The *K*th-Best Approach for Linear Bilevel Multifollower Programming with Partial Shared Variables among Followers

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Abstract - In a real world bilevel decision-making, the lower level of a bilevel decision usually involves multiple decision units. This paper proposes the *K*th-best approach for linear bilevel multifollower programming problems with shared variables among followers. Finally a numeric example is given to show how the *K*th-best approach works.

Keywords: Linear bilevel programming, mulitfollower, *K*th-best approach, Von Stackelberg game

1 Introduction

Bilevel programming (BLP) was motivated by the game theory of Von Stackelberg [1] in the context of unbalanced economic markets [2].The majority of research on BLP has centered on the linear version of the problem in which only one follower is involved. There have been nearly two dozen algorithms, such as, the K^{th} best approach [3, 4], Kuhn-Tucker approach [5, 6, 7], complementarity pivot approach [8], penalty function approach [9, 10], proposed for solving linear BLP problems since the field being caught the attention of researchers in the mid-1970s. Kuhn-Tucker approach has been proven to be a valuable analysis tool with a wide range of successful applications for linear BLP [2, 6, 7].

Our previous work presented new theory overcame the fundamental deficiency of existing linear BLP theory [11, 12, 13, 14]. We proposed a comprenhensive framework for bilevel multifollower programming (BLMFP) problems and developed solution technology for linear BLMFP problems without shared variables among followers [15, 16, 17]. We also proposed an extended Kuhn-Tucker approach for linear BLMFP problems with shared variables among followers **[18].**This paper proposes an extended *K*th-best approach for linear BLMFP problems with partial shared variables among followers. Following the introduction, this paper reviews a model for linear BLMFP problems with partial shared variables among followers in Section2. The *K*th-best approach for this model is proposed in Section 3. A numeric example for this approach is given in Section 4. A conclusion and further study are given in Section 5.

2 Model Overview

2.1 A model for linear BLMFP problems with partial shared variables among followers

For $x \in X \subset R^n$, $y_i \in Y_i \subset R^{m_i}$, $z \in Z \subset R^m$ $F : X \times Y_1 \times \ldots \times Y_K \times Z \to R^1$, and $f_i : X \times Y_1 \times \ldots \times Y_K \times Z \to R^1$, $i = 1,2,\ldots,K$, a linear BLMFP problem where $K (\geq 2)$ followers are involved and there are shared partial decision variables, but separate objective functions and constraint functions among the followers is defined as follows [18]:

$$\min_{x \in X} F(x, y_1, \ldots, y_K, z) = cx + \sum_{s=1}^{K} d_s y_s + dz$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s + Bz \leq b$$

$$\min_{y_i \in Y_i, z \in Z} f_i(x, y_1, \ldots, y_K, z) = c_i x + \sum_{s=1}^{K} e_{is} y_s + e_i z \qquad (1)$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s + C_i z \leq b_i,$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $d \in R^m$, $e_{is} \in R^{m_i}$, $e_i \in R^m$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p \times n}$, $B_i \in R^{p \times m_i}$, $B \in R^{p \times m}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $C_i \in R^{q_i \times m}$, $i, s = 1,2,\ldots,K$.

Definition 1 A topological space is compact if every open cover of the entire space has a finite subcover. For example, $[a,b]$ is compact in R (the Heine-Borel theorem) [19].

2.2 Model transformation for linear BLMFP with partial shared variables among followers

The main idea to deal with linear BLMFP problems with partial shared variables among the followers is that an assumed third party controls the shared variable z. It means that the i^{th} follower controls the variable y_i ($i = 1,2,\ldots,K$), and a third party called a virtual follower: the $(K+1)^{th}$ follower controls the variable z. By using this splitting method, (1) can be rewritten as follows:

$$\min_{x\in X} F(x, y_1,\ldots, y_K, y_{K+1}) = cx + \sum_{s=1}^{K+1} d_s y_s$$

$$\text{subject to } Ax + \sum_{s=1}^{K+1} B_s y_s \le b \qquad (2)$$

$$\min_{y_i\in Y_i} f_i(x, y_1,\ldots, y_K, y_{K+1}) = c_i x + \sum_{s=1}^{K+1} e_{is} y_s$$

$$\text{subject to } A_i x + \sum_{s=1}^{K+1} C_{is} y_s \le b_i,$$

where $i = 1,\ldots, K, K+1$, $y_{K+1} = z$, $d_{K+1} = d$, $B_{K+1} = B$, $e_{l(K+1)} = e_l (l = 1,\ldots, K)$, $c_{K+1} = \sum_{s=1}^{K} c_s$, $e_{(K+1)j} = \sum_{s=1}^{K} e_{sj} (j = 1,\ldots, K+1)$, $C_{l(K+1)} = C_l (l = 1,\ldots, K)$, $A_{K+1} = (0)_{q_{K+1}}$, $C_{(K+1)l} = (0)_{q_{K+1}\times m_l} (l = 1,\ldots, K)$, $b_{K+1} = (0)_{q_{K+1}}$.

