Proc. ICCS-12, Doha, Qatar
December 19-22, 2012, Vol. 23, pp. 519-528

SHRINKAGE ESTIMATION OF ELLIPTICAL REGRESSION MODEL UNDER BALANCED LOSS FUNCTION

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ABSTRACT

For the multiple regression model with error vector following elliptically contoured distribution we propose Bayesian shrinkage estimators under balanced loss function. Comparing a set of competing estimators for the regression vector, it is shown that the shrinkage factor of the Stein estimator is robust with respect to the regression parameters and unknown density generator of elliptical models. The dominance relation of the estimators is also provided.

Key Words: Bayes estimator; Elliptically contoured distribution; Preliminary test estimator; Stein-type shrinkage estimator; Positive-rule shrinkage estimator.
2010 Mathematics Subject Classification: 62J05; 62J07; 62F10.

1. Introduction

Consider the following multiple regression model

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{1.1}$$

where $\boldsymbol{y}$ is an n-vector of responses, $\boldsymbol{X}$ is an $n \times p$ non-stochastic design matrix with full rank p, $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_p)'$ is p-vector of regression coefficients and $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_n)'$ is the n-vector of random noises distributed as any member of the elliptically contoured distributions (ECDs), $\mathcal{E}_n(\mathbf{0}, \sigma^2\mathbf{V}, g_n)$ for some un-structured known matrix $\mathbf{V} \in S(n)$, where $S(n)$ denotes the set of all positive definite matrices of order $(n \times n)$. The density of $\boldsymbol{\epsilon}$ is given by

$$f(\boldsymbol{\epsilon}) = d_n|\sigma^2\mathbf{V}|^{-\frac{1}{2}} g_n[2\sigma^2]^{-1}\boldsymbol{\epsilon}'\mathbf{V}^{-1}\boldsymbol{\epsilon}, \tag{1.2}$$

where $d_n^{-1} = \pi^{\frac{p}{2}}[\Gamma\left(\frac{p}{2}\right)]^{-1} \int_{\mathbb{R}^+} y^{\frac{p}{2}-1} g_n(y)\mathrm{d}y$ and for some density generator function $g_n(.)$. The existence of the density generator $g_n(x)$ is dependent on the condition (Fang et al., 1990) $\int_0^\infty x^{\frac{n}{2}-1} g_n(x)\mathrm{d}x < \infty$. If $g_n(.)$ does not depend on n, we use the notation g instead. In this paper, we consider the estimation problem under the following loss function

$$\begin{aligned} L_{\omega,\boldsymbol{\delta}_0}^{\boldsymbol{W}}(\boldsymbol{\delta};\boldsymbol{\beta}) &= \omega r\left(\|\boldsymbol{\beta}\|^2\right)(\boldsymbol{\delta} - \boldsymbol{\delta}_0)'\boldsymbol{W}(\boldsymbol{\delta} - \boldsymbol{\delta}_0) \\ &\quad +(1-\omega) r\left(\|\boldsymbol{\beta}\|^2\right)(\boldsymbol{\delta} - \boldsymbol{\beta})'\boldsymbol{W}(\boldsymbol{\delta} - \boldsymbol{\beta}), \end{aligned} \tag{1.3}$$

where $\omega \in [0, 1]$, $r(.)$ is a positive weight function, $\boldsymbol{W}$ is a weight matrix, and $\boldsymbol{\delta}_0$ is a target estimator. This loss is pioneered by Jozani et al. (2006) inspiring by Zellner's (1994) balanced loss function. This loss function takes both goodness of fit and error of estimation into account. The $\omega r\left(\|\boldsymbol{\beta}\|^2\right)(\boldsymbol{\delta}-\boldsymbol{\delta}_0)'(\boldsymbol{\delta}-\boldsymbol{\delta}_0)$ part of the loss is analogous to a penalty term for lack of smoothness in nonparametric regression. The weight ω in (1.3) calibrates the relative importance of these two criteria. Dey et al. (1999) also considered issues of admissibility and dominance, under the loss (1.3) ignoring the term $r(.)$ when $\boldsymbol{W}=\boldsymbol{I}_p$. For the case $\omega=0$, we will simply write $L_0^{\boldsymbol{W}}(\boldsymbol{\delta};\boldsymbol{\beta})$ as the quadratic loss function. Of course, duty of the weight function $r(.)$ is clearly apparent in the Bayesian viewpoint. In this paper, we take it into consideration for the sake of generality. As it can be seen later, the structure of $r(.)$ does not alter the whole superiority conclusions.

This paper aims at the estimation of the regression parameter vector, $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_p)'$ when it is suspected that $\boldsymbol{\beta}$ may belong to any sub-space defined by $\boldsymbol{H}\boldsymbol{\beta}=\boldsymbol{h}$ where $\boldsymbol{H}$ is a $q\times p$ matrix of constants and $\boldsymbol{h}$ is a q-vector of known constants with focus on the Stein-type shrinkage estimator of $\boldsymbol{\beta}$ in addition to preliminary test estimator (PTE).

Saleh (2006) presents an overview on the topic under normal and nonparametric theory covering many standard models. Other relevant works in the area include Arashi (2012), Khan (2008, 2000), Arashi et al. (2008), Hoque et al. (2009), and Khan and Saleh (1997).

