Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
DOI : 10.47974/JIM-1566

Finding exact solutions for selected nonlinear evolution differential equations

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Abstract

Burgers, Breaking Soliton and Boussinesq equations are applicable in different areas of physics. Searching their real solutions are particularly important. In this research the modified Kudryashov method is applied for finding exact travelling wave solutions of the Burgers, Boussinesq and Breaking soliton equation. To visualize the dynamics, we present the solutions in the form of graphs in 2D and 3D plots. The solutions summarize that, the Kudryashov modified method is effectual for solving nonlinear evolution differential equations in exact form.

Subject Classification: *Primary 35Qxx, Secondary 35Cxx.*

Keywords: *Kudryashov method, Nonlinear partial differential equations.*

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1. Introduction

Nonlinear partial differential equations and their solutions often simulate real world physical problems. Even when not directly applicable such solutions can highlight general properties and characteristics of the equations. In the last years, a number of new exact methods for solving nonlinear PDEs have been developed, like the sine-cosine technique [1, 2], the tanh technique [3], the homotopy analysis method [4]. The (G′/G) expansion method discussed by Wang et al. [5] is largely utilized to find the real solutions of nonlinear evolution equations, as well as the simplest equation method by Kudryashov [6], the modified Kudryashov technique [7–9] and the generalized Kudryashov method [10, 11]. The He's Exp-technique [12] was successfully applied to nonlinear evolution equations [13, 14]. The modified Kudryashov method [8] can more effectually obtain solitary wave solutions for higher-order nonlinear equations than further existing techniques.

In this article, we find exact solutions of the (1+1)-D Burgers equation, (2+1)-D Burgers equation, and (1+1)-D and (2+1)-D Boussinesq equation and soliton equation by using the modified Kudryashov technique. The Boussinesq equation describes, for example, the behavior of the waves in shallow water and the sound waves of iron in plasma. The Burgers equation applies to fluid mechanics, shock waves, acoustic transmission, traffic flow, and the Breaking Soliton equation to the Riemann wave model. The traveling wave solutions are of specific focus to us in this paper. Following the introduction in Section 1, we discussed the modified Kudryashov method in Section 2 and solved the concerned equations using the method in Section 3. The exact solutions are illustrated graphically in 2D and 3D. Finally, we present the conclusions in Section 4.

2. Kudryashov Method

The steps involved in the Kudryashov method for solving partial differential equations are discussed in [8, 9]. We have a general PDE (partial differential equation),

$$L_1(\theta, \theta_t, \theta_x, \theta_{tt}, \theta_{xx}, \dots) = 0. \tag{1}$$

Where the wave solutions are

$$\theta(x,t) = \theta(\gamma), \gamma = \alpha x + \beta t,$$

The PDE L_1 is transformed into an ordinary differential equation (ODE)

$$L_2\left(\theta, \beta\theta_\gamma, \alpha\theta_\gamma, \beta^2\theta_{\gamma\gamma}, \alpha^2\theta_{\gamma\gamma}, \dots\right) = 0. \tag{2}$$

To find the control terms, we put

$$\theta(\gamma) = \gamma^{-p}, p > 0$$

into all monomials of Eq.(2). By using the homogeneous balance method, we compare two, or it can be more terms, by choosing the smallest exponents in Eq.(2) to find the maximum value of p (denoted by N) termed as a pole of the equation. Now, we want the solution in the form

$$\theta(\gamma) = a_0 + a_1 Q^1 + a_2 Q^2 + \cdots + a_n Q^n, \tag{3}$$

where a_0, a_1, a_2, ... are constants and

$Q(\gamma) = \frac{1}{(1+e^{\gamma})}$. (4) Where Q(γ) is the solution of the ODE

$$Q_{\gamma} = Q^2 - Q. \tag{5}$$

We find the required derivatives $\theta_{\gamma}, \theta_{\gamma\gamma}, \theta_{\gamma\gamma\gamma}, \ldots$ of the function $\theta(\gamma)$ by using Eq.(3), (4) and (5). By differentiating the function $\theta(\gamma)$, we get

$$\theta_{\gamma} = \sum_{n=0}^{N} a_n n Q^n (Q-1),$$

$$\theta_{\gamma\gamma} = \sum_{n=0}^{N} a_n n Q^n (Q-1)[(n+1)Q - n], \tag{6}$$

and so on.

