of those factors taken into account and tested it against data on the Amoco Cadiz event. The data from this accident were also used by Cuesta *et al.* [8] to test their numerical model. Based on a two-dimensional depth-averaged barotropic model, Noye *et al.* [9] presented a nice numerical model for the oil slick movement in the Northern Spencer Gulf of South Australia. Another big accident, the grounding of the tanker Braer, also stimulated considerable modelling efforts, although the results were not quite satisfactory. Turrell [10] and Proctor *et al.* [11] analysed these results and indicated possible ways of improvement of the models. Relatively, there are very few papers in the literature that address the confinement of spilled oil with floating booms, let alone the modelling of the flow near a boom.

A very practical approach of confining the oil spill on the surface of an ocean or river is to use mechanical barriers, such as floating booms. However, the confined oil may leak beneath the floating boom if either the towing speed of the boom or the amount of the confined oil or both exceeds certain critical values. The lack of quantitative understanding of the relationship between the submerged depth of the boom and these critical values requires some mathematical modelling of this practical engineering problem.

A single layer of fluid passing underneath a barrier is a classical fluid mechanics problem (frequently referred to as "flows under slice gates"), which can be modelled using the potential theory. Results for certain ranges of the Froude numbers and the length of the barrier are obtained by Gurevich [12], Benjamin [13], Budden and Norbury [14] and Vanden-Broeck and Keller [15]. Recently, Asavanant and Vanden-Broeck (AV) [16] managed to find the solutions for the full range of Froude numbers and barrier lengths.

The present work was motivated by the success of simple models used in flows under slice gates. The aim of the current research is to attempt some mathematical modelling for a fairly complicated problem in reality. The present paper is a first stage of the research aimed at formulating a simple mathematical model, which takes the previously published results as a special case for the purpose of validation of the model. By placing an amount of oil, which is assumed to be motionless as a start, we present here a two-dimensional two-layer flow model near a vertical barrier. Based on a few assumptions, we formulated our model as a system of nonlinear integral equations, which were solved numerically. The approach used to cast the system of integral equations is similar to that developed by Zhang and Zhu [17] for the flow above a bottom obstruction.

The remaining of the paper will be subdivided to several sections, with Section 2 describing the model, Section 3 explaining the numerical discretization, Section 4 showing the validation of the numerical solution and Section 5 presenting results of computations. Our concluding remarks are presented in the final section of the paper.

2. Model Formulation.

In order to capture certain aspects of the underlying physics for the problem of confining a given volume of oil with floating booms and to simplify our modelling effort, we began our modelling exercises with a simple two-dimensional model. This assumption is justifiable as the length of a boom is usually quite large and local curvatures are very small; the flow is predominantly two dimensional near the vertex of a floating boom towed by two tug boats.

As sketched in Fig. 1, we shall consider an oil-water model with the water flowing under a motionless patch of oil near a vertical barrier. For the convenience of formulation, the coordinate system is moving with the barrier at a constant velocity. Therefore, under the moving coordinate system, an equivalent uniform flow at far upstream with a horizontal velocity V_∞ flows towards the barrier, corresponding to a uniform towing speed of the boom when the ocean water is of no currents. The assumption of oil being motionless is reasonable when the towing speed (or upstream velocity hereafter) is small. Furthermore, for the same reason, as we expect that the deformation of the upstream free surface will be small if no waves are taken into consideration, a rigid lid has been placed above the water and oil surface on the upstream starting from the barrier. Far downstream, we also restrict ourselves to the cases where there are no waves generated and free surface asymptotically approaches a horizontal line.

With the further assumption that water is incompressible and flow is irrotational (with the tangential stress on the interface between the oil and water being ignored), the water flow is governed by the Laplace equation subject to appropriate kinematic and dynamic boundary conditions on the downstream free surface and zero-normal-velocity conditions on all the other rigid walls. On the interface between the water and oil, the continuity of pressure is demanded.

Such a differential system is highly non-linear because of the nonlinear dynamical boundary conditions (Bernoulli equation) on the interface and free surface and above all, the unknown positions of the free surface and interface themselves. Recently Zhang and Zhu [17] converted a similar differential system into an integral equation system and argued that their approach was quite efficient for the calculation of this kind of problems. Therefore, we shall use the same approach for the current problem.