2. Preliminaries for Bayesian estimation

It is easy to show that the unrestricted estimator (UE) of $\boldsymbol{\beta}$ and σ^2 are

$$\tilde{\boldsymbol{\beta}}=(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{V}^{-1}\boldsymbol{y}=\boldsymbol{C}^{-1}\mathbf{X}'\boldsymbol{V}^{-1}\boldsymbol{y}, \qquad \boldsymbol{C}=\mathbf{X}'\mathbf{V}^{-1}\mathbf{X}. \tag{2.4}$$

$$\tilde{\sigma}^2=n^{-1}(\boldsymbol{y}-\mathbf{X}\tilde{\boldsymbol{\beta}})'\mathbf{V}^{-1}(\boldsymbol{y}-\mathbf{X}\tilde{\boldsymbol{\beta}}), \text{ and} \tag{2.5}$$

$$S^2=(\boldsymbol{y}-\mathbf{X}\tilde{\boldsymbol{\beta}})'\mathbf{V}^{-1}(\boldsymbol{y}-\mathbf{X}\tilde{\boldsymbol{\beta}})/(n-p) \tag{2.6}$$

is an unbiased estimator of $\sigma_\epsilon^2=-2\psi'(0)\sigma^2$, where $\psi'(0)$ is the first derivative of characteristic generator of the elliptical model at point zero. Using the invariant theory due to Jeffreys (1961), we define the following prior of ignorance

$$\pi(\boldsymbol{\beta},\sigma^2)\propto \text{constant}, \quad \pi(\sigma^2)\propto\sigma^{-2}. \tag{2.7}$$

Assume in the multiple regression model (1.1), $\boldsymbol{\epsilon}\sim\mathcal{E}_n(\mathbf{0},\sigma^2\boldsymbol{V},\boldsymbol{g})$, where $\boldsymbol{V}\in\mathcal{S}_n$. Then w.r.t. the prior distribution given by (2.7), the posterior distribution of $\boldsymbol{\beta}$ is multivariate Student's t distribution, denoted by $\boldsymbol{\beta}|(\mathbf{X},\boldsymbol{y})\sim t_p(\tilde{\boldsymbol{\beta}},\boldsymbol{\Sigma},m)$, where $\boldsymbol{\Sigma}=S^2\boldsymbol{C}^{-1}$, with *pdf*

$$f(\boldsymbol{\beta}|\mathbf{X},\boldsymbol{y})=|\boldsymbol{\Sigma}|^{-\frac{1}{2}}[c(m,p)\pi^{\frac{p}{2}}]^{-1}\left[1+\frac{1}{m}(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}-\tilde{\boldsymbol{\beta}})\right]^{-\frac{n}{2}},$$

where $c(m,p)=m^{\frac{p}{2}}\Gamma\left(\frac{m}{2}\right)[\Gamma\left(\frac{n}{2}\right)]^{-1}$, and $m=n-p$. Thus the Bayes estimator is the posterior mean given by

$$\hat{\boldsymbol{\beta}}_B=\tilde{\boldsymbol{\beta}}. \tag{2.8}$$

Under (1.1), the distribution of Bayes estimator is $\mathcal{E}_p(\boldsymbol{\beta}, \sigma^2\boldsymbol{C}^{-1}, \boldsymbol{g})$. Thus the 1st moment of $\hat{\boldsymbol{\beta}}_B$ is the zero vector and the 2nd central moment is $E(\hat{\boldsymbol{\beta}}_B - \boldsymbol{\beta})'(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \sigma_{\epsilon}^2 \boldsymbol{tr}(\boldsymbol{C}^{-1})$.

Since the Bayes estimator is nothing more than that of the classical least square estimator of β, one may ask what would be the benefit of putting prior on the model? The answer is that the role of the prior distribution is obvious in the loss function dealing with the function $r(\|\boldsymbol{\beta}\|^2)$.

For the elliptically contoured family distributions the function $r(.)$ is given by

$$r(\|\boldsymbol{\beta}\|^2) = g(\|\boldsymbol{\beta}\|^2). \tag{2.9}$$

Actually by taking this assumption, the loss function relates to the density generator of the base model and therefore the prior information has direct impact on the model understudy. We note that $r(.)$ can be independent of $g(.)$.

Overall, what we need is to compute $E\left[r(\|\boldsymbol{\beta}\|^2)\right]$ which by making use of (2.7), and taking the constant to be 1, is given by

$$E\left[r(\|\boldsymbol{\beta}\|^2)\right] = \int_{\mathbb{R}^p} g(\|\boldsymbol{\beta}\|^2)\mathrm{d}\boldsymbol{\beta} = \frac{\pi^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)} \int_{\mathbb{R}+} y^{\frac{p}{2}-1} g(y)\mathrm{d}y = d_n^{-1}, \tag{2.10}$$

where d_n is the normalizing constant in (1.2).

3. Shrinkage Estimators

The restricted estimator (RE) is given by

$$\hat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} - \boldsymbol{C}^{-1}\boldsymbol{H}'\boldsymbol{V}_1(\boldsymbol{H}\tilde{\boldsymbol{\beta}} - \boldsymbol{h}), \quad \boldsymbol{V}_1 = [\boldsymbol{H}\boldsymbol{C}^{-1}\boldsymbol{H}']^{-1}. \tag{3.11}$$

By making use of (1.1) one can easily see that $\hat{\boldsymbol{\beta}} \sim \mathcal{E}_p\left(\boldsymbol{\beta} - \boldsymbol{\delta}, \sigma^2\boldsymbol{V}_2, \boldsymbol{g}\right)$ for $\boldsymbol{\delta} = \boldsymbol{C}^{-1}\boldsymbol{H}'\boldsymbol{V}_1(\boldsymbol{H}\boldsymbol{\beta} - \boldsymbol{h})$ and $\boldsymbol{V}_2 = \boldsymbol{C}^{-1}(\boldsymbol{I}_p - \boldsymbol{H}'\boldsymbol{V}_1\boldsymbol{H}\boldsymbol{C}^{-1})$. Similarly, under $H_0 : \boldsymbol{H}\boldsymbol{\beta} = \boldsymbol{h}$, the following estimator is unbiased for σ_{ϵ}^2.

$$S^{*2} = (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})/(n - p + q), \tag{3.12}$$

from least squares theory.