This simple transformation has shown that solving the linear BLMFP (1) is equivalent to solving (2). There are K followers that share the variable z for the linear BLMFP (1). However, (2) has $K+1$ followers and is the linear BLMFP without shared variables among the followers. We can also find that all the variables of the followers parameterise into the objective functions and constraint functions of the followers.

2.3 Definition of solution

For simplification and convenience, we write model (2) as follows:

$$\min_{x \in X} F(x, y_1, \ldots, y_K) = cx + \sum_{s=1}^{K} d_s y_s \tag{3a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{3b}$$

$$\min_{y_i \in Y_i} f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{3c}$$

$$\text{subject to } A_i x + \sum_{s=1}^{K} C_{is} y_s \le b_i , \tag{3d}$$

where $c \in R^n$, $c_i \in R^n$, $d_i \in R^{m_i}$, $e_{is} \in R^{m_s}$, $b \in R^p$, $b_i \in R^{q_i}$, $A \in R^{p \times n}$, $B_i \in R^{p \times m_i}$, $A_i \in R^{q_i \times n}$, $C_{is} \in R^{q_i \times m_s}$, $i, s = 1,2,\ldots, K$.

The formulation (3) is the same as (2) except the number of followers. They have the same solution algorithms. Corresponding to (3), [18] give following basic definition.

Definition 2

(a) Constraint region:

$$S = \{(x, y_1, \ldots, y_K) \in X \times Y_1 \times \ldots \times Y_k , Ax + \sum_{s=1}^{K} B_s y_s \le b,$$

$$A_i x + \sum_{s=1}^{K} C_{is} y_s \le b_i , i = 1,2,\ldots, K\} .$$

The constraint region refers to all possible combinations of choices that the leader and followers may make.

(b) Projection of S onto the leader's decision space:

$$S(X) = \{x \in X : \exists y_i \in Y_i, Ax + \sum_{s=1}^{K} B_s y_s \le b, A_i x + \sum_{s=1}^{K} C_{is} y_s \le b_i, i = 1,2,\ldots,K\}.$$

(c) Feasible set for each follower $\forall x \in S(X)$:

$$S_i(x) = \{y_i \in Y_i : (x, y_1, \ldots, y_K) \in S\}.$$

The feasible region for each follower is affected by the leader's choice of x, and the allowable choices of each follower are the elements of S.

(d) Each follower's rational reaction set for $x \in S(X)$:

$$P_i(x) = \{y_i \in Y_i : y_i \in \arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i) : \hat{y}_i \in S_i(x)]\},$$

where $i = 1,2,\ldots,K$, $\arg\min[f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i) : \hat{y}_i \in S_i(x)] =$

$$\{y_i \in S_i(x) : f_i(x, y_1, \ldots, y_K) \le f_i(x, \hat{y}_i, y_j, j = 1,2,\ldots,K, j \ne i), \hat{y}_i \in S_i(x)\}.$$

The followers observe the leader's action and simultaneously react by selecting y_i from their feasible set to minimize their objective functions.

(e) Inducible region:

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, y_i \in P_i(x), i = 1,2,\ldots,K\}.$$

Thus in terms of the above notations, (3) can be written as

$$\min\{F(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in IR\} \quad (4)$$

Shi proposed the following theorem to characterize the condition under which there is an optimal solution for (3) [18].

Theorem 1 If S is nonempty and compact, there exists an optimal solution for a linear BLMFP problem.

3 An Extended *K*th-best Algorithm for Linear Bilevel Multifollower Programming with Partial Shared Variables among Followers

3.1 Properties of Linear Bilevel Multifollower Programming with Partial Shared Variables among Followers

Theorem 2 The inducible region of the model (3) can be written equivalently as a piecewise linear equality constraint comprised of supporting hyperplanes of constraint region S.