3. Applications of the Kudryashov method

3.1 *The (1+1) D Boussinesq equation*

This equation is modeled for long water waves and has applications in hydrodynamics and explained in Figure 1.

$$\theta_{tt} - \theta_{xx} - \frac{1}{2}(\theta^2)_{xx} - q\theta_{xxxx} = 0. \tag{7}$$

By using $\theta(x, t) = \theta(\gamma)$ and $\gamma = \alpha x + \beta t$, we can reduce Eq.(7) into a nonlinear ODE as follows,

$$(\beta^2 - \alpha^2)\theta_{\gamma\gamma} - \frac{1}{2}\alpha^2(\theta^2)_{\gamma\gamma} - q\alpha^4\theta_{\gamma\gamma\gamma\gamma} = 0. \tag{8}$$

Here, the pole of Eq.(8) is equal to $N = 2$ so,

$$\theta(\gamma) = a_0 + a_1 Q^1 + a_2 Q^2,$$

$$\theta_{\gamma}(\gamma) = Q(Q-1)(a_1 + 2a_2 Q),$$

$$\theta_{\gamma\gamma}(\gamma) = Q(Q-1)(6a_2 Q^2 + 2a_1 Q - 4a_2 Q - a_1). \tag{9}$$

The following system of equations is obtained by substituting the results from Eq.(9) in Eq. (8).

Q^6: $-120a_2\alpha^4 q - 10{a_2}^2\alpha^2 = 0$

Q^5: $-24a_1\alpha^4 q + 336a_2\alpha^4 q - 12a_1a_2\alpha^2 + 18{a_2}^2\alpha^2 = 0$

Q^4: $60a_1\alpha^4 q - 330a_2\alpha^4 q - 6a_0a_2\alpha^2 - 3{a_1}^2\alpha^2 + 21a_1a_2\alpha^2 - 8{a_2}^2\alpha^2 - 6a_2\alpha^2 + 6a_2\beta^2 = 0$

Q^3: $-50a_1\alpha^4 q + 130a_2\alpha^4 q - 2a_0a_1\alpha^2 + 10a_0a_2\alpha^2 + 5{a_1}^2\alpha^2 - 9a_1a_2\alpha^2 - 2a_1\alpha^2 + 2a_1\beta^2 + 10a_2\alpha^2 - 10a_2\beta^2 = 0$

Q^2: $15a_1\alpha^4 q - 16a_2\alpha^4 q + 3a_0a_1\alpha^2 - 4a_0a_2\alpha^2 - 2{a_1}^2\alpha^2 + 3a_1\alpha^2 - 3a_1\beta^2 - 4a_2\alpha^2 + a_2\beta^2 = 0$

Q: $-a_1\alpha^4 q - a_0a_1\alpha^2 - a_1\alpha^2 + a_1\beta^2 = 0$

Solving the above system of equations, we get the following values,

$$a_0 = -\frac{q\alpha^4 + \alpha^2 - \beta^2}{\alpha^2}, a_1 = 12\alpha^2 q, a_2 = -12\alpha^2 q, \beta = \beta$$

The exact solution is, $\theta(\gamma) = a_0 + a_1 Q^1 + a_2 Q^2$

$$\Theta(\gamma) = -\frac{q\alpha^4 + \alpha^2 - \beta^2}{\alpha^2} + 12\alpha^2 q \frac{1}{(1+e^{\gamma})} - 12\alpha^2 q \frac{1}{(1+e^{\gamma})^2}$$

$$\Theta(\mathrm{x,t}) = -\frac{q\alpha^4 + \alpha^2 - \beta^2}{\alpha^2} + 12\alpha^2 q \frac{1}{(1+e^{\alpha x+\beta t})} - 12\alpha^2 q \frac{1}{(1+e^{\alpha x+\beta t})^2}$$

3.2 *The (2+1)-D Boussinesq equation*

Consider the Boussinesq-type equation discussed in [15]

$$\theta_{tt} - \theta_{xx} - \theta_{yy} - (\theta^2)_{xx} - \theta_{xxxx} = 0 \tag{10}$$

Looking for the wave solution, $\theta(x,y,t) = \theta(\gamma), \gamma = \alpha x + \beta y + ct$ we reduce Eq. (10) into the nonlinear ODE as explained in Figure 2.