Let $w = \{\boldsymbol{\beta} : \boldsymbol{\beta} \in \mathbb{R}^p, \boldsymbol{H}\boldsymbol{\beta} = \boldsymbol{h}, \sigma > 0, \boldsymbol{V} \in S(n)\}$ and $\Omega = \{\boldsymbol{\beta} : \boldsymbol{\beta} \in \mathbb{R}^p, \sigma > 0, \boldsymbol{V} \in S(n)\}$. Then to remove the uncertainly in the suspected value of $\boldsymbol{h}$, we test $H_0 : \boldsymbol{H}\boldsymbol{\beta} = \boldsymbol{h}$ (where $q < p$) against $H_a : \boldsymbol{H}\boldsymbol{\beta} \neq \boldsymbol{h}$, using Corollary 1 from Anderson et al. (1986), which gives the likelihood ratio test statistic

$$\mathcal{L}_n = (\boldsymbol{H}\tilde{\boldsymbol{\beta}} - \boldsymbol{h})'\boldsymbol{V}_1(\boldsymbol{H}\tilde{\boldsymbol{\beta}} - \boldsymbol{h})/(qS^2). \tag{3.13}$$

Under H_0, the pdf of $\mathcal{L}_n$ is given by

$$\boldsymbol{g}_{q,m}^*(\mathcal{L}_n) = \left(\frac{q}{m}\right)^{\frac{q}{2}} \mathcal{L}_n^{\frac{q}{2}-1}\left[B\left(\frac{q}{2}, \frac{m}{2}\right)\left(1 + \frac{q}{m}\mathcal{L}_n\right)^{\frac{1}{2}(q+m)}\right]^{-1} \tag{3.14}$$

which is the central F-distribution with (q, m) degrees of freedom.

In many practical situations, along with the model one may suspect that $\boldsymbol{\beta}$ belongs to the sub-space defined by $\boldsymbol{H\beta} = \boldsymbol{h}$. In such situation one combines the estimate of $\boldsymbol{\beta}$ and the test-statistic to obtain shrinkage estimators as in Saleh (2006). The preliminary test estimator (PTE) of $\boldsymbol{\beta}$ which is a convex combination of $\tilde{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$:

$$\hat{\boldsymbol{\beta}}^{PT} = \tilde{\boldsymbol{\beta}} I(\mathcal{L}_n \geq F_\alpha) + \hat{\boldsymbol{\beta}} I(\mathcal{L}_n < F_\alpha), \tag{3.15}$$

where $I(A)$ is the indicator function of the set A and F_α is the upper α^{th} percentile of the central F-distribution with (q, m) d.f. The PTE depends on α $(0 < \alpha < 1)$, the level of significance and also it yields the extreme results, namely $\tilde{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}$ depending on the outcome of the test. Therefore we define Stein-type shrinkage estimator (SE) of $\boldsymbol{\beta}$, as

$$\hat{\boldsymbol{\beta}}^{S} = \hat{\boldsymbol{\beta}} + (1 - d\mathcal{L}_n^{-1})(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) = \tilde{\boldsymbol{\beta}} - d\mathcal{L}_n^{-1}(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}), \tag{3.16}$$

where

$$d = (q-2)m/[q(m+2)] \text{ and } q \geq 3. \tag{3.17}$$

The SE has the disadvantage that it has strange behavior for small values of $\mathcal{L}_n$. Also, the shrinkage factor $(1 - d\mathcal{L}_n^{-1})$ becomes negative for $\mathcal{L}_n < d$. Hence we define a better estimator namely the positive-rule shrinkage estimator (PRSE) of $\boldsymbol{\beta}$ as

$$\hat{\boldsymbol{\beta}}^{S+} = \hat{\boldsymbol{\beta}} + (1 - d\mathcal{L}_n^{-1}) I[\mathcal{L}_n > d](\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}). \tag{3.18}$$

4. Properties of the estimators

The bias of the unrestricted (LSE) and restricted estimators are given by

$$\boldsymbol{B}_1(UE) = E[\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}] = \boldsymbol{0}, \text{ and } \boldsymbol{B}_2(RE) = E[\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}] = -\boldsymbol{\delta}, \text{ respectively}. \tag{4.19}$$

Following Arashi et al. (2012) the bias of the PTE becomes

$$\begin{aligned} \boldsymbol{B}_3(PT) &= E(\hat{\boldsymbol{\beta}}^{PT} - \boldsymbol{\beta}) = E[\tilde{\boldsymbol{\beta}} - I(\mathcal{L}_n \leq F_\alpha)(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) - \boldsymbol{\beta}] \\ &= -\boldsymbol{C}\boldsymbol{H}'\mathbf{V}_1^{1/2}\, E[I(\mathcal{L}_n \leq F_\alpha)\mathbf{V}_1^{1/2}(\boldsymbol{H}\tilde{\boldsymbol{\beta}} - \boldsymbol{h})] = -\boldsymbol{\delta} G_{q+2,m}^{(2)}\left(F_\alpha; \Delta_*^2\right). \end{aligned} \tag{4.20}$$

where

$$G_{q+2i,m}^{(2-h)}(l_\alpha, \Delta_*^2) = \sum_{r=0}^{\infty} K_r^{(h)}(\Delta_*^2) I_{l_\alpha}\left[\frac{q+2i}{2} + r, \frac{m}{2}\right],$$