The problem can be non-dimensionalized with a Froude number based on the upstream velocity V_∞ and upstream depth H as

$$F = V_\infty/\sqrt{gH}.$$

¿From now on, all the quantities are non-dimensional unless stated otherwise.

For the convenience of formulating our integral equation system, we can introduce a complex potential $f = \phi + i\psi$ with ϕ being a potential function and ψ being a stream function. The potential f is an analytical function of the complex variable $z = x + iy$ with x and y being the horizontal and vertical coordinates of the coordinate system shown in Fig.1. Since the total discharge can be non-dimensionalized to unity, the bottom and top boundaries are two streamlines with $\psi = -1$ and $\psi = 0$ respectively. The point of zero potential is chosen to be the lowest point E of the barrier (see Fig.1).

The conformal mapping

$$\zeta = e^{-\pi f} \tag{1}$$

transforms the region occupied by the water to the upper half of the ζ-plane as shown in Fig.2. The bottom and the free surface are respectively mapped onto the negative and positive real axes.

Upon introducing the logarithmic velocity variable $\Omega \equiv \delta + i\tau$ via

$$u - iv \equiv \frac{df(z(\zeta))}{dz} = \exp[-i\Omega(\zeta)], \tag{2}$$

we can show that δ represents the direction of the velocity and e^{τ} the magnitude of the velocity. Assuming $\Omega \to 0$ (physically, this corresponds to the case where the flow far upstream is parallel to the bottom and velocity is uniform) as $|\zeta| \to \infty$, we can apply the Cauchy's integral formula to the upper half of the ζ-plane:

$$\Omega(\zeta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Omega(\xi)}{\xi - \zeta}\, d\xi, \qquad \mathrm{Im}\zeta > 0, \tag{3}$$

where the integral is understood to be in the sense of the Cauchy principal value.

Since (1) maps the far-upstream area to the region of large $|\zeta|$, Eq. (3) thus implies uniformity far upstream. On the other hand, if the mapping

$$\zeta = e^{\pi f} \tag{4}$$

was adopted, Eq. (3) would imply uniformity far downstream. In that case, the semi-circle similar to that in Fig.2 would lie in the lower half of the ζ-plane and correspond to the downstream area. Except in the comparison of our model with that of AV [16] where the downstream velocity was chosen to be uniform, the mapping (1) is employed in our paper.

For real ζ, Eq. (3) becomes

$$\Omega(\zeta) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\Omega(\xi)}{\xi - \zeta}\, d\xi, \qquad \mathrm{Im}\zeta = 0, \tag{5}$$

which can be separated into real and imaginary parts as

$$\tau(\zeta) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\delta(\xi)}{\xi - \zeta}\, d\xi, \quad \delta(\zeta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tau(\xi)}{\xi - \zeta}\, d\xi. \tag{6}$$

On the interface and the free surface, the Bernoulli equation must be satisfied. After non-dimensionalizing based on g, V_∞, H and the density of the heavy fluid ρ_1 (the density of the light fluid is denoted as ρ_2), the Bernoulli equation assumes, in terms of the new variable ζ, the form

$$P(\zeta) + \frac{F^2}{2}\left(e^{2\tau(\zeta)} - 1\right) + y(\zeta) = 0, \tag{7}$$

where $P(\zeta)$ and $y(\zeta)$ are dimensionless pressure and interface elevation, which are defined respectively as $P = \bar{P}/(\rho_1 g H)$, $y = \bar{y}/H$, in terms of the corresponding dimensional variables.

With the assumption that the oil is motionless, the pressure in it must be hydrostatic. That is, it has no horizontal gradient and linearly varies vertically. The continuity of the pressure on the interface demands that

$$\bar{P}(\bar{y} + \Delta\bar{y}) = \bar{P}(\bar{y}) - \rho_2 g \Delta\bar{y},$$

or

$$\frac{\Delta\bar{P}}{\Delta\bar{y}} = -\rho_2 g,$$

which writes in non-dimensional form as

$$-\frac{dP}{dy} = \frac{\rho_2}{\rho_1} \equiv \alpha. \tag{8}$$

Now, differentiating (7) with respect to ζ yields

$$\frac{dP}{d\zeta} = -\frac{dy}{d\zeta} - F^2 e^{2\tau}\frac{d\tau}{d\zeta}. \tag{9}$$