$$(c^2 - \alpha^2 - \beta^2)\theta_{\gamma\gamma} - \alpha^2(\theta^2)_{\gamma\gamma} - \alpha^4\theta_{\gamma\gamma\gamma\gamma} = 0. \tag{11}$$

By integrating Eq.(11) twice we get

$$(c^2 - \alpha^2 - \beta^2)\theta - \alpha^2\theta^2 - \alpha^4\theta_{\gamma\gamma} = C_1\gamma + C_2 \tag{12}$$

We choose constants C_1, C_2 equal to zero, Eq.(12) becomes,

$$(c^2 - \alpha^2 - \beta^2)\theta - \alpha^2\theta^2 - \alpha^4\theta_{\gamma\gamma} = 0. \tag{13}$$

The pole of Eq.(13) is $N = 2$ so

$$\theta(\gamma) = a_0 + a_1 Q^1 + a_2 Q^2$$

$$\theta_\gamma(\gamma) = Q(Q-1)(a_1 + 2a_2 \mathrm{Q})$$

$$\theta_{\gamma\gamma}(\gamma) = Q(Q-1)(6a_2 Q^2 + 2a_1 \mathrm{Q} - 4a_2 \mathrm{Q} - a_1) \tag{14}$$

By substituting Eq. (14) in Eq. (13), we have the system of equations

$$(-\alpha^2 {a_2}^2 - 6\alpha^4 a_2)Q^4 = 0, (-2a_1 a_2\alpha^2 - 2a_1\alpha^4 + 10a_2)Q^3 = 0,$$
$$[a_2(c^2 - \alpha^2 - \beta^2) - \alpha^2 {a_1}^2 - 2a_0 a_2\alpha^2 - 4a_2\alpha^4 + 3a_1\alpha^4]Q^2 =$$
$$0, [a_1(c^2 - \alpha^2 - \beta^2) - 2a_0 a_1\alpha^2 - a_1\alpha^4]Q = 0, [a_0(c^2 - \alpha^2 - \beta^2) -$$
$${a_0}^2\alpha^2] = 0.$$

Solving the system we find, Set 1: $a_0 = -\alpha^2,\ a_1 = 6\alpha^2,\ a_2 = -6\alpha^2, \beta = \sqrt{(\alpha^4 + c^2 - \alpha^2)}$ Set 2: $a_0 = 0,\ a_1 = 6\alpha^2,\ a_2 = -6\alpha^2, \beta = \sqrt{(-\alpha^4 + c^2 - \alpha^2)}$ An exact solution for Set 1 is $\theta(\gamma) = a_0 + a_1 Q^1 + a_2 Q^2$

that is

$$\Theta(\mathrm{x,y,t}) = -\alpha^2 + 6\alpha^2 \frac{1}{\{1 + e^{[\alpha x + (\sqrt{-\alpha^4 + c^2 - \alpha^2})y + ct]}\}} - 6\alpha^2 \frac{1}{\{1 + e^{[\alpha x + (\sqrt{-\alpha^4 + c^2 - \alpha^2})y + ct]}\}^2}$$

An exact solution for Set 2 is,

$$\theta(\gamma) = a_0 + a_1 Q + a_2 Q^2,$$

that is

$$\Theta(\mathrm{x,y,t}) = 6\alpha^2 \frac{1}{\{1 + e^{[\alpha x + (\sqrt{-\alpha^4 + c^2 - \alpha^2})y + ct]}\}} - 6\alpha^2 \frac{1}{\left\{1 + e^{[\alpha x + (\sqrt{-\alpha^4 + c^2 - \alpha^2})y + ct]}\right\}^2}.$$

The Set 1 is shown in Fig.2. We reviewed all the acquired solutions by re put them in the original equation using MAPLE 2017.