$l_\alpha = \frac{qF_{q,m}(\alpha)}{m+qF_{q,m}(\alpha)}$, $I_x[a,b] = \int_0^x u^{a-1}(1-u)^{b-1}du$ is the incomplete beta function and

$$K_r^{(h)}(\Delta_*^2) = [-2\psi'(0)]^r \left(\frac{\Delta_*^2}{2}\right)^r \int_0^\infty \frac{(t^{-1})^{-r+h}}{r!}\, e^{\frac{-t\Delta_*^2[-2\psi'(0)]}{2}} W(t)dt.$$

The bias of the SE is

$$\begin{aligned} \boldsymbol{B}_4(S) &= E(\hat{\boldsymbol{\beta}}^{S} - \boldsymbol{\beta}) = E[\tilde{\boldsymbol{\beta}} - d\mathcal{L}_n^{-1}(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}) - \boldsymbol{\beta}] \\ &= -d\boldsymbol{C}^{-1}\boldsymbol{H}'\boldsymbol{V}_1^{1/2} E[\mathcal{L}_n^{-1}\boldsymbol{V}_1^{1/2}(\boldsymbol{H}\tilde{\boldsymbol{\beta}} - \boldsymbol{h})] = -dq\boldsymbol{\delta} E^{(2)}[\chi_{q+2}^{*-2}(\Delta_*^2)], \end{aligned} \tag{4.21}$$

where

$$E^{(2-h)}[\chi_{q+s}^{*^{-2}}(\Delta_*^2)] = \sum_{r\geq 0}\frac{1}{r!}K_r^{(h)}(\Delta_*^2)(q+s-2+2r)^{-1},$$

and that of the PRSE is

$$\begin{aligned} \boldsymbol{B}_5(S+) &= E(\hat{\boldsymbol{\beta}}^S-\boldsymbol{\beta}) - E[I(\mathcal{L}_n \leq d)(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})] + dE[\mathcal{L}_n^{-1}I(\mathcal{L}_n \leq d)(\tilde{\boldsymbol{\beta}}-\hat{\boldsymbol{\beta}})] \\ &= -dq\boldsymbol{\delta}E_N^{(2)}[\chi_{q+2}^{*^{-4}}(\Delta_*^2)] + \boldsymbol{\delta}G_{q+2,m}^{(2)}(d;\Delta_*^2) \\ &\quad + \frac{qd}{q+2}\,\boldsymbol{\delta}E^{(2)}\left[F_{q+2,m}^{-1}(\Delta_*^2)I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right], \end{aligned} \tag{4.22}$$

where

$$\begin{aligned} &E^{(2-h)}[F_{q+s,n-p}^{-j}(\Delta_*^2)I(F_{q+s,n-p}(\Delta_*^2) < d_1)] \\ &= \sum_{r=0}^{\infty} K_r^{(h)}(\Delta_*^2)\left(\frac{q+s}{n-p}\right)^j \frac{B(\frac{q+s+2r-2j}{2},\frac{m+2j}{2})}{B(\frac{q+s+2r}{2},\frac{m}{2})} I_{x'}\left[\frac{q+s+2r-2j}{2},\frac{m+2j}{2}\right], \end{aligned}$$

in which $d_1 = \frac{dq}{q+2}$, and $x' = \frac{dq}{m+dq}$. Note that as the non-centrality parameter $\Delta_*^2 \to \infty$, $\boldsymbol{B}_1 = \boldsymbol{B}_3 = \boldsymbol{B}_4 = \boldsymbol{B}_5 = \mathbf{0}$ while $\boldsymbol{B}_2$ becomes unbounded. However, under $H_0 : \boldsymbol{H\beta} = \boldsymbol{h}$, because $\boldsymbol{\delta} = \mathbf{0}$, $\boldsymbol{B}_1 = \boldsymbol{B}_2 = \boldsymbol{B}_3 = \boldsymbol{B}_4 = \boldsymbol{B}_5 = \mathbf{0}$.

The risk function for any estimator $\boldsymbol{\beta}^*$ of $\boldsymbol{\beta}$ under balanced loss function is

$$\boldsymbol{R}_{\omega,\boldsymbol{\delta}_0}^{\boldsymbol{W}}(\boldsymbol{\beta}^*;\boldsymbol{\beta}) = E_{\boldsymbol{\beta}}\left\{E_{\chi}[L_{\omega,\boldsymbol{\delta}_0}^{\boldsymbol{W}}(\boldsymbol{\beta}^*;\boldsymbol{\beta})]|\boldsymbol{\beta}\right\}. \tag{4.23}$$

Using the above definition we find the risk function (4.23) when $\boldsymbol{\delta}_0 = \tilde{\boldsymbol{\beta}}$, as the target estimator, and $\boldsymbol{W} = \boldsymbol{C}$, given by (2.1), evaluate the risks of the five different estimators. For the case $\omega = 0$, we will simply write $\boldsymbol{R}_0^{\boldsymbol{W}}(\boldsymbol{\beta}^*;\boldsymbol{\beta})$.

For the risk of the Bayes estimator, from $\boldsymbol{R}_{\omega,\tilde{\boldsymbol{\beta}}}^{\boldsymbol{C}}(.;\boldsymbol{\beta})$ given in (4.23), we have

$$\begin{aligned} \boldsymbol{R}_{\omega,\tilde{\boldsymbol{\beta}}}^{\boldsymbol{C}}(\tilde{\boldsymbol{\beta}};\boldsymbol{\beta}) &= (1-\omega)E_{\boldsymbol{\beta}}\left\{r\left(\|\boldsymbol{\beta}\|^2\right)E[(\tilde{\boldsymbol{\beta}}-\boldsymbol{\beta})'\boldsymbol{C}(\tilde{\boldsymbol{\beta}}-\boldsymbol{\beta})]|\boldsymbol{\beta}\right\} \\ &= p\,\sigma_\epsilon^2(1-\omega)E_{\boldsymbol{\beta}}\left\{r\left(\|\boldsymbol{\beta}\|^2\right)\right\} = p\,d_n^{-1}\sigma_\epsilon^2(1-\omega). \end{aligned} \tag{4.24}$$