The relation (8) can be rewritten as

$$\frac{dP}{d\zeta}\frac{d\zeta}{dy} = -\alpha. \tag{10}$$

Eliminating $dP/d\zeta$ from (9) and (10) leads to

$$F^2 e^{2\tau}\frac{d\tau}{d\zeta}\frac{d\zeta}{dy} = \alpha - 1, \qquad \zeta_A < \zeta < \zeta_B. \tag{11}$$

It is easy to show that from (2), we have, on the real axis of the ζ-plane,

$$\frac{dx}{d\zeta} = -\frac{e^{-\tau}}{\pi\zeta}\cos\delta, \quad \frac{dy}{d\zeta} = -\frac{e^{-\tau}}{\pi\zeta}\sin\delta, \tag{12}$$

which are purely kinematic conditions on the free surface and interface.

Taking into account that $dy/d\zeta$ is expressed by the second Eq. in (12), Eq. (11) can be integrated to represent τ in terms of δ on the interface:

$$\tau(\zeta) = \frac{1}{3}\ln\left(\exp(3\tau_B) - \frac{3(1-\alpha)}{\pi F^2}\int_{\zeta}^{\zeta_B}\frac{\sin\delta(\xi)}{\xi}\,d\xi\right), \qquad \zeta_A < \zeta < \zeta_B, \tag{13}$$

where τ_B is the value of τ at the point B.

For the free surface we have, instead of (11), the equation

$$F^2 e^{2\tau} \frac{d\tau}{d\zeta}\frac{d\zeta}{dy} = -1, \qquad 0 < \zeta < 1, \tag{14}$$

which is obtained from (11) by setting $\alpha = 0$. Integrating (14) yields

$$\tau(\zeta) = \frac{1}{3}\ln\left(\exp(3\tau_D) + \frac{3}{\pi F^2}\int_0^{\zeta}\frac{\sin\delta(\xi)}{\xi}\,d\xi\right), \qquad 0 < \zeta < 1, \tag{15}$$

where τ_D denotes the magnitude of τ far downstream.

On the other hand, the first equation in (6) can be written as

$$\tau(\zeta) = -\frac{1}{\pi}\left(\int_0^1 \frac{\delta(\xi)}{\xi-\zeta}\,d\xi - \frac{\pi}{2}\ln\left|\frac{\zeta_A-\zeta}{\zeta-1}\right| + \int_{\zeta_A}^{\zeta_B}\frac{\delta(\xi)}{\xi-\zeta}\,d\xi\right), \qquad \zeta > 0 \tag{16}$$

in view of $\delta \equiv -\pi/2$ for $1 < \zeta < \zeta_A$ (vertical barrier) and $\delta \equiv 0$ for $\zeta > \zeta_B$ (rigid lid) and for $\zeta < 0$ (bottom).

Eliminating $\tau(\zeta)$ from (16) and (13), we obtain an integral equation for the unknown function $\delta(\xi)$ in the range $\zeta_A < \zeta < \zeta_B$:

$$-\frac{1}{\pi}\left(\int_0^1 \frac{\delta(\xi)}{\xi-\zeta}\,d\xi - \frac{\pi}{2}\ln\left|\frac{\zeta_A-\zeta}{\zeta-1}\right| + \int_{\zeta_A}^{\zeta_B}\frac{\delta(\xi)}{\xi-\zeta}\,d\xi\right) = \frac{1}{3}\ln\left(\exp(3\tau_B) - \frac{3(1-\alpha)}{\pi F^2}\int_{\zeta}^{\zeta_B}\frac{\sin\delta(\xi)}{\xi}\,d\xi\right). \tag{17}$$

Similarly, eliminating $\tau(\zeta)$ from (16) and (15) gives us another integral equation for $\delta(\zeta)$ in the range $0 < \zeta < 1$:

$$-\frac{1}{\pi}\left(\int_0^1 \frac{\delta(\xi)}{\xi-\zeta}\,d\xi - \frac{\pi}{2}\ln\left|\frac{\zeta_A-\zeta}{\zeta-1}\right| + \int_{\zeta_A}^{\zeta_B}\frac{\delta(\xi)}{\xi-\zeta}\,d\xi\right) = \frac{1}{3}\ln\left(\exp(3\tau_D) + \frac{3}{\pi F^2}\int_0^{\zeta}\frac{\sin\delta(\xi)}{\xi}\,d\xi\right). \tag{18}$$