3.3 *The (2 + 1) D Breaking soliton equation*

$$\theta_{xxxy} - 2\theta_y\theta_{xx} - 4\theta_x\theta_{xy} + \theta_{xt} = 0 \tag{15}$$

By using $\theta(x,y,t) = \theta(\gamma)$ and $\gamma = \alpha x + \beta y - ct$, we can reduce Eq.(15) into nonlinear ODE as follows. Figure 3, explain the graphic solution of equation 15.

$$\beta\alpha^3\theta_{\gamma\gamma\gamma\gamma} - 6\beta\alpha^2\theta_\gamma\theta_{\gamma\gamma} - c\alpha\theta_{\gamma\gamma} = 0 \tag{16}$$

By integrating Eq.(16) w.r.t γ neglecting the constant generated due to integration, we have (16)

$$\beta\alpha^3\theta_{\gamma\gamma\gamma} - 3\beta\alpha^2(\theta_\gamma)^2 - c\alpha\theta_\gamma = 0 \tag{17}$$

Here, for Eq. (16) $N = 1$ so,

$$\theta = a_0 + a_1 Q$$
$$\theta_\gamma = -a_1 Q + a_1 \mathrm{Q}^2$$
$$\theta_{\gamma\gamma} = a_1 Q - 3a_1 \mathrm{Q}^2 + 2a_1 \mathrm{Q}^3,$$
$$\theta_{\gamma\gamma\gamma} = -a_1 Q + 7a_1 \mathrm{Q}^2 - 12a_1 \mathrm{Q}^3 + 6a_1 \mathrm{Q}^4. \tag{18}$$

by putting Eq.(18) in Eq. (17) we have the following system of equations

$$(-3\beta\alpha^2 {a_1}^2 + 6\beta\alpha^3 a_1)\mathrm{Q}^4 = 0$$
$$(6\beta\alpha^2 a^2{}_1 - 12\alpha^3\beta a_1)Q^3 = 0$$
$$(-3\beta\alpha^2 a^2{}_1 + 7\alpha^3\beta a_1 - \alpha c a_1)Q^2 = 0$$
$$(-\alpha^3\beta a_1 + \alpha c a_1)Q = 0$$

By solving the system of equations above, we obtain, $a_1 = 2\alpha$, $\omega = \omega$, $c = \beta\alpha^2$

Exact solution is, $\theta(\gamma) = a_0 + a_1 Q^1$

$$\Theta(\gamma) = a_0 + 2\alpha \frac{1}{(1+e^\gamma)}$$

$$\Theta(\mathrm{x,y,t}) = a_0 + 2\alpha \frac{1}{(1+e^{\alpha x + \beta y - \beta\alpha^2 t})} \tag{19}$$

3.4 *The (1+1)-D Burgers equation*

Here, we use the Kudryashov method to get exact solutions of the (1+1)-D Burgers equation,

$$\theta_t + \delta\theta\theta_x - \xi\theta_{xx} = 0, \tag{20}$$

where δ and ξ are parameters. The equation describes for example, shock waves at large values of Reynolds numbers as explained in Figure 4. Note that interesting exact solutions were found relatively recently in [16].

Looking for the travelling wave $\theta(x,t) = \theta(\gamma)$ and $\gamma = \alpha x + t,$ we reduce Eq. (20) into a nonlinear ODE,

$$\theta_\gamma + \delta\alpha\theta\theta_\gamma - \xi\alpha^2\theta_{\gamma\gamma} = 0,. \quad (21)$$

The pole of Eq.(21) is $N = 1$ so,

$$\theta = a_0 + a_1 Q,$$
$$\theta_\gamma = -a_1 Q + a_1 Q^2$$
$$\theta_{\gamma\gamma} = a_1 Q - 3a_1 Q^2 + 2a_1 Q^3, \quad (22)$$

Inserting (22) into Eq.(21) we get,

$$(\delta\alpha a^2{}_1 - 2a_1\xi\alpha^2)Q^3 + (a_1 + a_0a_1\delta\alpha - a^2{}_1\delta\alpha + 3a_1\xi\alpha^2)Q^2 + (-a_1 - a_0a_1\delta\alpha - a_1\xi\alpha^2)Q = 0. \quad (23)$$

By equating the coefficients to 0, we get the equation's system,

$$\delta\alpha a^2{}_1 - 2a_1\xi\alpha^2 = 0,$$
$$a_1 + a_0a_1\delta\alpha - a^2{}_1\delta\alpha + 3a_1\xi\alpha^2 = 0,$$
$$- a_1 - a_0a_1\delta\alpha - a_1\xi\alpha^2 = 0. \quad (24)$$