Noting $\boldsymbol{V}_1^{\frac{1}{2}}(\boldsymbol{H}\tilde{\boldsymbol{\beta}}-\boldsymbol{h}) \sim \mathcal{E}_q(\boldsymbol{V}_1^{\frac{1}{2}}(\boldsymbol{H\beta}-\boldsymbol{h}),\sigma^2\boldsymbol{I}_q,\boldsymbol{g})$, the risk of the RE is

$$\begin{aligned} \boldsymbol{R}_{\omega,\tilde{\boldsymbol{\beta}}}^{\boldsymbol{C}}(\hat{\boldsymbol{\beta}};\boldsymbol{\beta}) &= \omega E_{\boldsymbol{\beta}}\left\{r\left(\|\boldsymbol{\beta}\|^2\right)E[(\boldsymbol{H}\tilde{\boldsymbol{\beta}}-\boldsymbol{h})'\boldsymbol{V}_1(\boldsymbol{H}\tilde{\boldsymbol{\beta}}-\boldsymbol{h})]|\boldsymbol{\beta}\right\} \\ &\quad +(1-\omega)E_{\boldsymbol{\beta}}\left\{r\left(\|\boldsymbol{\beta}\|^2\right)E[(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})'\boldsymbol{C}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})]|\boldsymbol{\beta}\right\} \\ &= \boldsymbol{R}_{\omega,\tilde{\boldsymbol{\beta}}}^{\boldsymbol{C}}(\tilde{\boldsymbol{\beta}};\boldsymbol{\beta}) - q\,d_n^{-1}\sigma_\epsilon^2 + (1-\omega)d_n^{-1}\theta, \end{aligned} \tag{4.25}$$

where $\theta = \boldsymbol{\delta}'\boldsymbol{C}\boldsymbol{\delta} = (\boldsymbol{H\beta}-\boldsymbol{h})'\boldsymbol{V}_1(\boldsymbol{H\beta}-\boldsymbol{h})$. Note that $\boldsymbol{R} = \boldsymbol{C}_1^{-1/2}\boldsymbol{H}'\boldsymbol{V}_1\boldsymbol{H}\boldsymbol{C}_1^{-1/2}$ is a symmetric idempotent matrix of rank $q \leq p$. Thus, there exists an orthogonal matrix $\boldsymbol{Q}$ $(\boldsymbol{Q}'\boldsymbol{Q} = \boldsymbol{I}_p)$ such that $\boldsymbol{QRQ}' = \begin{bmatrix} \boldsymbol{I}_q & 0 \\ 0 & 0 \end{bmatrix}$. Now we define random vector $\boldsymbol{w} =$

$QC_1^{1/2}\tilde{\beta}-QC_1^{-1/2}H'V_1h$, then $w \sim \mathcal{E}_p(\eta, \sigma^2 I_p, g)$, where $\eta = QC_1^{1/2}\beta - QC_1^{-1/2}H'V_1h$. Partitioning the vector $w = (w_1', w_2')'$ and $\eta = (\eta_1', \eta_2')'$ where w_1 and w_2 are sub-vectors of order q and $p-q$ respectively, we can represent the test statistic $\mathcal{L}_n$ given by (2.6) as

$$\mathcal{L}_n = w_1'w_1/(qS^2), \text{ and } \theta = \eta_1'\eta_1. \tag{4.26}$$

For the risk of the PTE, note $\hat{\beta}-\tilde{\beta} = C^{-1}H'V_1HC^{-\frac{1}{2}}w$, then simplifications yield

$$\begin{aligned} R^{C}_{\omega,\tilde{\beta}}(\hat{\beta}^{PT};\beta) &= \omega E_\beta\left\{r\left(\|\beta\|^2\right)E[I(\mathcal{L}_n < F_\alpha)(\hat{\beta}-\tilde{\beta})'C(\hat{\beta}-\tilde{\beta})]|\beta\right\} \\ &\quad +(1-\omega)E_\beta\left\{r\left(\|\beta\|^2\right)E[(\hat{\beta}^{PT}-\beta)'C(\hat{\beta}^{PT}-\beta)]|\beta\right\} \\ &= R^{C}_{\omega,\tilde{\beta}}(\tilde{\beta};\beta) - (1-2\omega)q\sigma_\epsilon^2 d_n^{-1}G^{(1)}_{q+2,m}\left(F_\alpha;\Delta_*^2\right) \\ &\quad +2\theta(1-\omega)d_n^{-1}\left[2G^{(2)}_{q+2,m}\left(F_\alpha;\Delta_*^2\right) - G^{(2)}_{q+4,m}\left(F_\alpha;\Delta_*^2\right)\right]. \end{aligned} \tag{4.27}$$

On simplifications, the risk of the SE becomes

$$\begin{aligned} R^{C}_{\omega,\tilde{\beta}}(\hat{\beta}^{S};\beta) &= R^{C}_{\omega,\tilde{\beta}}(\tilde{\beta};\beta) + q\,d_n^{-1}\left\{\left[d^2\omega - 2d(1-\omega)\right]E^{(1)}[\chi^{*^{-2}}_{q+2}(\Delta_*^2)]\right. \\ &\quad \left.+d^2(1-\omega)E^{(1)}[\chi^{*^{-4}}_{q+2}(\Delta_*^2)]\right\} + \theta\,d_n^{-1} \\ &\quad \times\left\{\left[d^2\omega - 2d(1-\omega)\right]E^{(2)}[\chi^{*^{-2}}_{q+4}(\Delta_*^2)] - 2d(1-\omega)\right. \\ &\quad \left.\times E^{(2)}[\chi^{*^{-2}}_{q+2}(\Delta_*^2)] + d^2(1-\omega)E^{(2)}[\chi^{*^{-4}}_{q+4}(\Delta_*^2)]\right\}, \end{aligned} \tag{4.28}$$

where

$$E^{(2-h)}[\chi^{*^{-4}}_{q+s}(\Delta_*^2)] = \sum_{r\geq 0}\frac{1}{r!}K_r^{(h)}(\Delta_*^2)(q+s-2+2r)^{-1}(q+s-4+2r)^{-1}.$$