The amount of oil can also be expressed as an integral with respect to ξ:

$$q = \left|\int_{y_A}^{y_B} x(y)\,dy\right| = \left|\int_{\zeta_A}^{\zeta_B} x(\xi)\frac{\exp(-\tau(\xi))\sin\delta(\xi)}{\pi\xi}\,d\xi\right| \tag{19}$$

and the submerged depth of the barrier, l, is

$$l = \left|\int_1^{\zeta_B}\frac{\exp(-\tau(\xi))\sin\delta(\xi)}{\pi\xi}\,d\xi\right|. \tag{20}$$

Eqs. (12) and (17)-(20) form a closed integral equation system. With a given set of parameters α, F, q, l, these equations are to be solved to yield a complete information about the flow, in particular, the values of τ_B, τ_D, ζ_A, ζ_B, $\delta(\zeta)$ values for $\zeta_A < \zeta < \zeta_B$ and $0 < \zeta < 1$ and the positions of the free surface and interface. This is what we called a direct approach. Alternatively, one can employ a so-called indirect approach, in which α, F, τ_B, ζ_B are taken as the input parameters and the volume of oil q and submerged depth of the barrier l become a result of the calculation. Physically, the direct approach is more natural. However, computationally the indirect approach has a clear advantage, because for given τ_B and ζ_B, one only needs to solve a smaller set of integral Eqs. (17) and (18) with the limits being fixed. Then, the values of q and l can be determined using (19) and (20) respectively. For simplicity of computation, we decided to tackle the problem in this work with this indirect approach.

For the indirect approach, there are some special cases that need to be discussed and consequently treated separately. These special cases arise from the asymptotic behaviour of Eq. (17) near two contact points.

1) Contact point A.

One can show that from the Bernoulli equation and the condition of motionlessness of the oil it follows that the interface re-attaches the barrier tangentially. Indeed, the Bernoulli equation, in dimensional form, gives

$$\bar{P}_B/\rho_1 + V_B^2/2 + g\bar{y}_B = \bar{P}_A/\rho_1 + V_A^2/2 + g\bar{y}_A,$$

where V, $\bar{P}$ and $\bar{y}$ are velocity, pressure and coordinate at the corresponding points denoted by the subscripts, respectively. Therefore,

$$(V_A^2 - V_B^2)/2 = (\bar{P}_B - \bar{P}_A)/\rho_1 + g(\bar{y}_B - \bar{y}_A). \tag{21}$$

Recall that the pressure inside the oil is hydrostatic:

$$\bar{P}_A = \bar{P}_B + \rho_2 g(\bar{y}_B - \bar{y}_A). \tag{22}$$

Substituting (22) into (21) we obtain

$$(V_A^2 - V_B^2)/2 = g(\bar{y}_B - \bar{y}_A)(1 - \rho_2/\rho_1) > 0. \tag{23}$$

The equation (23) asserts that $V_A \neq 0$, that is the point A is not a stagnation point. Hence,

$$\delta_A = -\pi/2. \tag{24}$$

2) Contact point B.

Unlike the reattachment point A, there is no *a priori* information about whether the detaching point B is a stagnation point or not. Therefore, both cases should be considered. Although the case where point B is a stagnation point is only a special case in general, a special treatment is sensible as part of our indirect approach with τ_B (and hence, velocity at the point B) being a given parameter.

If the streamline is tangent to the rigid lid at the point B, then we simply set $\delta_B = 0$ and let τ_B be one of input parameters. On the other hand, if it is a stagnation point, the slope δ_B can be found from the following asymptotic analysis.