We solve the system usingMaple to get

$$a_0 = -\frac{\xi k^2+1}{\delta k}, a_1 = \frac{2\xi k}{\delta}$$

By putting the values of z, a_0, a_1 into

$$\Theta(\gamma) = a_0 + a_1 \frac{1}{(1+e^\gamma)}, \quad (25)$$

we get

$$\Theta(x,t) = -\frac{\xi k^2+1}{\delta k} + \frac{2\xi k}{\delta}\frac{1}{(1+e^{kx+t})} \quad (26)$$

3.5 *The (2+1)-D Burgers equation*

Here, we obtain exact travelling wave solutions of 2-D Burgers equation. We consider (2+1)- D Burgers equation as,

$$\theta_t + \theta\theta_x - \theta_{xx} - \theta_{yy} = 0 \quad (27)$$

By using $\theta(x,y,t) = \theta(\gamma)$ and $\gamma = \alpha x + y - ct,$ we can reduce Eq. (27) into a nonlinear ODE as follows. Figure 5, explain the 3D graphic solution of equation 27.

$$c\theta_\gamma + \alpha\theta\theta_\gamma + (\alpha^2 + 1)\theta_{\gamma\gamma} = 0. \quad (28)$$

Here, the pole of Eq. (28) is $N = 1$, after substituting v and its derivatives in Eq. (28) we have the following system of equations

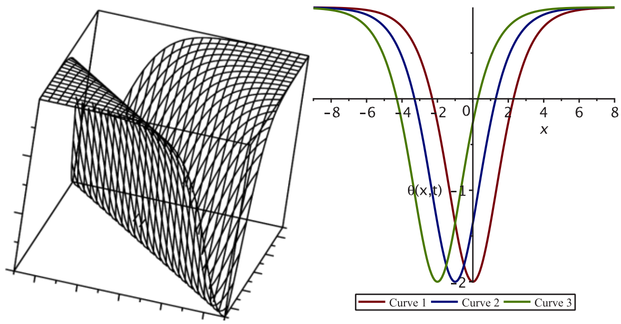
$$\alpha a^2{}_1 + 2a_1(\alpha^2 + 1) = 0$$
$$ca_1 + \alpha a_0a_1 - \alpha a^2{}_1 - 3(\alpha^2+1)a_1 = 0$$
$$-ca_1 + (\alpha^2+1)a_1 - \alpha a_0a_1 = 0$$
$$a_0 = \frac{\alpha^2 - c + 1}{\alpha}, a_1 = -\frac{2\alpha^2+2}{\alpha}$$

By putting the values of γ, a_0, a_1 into

$$\Theta(\gamma) = a_0 + a_1\frac{1}{(1+e^\gamma)} \quad (30)$$

We get

$$\Theta(x, y, t) = \frac{\alpha^2 - c + 1}{\alpha} - \frac{2\alpha^2 + 2}{\alpha} \frac{1}{(1 + e^{\alpha x + y - ct})} \quad (31)$$



(a) **(b)**

Figure 1

(a) 3D Graphical solution of Eq.(7), represents (1 + 1) D Boussinesq equation for $\alpha = 2, \beta = 1, q = 1$ $-4 \leq x \leq 4$ and $-4 \leq t \leq 4$, (b) 2D plot of θ versus x at different times, solitons are moving from right to left.

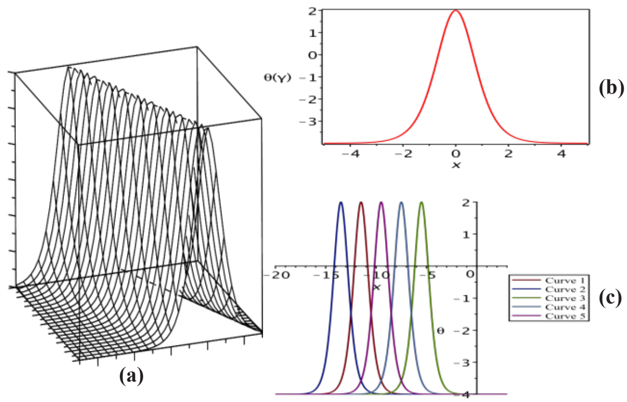
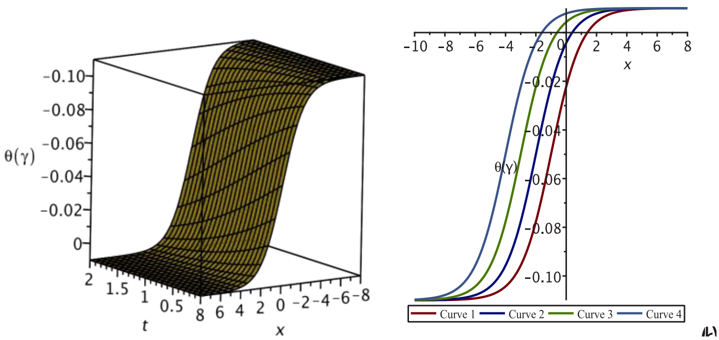