Finally, for the risk of PRSE, after some simplifications, we obtain

$$\begin{aligned} R^{C}_{\omega,\tilde{\beta}}(\hat{\beta}^{S+};\beta) &= R^{C}_{\omega,\tilde{\beta}}(\hat{\beta}^{S};\beta) \\ &\quad -d_n^{-1}\sigma_\epsilon^2\left\{qE^{(1)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right)^2 I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right]\right. \\ &\quad \left.+\frac{\theta}{\sigma_\epsilon^2}E^{(2)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right)^2 I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right]\right\} \\ &\quad -2d_n^{-1}\theta E^{(2)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right) I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right]. \end{aligned} \tag{4.29}$$

5. Performance comparison

This section provides risk analysis of the above estimators with the weight matrix $\boldsymbol{C}$. From equations (4.24) and (4.25) the risk difference of the UE and RE is given by

$$\mathcal{D}_{21} = \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}};\boldsymbol{\beta}) - \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\hat{\boldsymbol{\beta}}}(\tilde{\boldsymbol{\beta}};\boldsymbol{\beta}) = d_n^{-1}\left[(1-\omega)\theta - q\sigma_\epsilon^2\right]. \tag{5.30}$$

Then it can be directly concluded that $\hat{\boldsymbol{\beta}}$ performs better than $\tilde{\boldsymbol{\beta}}$ that is, $\hat{\boldsymbol{\beta}}$ dominates $\tilde{\boldsymbol{\beta}}$ ($\hat{\boldsymbol{\beta}} \succeq \tilde{\boldsymbol{\beta}}$) provided $0 \leq \theta \leq \frac{q\sigma_\epsilon^2}{1-\omega}$, for $\omega \neq 1$ since $d_n > 0$.

For comparing the $\hat{\boldsymbol{\beta}}^{PT}$ and $\tilde{\boldsymbol{\beta}}$, the risk difference is

$$\begin{aligned}\mathcal{D}_{13} &= \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\tilde{\boldsymbol{\beta}};\boldsymbol{\beta}) - \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}) = (1-2\omega)q\, d_n^{-1}\sigma_\epsilon^2 G^{(1)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right) \\ &\quad -2\theta\, d_n^{-1}(1-\omega)[2G^{(2)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right) - G^{(2)}_{q+4,m}\left(F_\alpha; \Delta_*^2\right)].\end{aligned} \tag{5.31}$$

The right hand side of (5.31) is nonnegative i.e. $\hat{\boldsymbol{\beta}}^{PT} \succeq \tilde{\boldsymbol{\beta}}$ for $\omega \neq 1$ whenever

$$\theta \leq \frac{(1-2\omega)q\sigma_\epsilon^2 G^{(1)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right)}{2(1-\omega)\left[2G^{(2)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right) - G^{(2)}_{q+4,m}\left(F_\alpha; \Delta_*^2\right)\right]}. \tag{5.32}$$

Moreover, under $H_0 : \boldsymbol{H\beta} = \boldsymbol{h}$, because of $\theta = 0$, $\hat{\boldsymbol{\beta}}^{PT} \succeq \tilde{\boldsymbol{\beta}}$ for values ω such that $\omega \leq \frac{1}{2}$. Now we compare $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}^{PT}$ by the risk difference

$$\begin{aligned}\mathcal{D}_{23} &= \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}};\boldsymbol{\beta}) - \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}) \\ &= -q\, d_n^{-1}\sigma_\epsilon^2[1-(1-2\omega)G^{(1)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right)] + \theta\, d_n^{-1}(1-\omega)[1-2G^{(2)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right) \\ &\quad +G^{(2)}_{q+4,m}\left(F_\alpha; \Delta_*^2\right)].\end{aligned} \tag{5.33}$$

Thus $\hat{\boldsymbol{\beta}}^{PT} \succeq \hat{\boldsymbol{\beta}}$ whenever

$$\theta \geq \frac{q\sigma_\epsilon^2\left[1-(1-2\omega)G^{(1)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right)\right]}{(1-\omega)\left[1-2G^{(2)}_{q+2,m}\left(F_\alpha; \Delta_*^2\right) + G^{(2)}_{q+4,m}\left(F_\alpha; \Delta_*^2\right)\right]}, \tag{5.34}$$

and vice versa. However, under H_0, the dominance order of $\tilde{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}^{PT}$ is as follows

$$\hat{\boldsymbol{\beta}} \succeq \hat{\boldsymbol{\beta}}^{PT} \succeq \tilde{\boldsymbol{\beta}}, \quad \text{or} \quad \hat{\boldsymbol{\beta}}^{PT} \succeq \hat{\boldsymbol{\beta}} \succeq \tilde{\boldsymbol{\beta}}, \tag{5.35}$$

depending on the value α satisfying (5.5).

In order to determine the superiority of $\hat{\boldsymbol{\beta}}^{S}$ to $\tilde{\boldsymbol{\beta}}$, we give the following results.