Let us estimate the integral in the formula (16) as $\zeta \to \zeta_B$. Place ζ a short distance ε from $\zeta = \zeta_B$: $\zeta + \varepsilon = \zeta_B$. We also introduce some small fixed positive value ε_1 so that $\varepsilon \ll \varepsilon_1 \ll \zeta_B - \zeta_A$. Within small interval $\zeta_B - \varepsilon_1 < \zeta < \zeta_B$ the function $\delta(\zeta)$ approximately equals to its boundary value δ_B. Let us estimate the last integral in (16) which makes primary contribution to τ:

$$\int_{\zeta_A}^{\zeta_B} \frac{\delta}{\xi - \zeta}\, d\xi = \int_{\zeta_A}^{\zeta-\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi + \int_{\zeta-\varepsilon}^{\zeta+\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi \approx \int_{A}^{\zeta-\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi + \delta_B \int_{\zeta-\varepsilon}^{\zeta+\varepsilon} \frac{d\zeta}{\xi - \zeta}\, d\xi$$

$$= \int_{\zeta_A}^{\zeta-\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi.$$

Further, taking into account the above inequality, we have

$$\int_{\zeta_A}^{\zeta-\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi = \int_{\zeta_A}^{\zeta_B-\varepsilon_1} \frac{\delta}{\xi - \zeta}\, d\xi + \int_{\zeta_B-\varepsilon_1}^{\zeta-\varepsilon} \frac{\delta}{\xi - \zeta}\, d\xi \approx \int_{\zeta_A}^{\zeta_B-\varepsilon_1} \frac{\delta}{\xi - \zeta}\, d\xi + \delta_B \ln(\zeta - \xi)\Big|_{\zeta_B-\varepsilon_1}^{\zeta-\varepsilon}$$

$$= \int_{\zeta_A}^{\zeta_B-\varepsilon_1} \frac{\delta}{\xi - \zeta}\, d\xi + \delta_B[\ln\varepsilon - \ln(\zeta - \zeta_B + \varepsilon_1)]$$

$$\approx \int_{\zeta_A}^{\zeta_B-\varepsilon_1} \frac{\delta}{\xi - \zeta}\, d\xi + \delta_B[\ln\varepsilon - \ln(\varepsilon_1 - \varepsilon)]. \approx \delta_B \ln\varepsilon. \tag{25}$$

since ε_1 is fixed and $\varepsilon \ll \varepsilon_1$. Thus, when $\varepsilon \to 0$, the expression (16) gives

$$\tau \to -\frac{\delta_B}{\pi} \ln\varepsilon. \tag{26}$$

On the other hand, as $\varepsilon \to 0$ ($\zeta \to \zeta_B$) the equation (13) gives

$$\tau \to \frac{1}{3} \ln\left(-\int_{\zeta}^{\zeta_B} \frac{\sin\delta}{\xi}\, d\xi \right) \to \frac{1}{3} \ln\left(-\frac{\sin\delta_B}{\zeta_B} \int_{\zeta}^{\zeta_B} d\xi \right) \to \frac{1}{3} \ln\varepsilon. \tag{27}$$

Comparing (26) and (27) we get

$$\delta_B = -\pi/3. \tag{28}$$

Remarkably, this slope does not depend on the density ratio and the Froude number.

At the lowest point E of the barrier the flow is supposed to separate smoothly, that is $\delta_E = -\pi/2$.

Even with the indirect approach, which substantially reduced the computational effort, we still had to resort to numerical approaches for the solution of these coupled nonlinear integral equations as described in the next section.

3. Numerical discretization and solution approach.

Following Neissner's arguments [18], we adopted a scheme with piecewise linear interpolation for the unknown functions and the collocation points placed at the midpoint of each element. The discretization of the integral equation system (17) and (18) leads to a system of nonlinear algebraic equations. Care must be taken in the selection of the nodal points where the unknown function nodal values are to be defined. As shown by Zhang and Zhu [17] the uniform discretization in ϕ allows one to avoid crowding the nodal points towards upstream. A satisfactory accuracy is achieved when a set of uniform grids for the potential function ϕ rather than ζ is chosen, although the latter is independent variable in our integral equation system. After a few test runs, we found that 180 nodal points placed on each of the three sections (free surface, interface and barrier) were enough to produce convergent results.