Figure 2

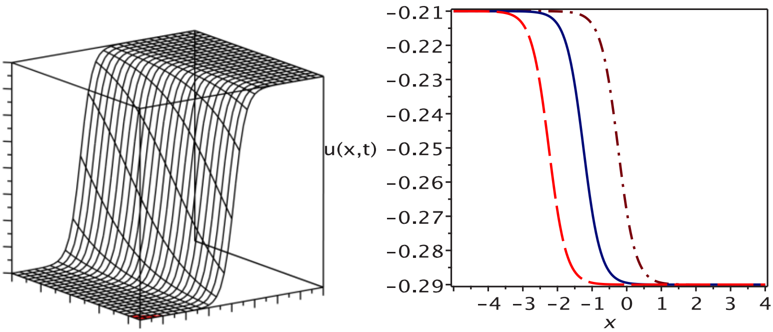
Bell shaped soliton of Boussinesq equation, (a)3D Graphical solution of Eq.(10), represents (2+1)-D Boussinesq equation for $\alpha = 2, \beta = 1$ $-4 \leq x \leq 4$ and $-4 \leq y \leq 4$, (b) 2D plot of u versus x at time t = 0 and (c) Moving solitons from right to left direction at different times when y=0.



(a) **(b)**

Figure 3

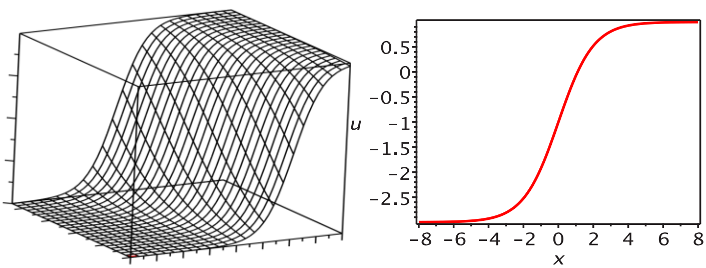
(a) 3D Graphical solution of Eq.(15), represents (2 + 1) D Breaking Soliton equation for $\omega = 1, k = 1, -8 \leq x \leq 4$ and $0 \leq y \leq 2$, (b) 2D plot of u versus x at different times, solitons are moving from right to left.



(a) **(b)**

Figure 4

Kinked shaped soliton of Burgers equation,(a) 3D Graphical solution of Eq.(20), represents (1+1)-D Burgers equation for α = 1,β 0.01,k= 2,−4 ≤ x ≤ 4 and −4 ≤ t ≤ 4, (b) 2D plot of u versus x at time t = 1, t = 5 and t = 9, this plot shows the travelling wave motion from right direction to the left with time.



(a) **(b)**

Figure 5

Kinked shaped soliton of Burgers equation (a) 3D Graphical solution of Eq.(27), represents (2+1)-D Burgers equation for k = 1,c = 1,−8 ≤ x ≤ 8 and −4 ≤ y ≤ 4 at t = 0. (b) 2D plot of u versus x at time t = 0.

4. Conclusions

We proved adequacy of the modified Kudryashov method for getting exact solutions of some nonlinear evolution PDEs. We obtained solitary wave solutions for the Burgers equation, Boussinesq equation and Breaking soliton equation. In future work we plan to use other versions of the Kudryshov method to solve more complicated nonlinear differential equations.

5. Acknowledgements

The second author would like to thank the Ministry of higher education and scientific research (Iraqi Government) and University of Thi-Qar; also thanks the University of Southern Queensland for accepting her in one-year sabbatical leave.

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