Theorem 5.1. *Consider the model* (1.1) *where the error-vector belongs to the ECD,* $\mathcal{E}_n(\mathbf{0}, \sigma^2\mathbf{V}, \boldsymbol{g})$. *Then the Stein-type shrinkage estimator,* $\hat{\boldsymbol{\beta}}^{S}$ *of* $\boldsymbol{\beta}$ *given by*

$$\hat{\boldsymbol{\beta}}^{S} = \tilde{\boldsymbol{\beta}} - d^*\mathcal{L}_n^{-1}(\tilde{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}})$$

uniformly dominates the Bayes estimator $\tilde{\beta}$ *with respect to the balanced loss function* $L_0^C(\delta;\beta)$ *and is minimax if and only if* $0 < d^* \leq \frac{2m}{m+2}$. *The largest reduction of the risk is attained when* $d^* = \frac{m}{m+2}$.

Following Srivastava and Bilodeau (1989), the risk difference of the SE and Bayes estimator under balanced loss function, is given by

$$\begin{aligned}\mathcal{D}_{41} &= E_\beta\left\{E(\hat{\beta}^S - \beta)'C(\hat{\beta}^S - \beta) - E(\tilde{\beta} - \beta)'C(\tilde{\beta} - \beta)|\beta\right\} \\ &= d_n^{-1}\left\{\frac{q^2(m+2)}{m}(d^*)^2 E_\tau\left(\frac{\tau^{-2}}{\dot{z}'C^{-1}\dot{z}}\right) - 2q^2 d^* E_\tau\left(\frac{\tau^{-2}}{\dot{z}'C^{-1}\dot{z}}\right)\right\},\end{aligned}$$

since $\left(\frac{mS^2}{\sigma^2}\right)\Big|\tau \sim \tau^{-1}\chi_m^2$ and $\tilde{\beta}'H'V_1H\tilde{\beta} \mid \tau \sim \tau^{-2}\sigma^4\chi_q^2(\dot{\delta})$, where $\dot{\delta} = \beta'H'V_1H\beta$, where E_N means getting expectation with respect to multivariate normal with covariance $\tau^{-1}\sigma^2 V$ and E_τ means getting expectation with respect to measure $dW(.)$.
Therefore, $\mathcal{D}_{41} \leq 0$ if and only if $0 < d^* \leq \frac{2m}{m+2}$ since $\int_0^\infty \frac{\tau^{-2}}{\dot{z}'C^{-1}\dot{z}}\, dW(\tau) > 0$.

Theorem 5.2. *Suppose in the model* (1.1), $\epsilon \sim \mathcal{E}_n(0, \sigma^2 V, g)$. *Then the Stein-type shrinkage estimator*

$$\hat{\beta}_*^S = \tilde{\beta} - d(1-\omega)\mathcal{L}_n^{-1}(\tilde{\beta} - \hat{\beta}) \tag{5.36}$$

uniformly dominates $\tilde{\beta}$ *under the balanced loss function* $L^C_{\omega,\tilde{\beta}}(\tilde{\beta};\beta)$.

Corollary 5.1. *Suppose in the model* (1.1), $\epsilon \sim \mathcal{E}_n(0, \sigma^2 V, g)$. *Then* $\hat{\beta}^S \succeq \tilde{\beta}$ *under the balanced loss function* $L^C_{\omega,\tilde{\beta}}(\tilde{\beta};\beta)$.

The proof directly follows from Theorem 5.2 for the special case $\omega = 0$. To compare $\hat{\beta}$ and $\hat{\beta}^S$, it is easy to show that

$$\begin{aligned}R_0^C(\hat{\beta}^S;\beta) &= R_0^C(\hat{\beta};\beta) + d_n^{-1}\Bigg(q\sigma_\epsilon^2 - \theta - dq^2\sigma_\epsilon^2\Big\{(q-2)E[\chi_{q+2}^{*-4}(\Delta_*^2)] \\ &\quad + \left[1 - \frac{(q+2)\theta}{2q\sigma_\epsilon^2\Delta_*^2}\right](2\Delta_*^2)E[\chi_{q+4}^{*-4}(\Delta_*^2)]\Big\}\Bigg). \end{aligned} \tag{5.37}$$

Under H_0, this becomes

$$R_0^C(\hat{\beta}^S;\beta) = R_0^C(\hat{\beta};\beta) + qd_n^{-1}\sigma_\epsilon^2(1-d) \geq R_0^C(\hat{\beta};\beta), \text{ with} \tag{5.38}$$

$$R_0^C(\hat{\beta};\beta) = R_0^C(\tilde{\beta};\beta) - qd_n^{-1}\sigma_\epsilon^2 \leq R_0^C(\tilde{\beta};\beta). \tag{5.39}$$

Therefore, $\hat{\beta} \succeq \hat{\beta}^S$ under H_0 with the balanced loss $L_0^C(\beta^*, \beta)$. Therefore under H_0, $\hat{\beta} \succeq \hat{\beta}^S$ with the balanced loss $L^C_{\omega,\tilde{\beta}}(\beta^*;\beta)$. However, as η_1 moves away from 0, θ increases and the risk of $\hat{\beta}$ becomes unbounded while the risk of $\tilde{\beta}^S$ remains below the risk of $\tilde{\beta}$; thus for similar reasons, $\tilde{\boldsymbol{\beta}}^{S}$ dominates $\hat{\boldsymbol{\beta}}$ outside an interval around the origin under the balanced loss $L^{C}_{\omega,\tilde{\boldsymbol{\beta}}}(\boldsymbol{\beta}^*;\boldsymbol{\beta})$. This scenario repeats when we compare $\hat{\boldsymbol{\beta}}^{S}$ and $\hat{\boldsymbol{\beta}}^{PT}$. Under H_0