The choice of collocation points at the midpoint of each element resulted in an unbalanced number of algebraic equations to that of the unknowns. An additional equation was needed for each interval on which the function δ is defined. For the interval $\zeta_A < \zeta < \zeta_B$, $\delta_B = -\pi/3$ was used in the case where the contact point B is a stagnation point and $\delta_B = 0$ in the case where it is a tangential point. Similarly, an additional equation $\delta_E = -\pi/2$ was used for the interval $0 < \zeta < 1$.

The resulting nonlinear algebraic equation system was solved through by calling an IMSL subroutine NEQNF (a variation of Newton's method of the nonlinear iteration). With the initial guess being linear curves connecting the points $(\delta_D = 0, \zeta_D = 0)$, $(\delta_E = -\pi/2,\ \zeta_E = 1)$, and $(\delta_A = -\pi/2,\ \zeta = \zeta_A)$, $(\delta_B = 0,\ \zeta = \zeta_B)$, we found that the rate of convergence was overall satisfactory.

For each set of F, ζ_B and τ_B we solved the system of equations using various ζ_A and τ_D. The calculations run until we have come to such values of ζ_A and τ_D that the conditions $\delta_A = -\pi/2$ and $\delta_D = 0$ are satisfied.

4. Model validation.

Any numerical model should be properly tested. As there is no exact solution for this highly nonlinear problem, we relied upon some very special cases to base our judgement on the numerical accuracy and overall convergence of the code.

First, we designed a simpler nonlinear integral equation which on one hand is similar to our equations and on the other hand, bears an analytical solution. The integral equation

$$\int_a^b \frac{f(x)}{x-x_0}\,dx = c\left(\frac{2}{b-a}\right)^2\left[(x_0-a-b)(b-a)+\frac{b^2-a^2}{2}+\left(x_0-\frac{a+b}{2}\right)^2\log\frac{b-x_0}{x_0-A}\right],$$

where a, b, c are constants, has an exact solution

$$f(x) = c\left(\frac{2}{b-a}\right)^2\left(x-\frac{a+b}{2}\right)^2.$$

Numerical results were obtained when we used the same code that we used to solve our system. Fig.3 shows an overall good agreement between the numerical solution and the exact solution. While the symmetry of the solution is well preserved, the only slightly large discrepancy is near the two ends of the integral as expected.

Very fortunately, we found that there were some published results, which we could regard as a special case of our problem and consequently test our code. As mentioned in the Sec. 1, in the paper of AV [16] the flow of a single fluid near a barrier is studied. If we let the volume of oil approach zero in our problem, our results should match with theirs as the two problems become the same.

Special care must be taken when the comparison is carried out as AV imposed a uniform-far-downstream velocity condition, rather than the conventional approach of imposing a uniform velocity condition for upstream. Therefore, while we temporarily removed the oil, we also replaced the mapping (1) by the mapping (4), which ensures a uniform far-downstream as stated in Sec. 2. For the indirect approach we chose, we also need to determine the appropriate value of ζ_B. This can be done using the following formula from the paper [16]:

$$\phi_B + i = -\frac{1}{\pi}\ln\frac{(1-b)^2}{4b} = -\frac{1}{\pi}\ln\frac{(1-b)^2}{4|b|}+i,$$

from which we obviously get

$$\phi_B = -\frac{1}{\pi}\ln\frac{(1-b)^2}{4|b|}.$$

Taking (4) into consideration, we can work out that $b=-0.5$ corresponds to

$$\zeta_B = \exp(\pi\phi_B) = 4|b|/(1-b)^2 = 0.8889.$$

Fig.4 exhibits the comparison of the free-surface profiles of our model (with $F^2 = 2.5$ and $\zeta_B = 0.8889$) and that of AV's model [16] (with $F^2 = 2.5$ and $b = -0.5$). Clearly, the agreement between the two is quite good.

Another aspect of a numerical model is its grid convergence. Fig.5 shows various solutions as grid is refined by consecutive doubling the total number of grids on the free surface and interface from 10 to 320 points. There is no doubt that as the number of the nodal points increases the curves asymptotically approach a limiting profile. Through these tests, we were also convinced that some intermediate number of grid points would suffice in the final calculation of our system of nonlinear equations. Thus, we used 180 grid points for each part of the water surface (oil-water interface, free surface and barrier) for all the results presented in the next section.