$$\boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{S};\boldsymbol{\beta}) = \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}) + qd_n^{-1}\sigma_\epsilon^2[1-\alpha-d] \geq \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}),$$

for all α such that $F^{-1}_{q+2,m}(d,0) \leq \frac{qF_\alpha}{q+2}$. This means, $\hat{\boldsymbol{\beta}}^{S}$ does not always dominate $\hat{\boldsymbol{\beta}}^{PT}$ under H_0. So under H_0, with α satisfying $F^{-1}_{q+2,m}(d,0) \leq \frac{qF_\alpha}{q+2}$, under the balanced loss function we have $\hat{\boldsymbol{\beta}} \succeq \hat{\boldsymbol{\beta}}^{PT} \succeq \hat{\boldsymbol{\beta}}^{S} \succeq \tilde{\boldsymbol{\beta}}$.. The risk difference of $\hat{\boldsymbol{\beta}}^{S+}$ and $\hat{\boldsymbol{\beta}}^{S}$ is given by

$$\begin{aligned}
\mathcal{D}_{54} &= \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}}^{S+};\boldsymbol{\beta}) - \boldsymbol{R}^{\boldsymbol{C}}_{\omega,\tilde{\boldsymbol{\beta}}}(\hat{\boldsymbol{\beta}}^{S};\boldsymbol{\beta}) = \\
&-d_n^{-1}\sigma_\epsilon^2\left\{qE^{(1)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right)^2 I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right]\right. \\
&\left.+\frac{\theta}{\sigma_\epsilon^2}E^{(2)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right)^2 I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right]\right\} \\
&-2d_n^{-1}\theta E^{(2)}\left[\left(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(\Delta_*^2)\right) I\left(F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}\right)\right].
\end{aligned}$$

The r.h.s. of the above equality is $-$ve since for $F_{q+2,m}(\Delta_*^2) \leq \frac{qd}{q+2}$, $(\frac{qd}{q+2}F_{q+2,m}(\Delta_*^2)-1) \geq 0$ and also the expectation of a positive random variable is positive. Thus $\hat{\boldsymbol{\beta}}^{S+} \succeq \hat{\boldsymbol{\beta}}^{S}$.

Remark 5.1. *The positive-rule shrinkage estimator $\hat{\boldsymbol{\beta}}^{S+}$ of $\boldsymbol{\beta}$ is minimax.*

Continue the comparisons under $L^{C}_0(\boldsymbol{\beta}^*;\boldsymbol{\beta})$. The results are the same for the balanced loss $L^{C}_{\omega,\tilde{\boldsymbol{\beta}}}(\boldsymbol{\beta}^*;\boldsymbol{\beta})$. To compare $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}^{S+}$, note under H_0, i.e., $\boldsymbol{\eta}_1 = 0$,

$$\begin{aligned}
\boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{S+};\boldsymbol{\beta}) &= \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}};\boldsymbol{\beta}) + qd_n^{-1}\sigma_\epsilon^2\left\{(1-d) - E\left[(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(0))^2\right.\right. \\
&\left.\left.\times I(F_{q+2,m}(0) \leq \frac{qd}{q+2})\right]\right\} \geq \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}};\boldsymbol{\beta}),
\end{aligned}$$

since $E\left[(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(0))^2 I(F_{q+2,m}(0) \leq \frac{qd}{q+2})\right] \leq E\left[(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(0))^2\right] = 1-d$. Thus under H_0, $\hat{\boldsymbol{\beta}} \succeq \hat{\boldsymbol{\beta}}^{S+}$. But, as $\boldsymbol{\eta}_1$ moves away from 0, θ increases and the risk of $\hat{\boldsymbol{\beta}}$ becomes unbounded while the risk of $\tilde{\boldsymbol{\beta}}^{S+}$ remains below the risk of $\tilde{\boldsymbol{\beta}}$; thus $\tilde{\boldsymbol{\beta}}^{S+}$ dominates $\hat{\boldsymbol{\beta}}$ outside an interval around the origin. When H_0 holds, $G^*_{q+2,m}(F_\alpha,0) = 1-\alpha$,

$$\begin{aligned}
\boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{S+};\boldsymbol{\beta}) &= \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}) + qd_n^{-1}\sigma_\epsilon^2\left\{1-\alpha-d - E\left[(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(0))^2\right.\right. \\
&\left.\left.\times I(F_{q+2,m}(0) \leq \frac{qd}{q+2})\right]\right\} \geq \boldsymbol{R}^{\boldsymbol{C}}_0(\hat{\boldsymbol{\beta}}^{PT};\boldsymbol{\beta}), \text{ for all } \alpha \text{ satisfying} \\
&E\left[(1-\frac{qd}{q+2}F^{-1}_{q+2,m}(0))^2 I(F_{q+2,m}(0) \leq \frac{qd}{q+2})\right] \leq 1-\alpha-d.
\end{aligned}$$

Thus, $\hat{\beta}^{S+}$ does not always dominates $\hat{\beta}^{PT}$ when the null-hypothesis H_0 holds.

Therefore the dominance order of five estimators under the balanced loss function $L^{C}_{\omega,\tilde{\beta}}(\beta^*;\beta)$ can be determine under following two categories

$$1.\quad \hat{\beta} \succeq \hat{\beta}^{PT} \succeq \hat{\beta}^{S+} \succeq \hat{\beta}^{S} \succeq \tilde{\beta},\ \text{and 2.}\quad \hat{\beta} \succeq \hat{\beta}^{S+} \succeq \hat{\beta}^{S} \succeq \hat{\beta}^{PT} \succeq \tilde{\beta}.$$

Acknowledgments

First author's research is supported by Shahrood University of Technology, Iran.

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