5. Numerical results and discussion.

For simplicity, the value of α has been fixed at 0.85 throughout this paper. This is a typical value for the oil-water density ratio. With this in mind, hereafter we shall no longer mention the value of α used in our calculation.

Since the case where B is a stagnation point is simpler, we shall present our results starting from this case. For the indirect approach, we only have three given parameters α, F, ζ_B in order to determine all characteristics of the flow including the unknown function $\delta(\zeta)$, the shape of free surface and interface, the values l and q.

A typical flow profile obtained numerically is shown in Fig.6. The slope of the oil-water interface varies monotonically from $-\pi/3$ at the detachment point B to $-\pi/2$ at the reattachment point A on the barrier. The cross-section shape of the trapped oil volume is virtually triangular. Similarly to the work by AV we plotted in Fig.7 the family of curves showing the submerged barrier depth l versus F for different values of ζ_B. The right-hand ends of the curves correspond to very small oil volumes, which, together with other geometric characteristics of the flow, is part of the solution in the indirect approach. Because of the smallness of q the employed numerical scheme loses accuracy at these ends and we were no longer able to produce reliable solutions. Fortunately, these parts of the solutions are of little interest to us anyway, as small q limit has very little practical significance.

On the other hand, there exist lower bounds of Froude numbers, F_m, below which the nonlinear iterations exhibited no convergence and consequently no solutions were found. Generally, these minimum Froude numbers tend to increase as l increases. Physically, existence of the lower bounds of Froude numbers suggests is that if the Froude number is too small, there is no solution with a free surface extending from the tip of the barrier downstream with an exiting angle tangent to the barrier. Some qualitative laboratory observations [19] seemed to support this argument; when Froude number is too small, the downstream free surface starts from the other side of the barrier and gradually increases its elevation, approaching the horizontal level far downstream as sketched in Fig.8.

Table 1 displays geometric parameters of the flow (see definitions in Fig.9) as functions of F and ζ_B. For each given F, the larger ζ_B the larger horizontal and vertical dimensions of the oil patch as well as its volume. The smallest achievable values of F take place at large ζ_B. While at large Froude numbers the slope of the interface is almost monotonic, in the range of small Froude numbers a bending point appears on the interface making it S-shaped (Fig.10). For example, in the numerical experiments with $\zeta_B = 1000$, $F = 0.21$, the slope at the bending point is $\delta \approx 0.45$, which is about half of the slope near the rigid lid.

For the case of B being a stagnation point we never observed the critical situation, i.e. the situation where oil approaches the lower tip of the barrier. Thus it is impossible for the oil to leak in the regime when B is a stagnation point.

If B is not a stagnation point, we need to specify all four parameters α, F, ζ_B, τ_B as the input for our indirect approach. A series of plots corresponding to fixed F and ζ_B but different values τ_B is shown in Fig.11. This series exhibits the downward movement of the reattachment point A as the total amount of oil increases from $3.30 \cdot 10^{-3}$ (in Fig. 11a) to $3.35 \cdot 10^{-2}$ (in Fig. 11d). Fig.11d shows the case where the oil is about to leak under the barrier. A further increase of oil volume results in no convergent solution with the restriction that the reattachment point A must remain on the barrier, indicating that the leak has taken place. Consequently, we can interpret the Froude number in the critical case when A touches the tip of the barrier as the critical Froude number, beyond which the leak underneath the barrier takes place. For this particular case, the model predicted critical Froude number being $F_{cr} = 2$ when the submerged barrier depth $l = 0.24$ and oil volume $q = 3.35 \cdot 10^{-2}$. Alternatively, one can perceive that if we had used a direct approach, we would have been able to hold the volume of the trapped oil and the submerged depth of the barrier as constants and increase the Froude number gradually until it reaches the critical value calculated here through our indirect approach.

Another qualitative and encouraging conclusion that can be drawn from this example is that the model seems to have predicted a reasonable critical Froude number ($F_{cr} = 2.0$, see Fig.11d) at a reasonable ratio of the barrier depth to the far-upstream depth of the flow (1:4 for Fig.11d). This point shows that even such an idealized model is capable of catching certain important aspects of the flow.

We already mentioned that the drawback of the indirect approach we adopted is that it is very difficult to obtain data corresponding to a fixed value of the submerged barrier depth or the amount of oil trapped. On the other hand, such an approach is not only numerically efficient but can indeed be practically useful as, theoretically, one can always tabulate all the solutions in terms of τ_B and ζ_B. Currently we are working on an improved model in which the direct approach is adopted.

As required by the nature of our indirect approach, we separated the cases based on the two types of flow velocity at the detachment point B. Ultimately, one should expect that the case with B being a stagnation point should be a special case when τ_B decreases from a finite negative value to $\tau_B = -\infty$. This limit corresponds to the velocity at the point B, which is equal to $\exp(\tau_B)$, going to zero, so that the point B is becoming asymptotically closer to the stagnation point. So it would be interesting to observe the behaviour of the solution as τ_B approaches $-\infty$. This was indeed verified. Shown in Fig.12 is a series of solutions with decreasing τ_B values. One can see that at $\tau_B = -10$, the solution already well coincides with the solution we obtained by setting τ_B to $-\infty$. The interesting part is that for any finite τ_B, no matter how small it is, the slope at B should always be zero. Then, it appears that the slope jumped to $-\pi/3$ when $\tau_B \to -\infty$. Indeed, in Fig.12 the two interfaces corresponding to the case when setting B as a stagnation point and when $\tau_B = -10$ are only different within a small vicinity of the contact point B, which can be hardly visualized on the figure.

We conclude this section with the remark that the proposed model is only a first step towards the design of an adequate model of the oil spill trapped by floating booms. As no friction is taken into account in this model, the oil volume is assumed to be pressed to the barrier by hydrodynamic pressure only. In the future, we plan to construct a more realistic model with friction taken into account as the next step of our modelling effort. By comparing the results on the models with and without friction it would be interesting to judge the role of pressure and friction acting on the oil patch.

6. Conclusion.

We considered an idealized friction-free model for the oil confined by the floating boom. Two situations are considered: one characterized by a stagnation contact point B

and the other by a tangential contact point B. For the first situation, it is shown that the angle formed by the oil-water interface with the rigid lid is 60 degrees regardless of Froude numbers and densities of the fluids. Flow profiles are calculated for a broad ranges of the Froude number and the potential-related values ζ_B. There are no leaks beneath the barrier when B is a stagnation point. For the second situation, the leakage beneath the barrier is possible as demonstrated by a sequence of flow profiles. For the critical case where oil is about to leak, reasonable proportions between the barrier depth and upstream depth of the flow were observed. In the future studies, we plan to focus on these critical cases to determine the critical Froude number as a function of the submerged depth of the barrier and oil volume.

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Figure captions

Fig.1. Scheme of the flow.

Fig.2. The ζ-plane.

Fig.3. Comparison with the analytic solution of the model equation.

Fig.4. Comparison between our solution (1) and the one of AV [16] (2).

Fig.5. Convergence of the solution. The curves 1-6 correspond respectively to 10,20, 40,80,160,320 nodes on the free surface.

Fig.6. The computed flow profile for (a) $\zeta_B = 10.$, $F = 0.5$.

Fig.7. Submerged depth of the barrier as a function of Froude number.

Table 1. Depth of the barrier (l), length (X) and thickness (Y) of the patch, and fall-down of the free surface (L) depending on F and ζ_B.

Fig.8. Sketch of the flow for very small Froude number.

Fig.9. Definition of geometric characteristics.

Fig.10. The profiles at minimal Froude numbers F_m. (a) $\zeta_B = 10.$, $F_m = 0.33$, (b) $\zeta_B = 100.$, $F_m = 0.23$, (c) $\zeta_B = 1000.$, $F_m = 0.21$.

Fig.11. Series of profiles for $F = 2.0$, $\zeta_B = 4.0$ showing approach to the critical situation: (a) $\tau_B = -2.0$, (b) $\tau_B = -1.0$, (c) $\tau_B = -0.6$, (d) $\tau_B = -0.3$.

Fig.12. As $\tau_B \to -\infty$, the solution with the tangential point B goes over to the solution with the stagnation point B: (1) $\tau_B = -0.5$, (2) $\tau_B = -1.0$, (3) $\tau_B = -2.0$, (4) $\tau_B = -10.0$, (5) $\tau_B = -\infty$.