Proof: Let us begin by writing the inducible region of Definition 2(e) explicitly as follower:

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, e_{ii} y_i = \min[e_{ii} \tilde{y}_i : B_i \tilde{y}_i \le b - Ax - \sum_{s=1, s\neq i}^{K} B_s y_s,$$

$$C_{ji} \tilde{y}_i \le b_i - A_i x - \sum_{s=1, s\neq i}^{K} C_{is} y_s,\ j = 1,2,\ldots,K,\ \tilde{y}_i \ge 0], i = 1,2,\ldots,K\}.$$

Let us define

$$b^{'} = (b, b_1, \ldots, b_K)^T,\ A^{'} = (A, A_1, \ldots, A_K)^T,\ B_i^{'} = (B_i, C_{1i}, \ldots, C_{Ki})^T,$$

where $i = 1,2,\ldots,K$. Now we have

$$IR = \{(x, y_1, \ldots, y_K) : (x, y_1, \ldots, y_K) \in S, e_{ii} y_i = \min[e_{ii} \tilde{y}_i : B_i^{'} \tilde{y}_i \leq b_i^{'} -$$

$$A^{'} x - \sum_{s=1, s \neq i}^{K} B_s^{'} y_s, \tilde{y}_i \geq 0], i = 1,2,\ldots, K\}.$$

Let us define

$$Q_i(x, y_j, j = 1,2,\ldots K, j \neq i) = \min[e_{ii} \tilde{y}_i : B_i^{'} \tilde{y}_i \leq b_i^{'} - A^{'} x - \sum_{s=1, s \neq i}^{K} B_s^{'} y_s, \tilde{y}_i \geq 0], \quad (5)$$

where $i = 1,2,\ldots, K$. For each value of $x \in S(X)$, the resulting feasible region to problem (3) is nonempty and compact. Thus, for Q_i, which is a linear program parameterized in x, y_j, $j = 1,2,\ldots, K$ and $j \neq i$, always has a solution. From duality theory we get

$$\max\{u(A^{'} x + \sum_{s=1, s \neq i}^{K} B_s^{'} y_s - b_i^{'}) : uB_i^{'} \geq -e_{ii}, u \geq 0\}, \quad (6)$$

which has the same optimal value as (5) at the solution u^*. Let $u^1, \ldots, u^s$ be a listing of all the vertices of the constraint region of (6) given by $U = \{u : uB_i^{'} \geq -e_{ii}, u \geq 0\}$. Because we know that a solution to (6) occurs at a vertex of U, we get the equivalent problem

$$\max\{u^l (A^{'} x + \sum_{s=1, s \neq i}^{K} B_s^{'} y_s - b_i^{'}) : u^l \in \{u^1, \ldots, u^s\}\},$$

which demonstrates that $Q_i(x, y_j, j = 1,2,\ldots, K, j \neq i)$, is a piecewise linear function. Rewriting IR as

$$IR = \{(x, y_1, \ldots, y_k) \in S : Q_i(x, y_j, j = 1,2,\ldots, K, j \neq i) - e_{ii} y_i = 0, i = 1,2,\ldots, K\} \quad (7)$$

yields desired result.

Corollary 1 The problem (3) is equivalent to minimizing F over a feasible region comprised of a piecewise linear equality constraint.

Proof: By (4) and 2, we have the desired result.

The each function Q_i defined by (5) is convex and continuous. In general, because we are minimizing a linear function $F = cx + \sum_{s=1}^{K} d_s y_s$ over IR, and because F is bounded below S by, say, $\min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\}$, the following can be concluded.

Corollary 2 A solution for the linear BLMFP problem occurs at a vertex of IR.

Proof: A linear BLMFP problem can be written as in (4). Since $F = cx + \sum_{s=1}^{K} d_s y_s$ is linear, if a solution exists, one must occur at a vertex of IR. The proof is completed.

Theorem 3 The solution $(x^*, y_1^*, \ldots, y_K^*)$ of the linear BLMFP problem occurs at a vertex of S.

Proof: Let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^r, y_1^r, \ldots, y_K^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y_1^*, \ldots, y_K^*) = \sum_{j=1}^{r} \alpha_j (x^j, y_1^j, \ldots, y_K^j)$, where $\sum_{j=1}^{r} \alpha_j = 1, \alpha_j \geq 0, j = 1,2,\ldots,\bar{r}$ and $\bar{r} \leq r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (3) at $(x^*, y_1^*, \ldots, y_K^*)$ in their piecewise linear form (7).

$$0 = Q_i(x, y_l^*, l = 1,2,\ldots,K, l \neq i) - e_{ii} y_i^*,\ i = 1,2,\ldots,K$$

Rewrite it as follows

$$0 = Q_i(\sum_j \alpha_j (x^j, y_l^j, l = 1,2,\ldots,K, l \neq i)) - e_{ii}(\sum_j \alpha_j y_i^j)$$

$$\leq \sum_j \alpha_i Q_i(x^j, y_l^j, l = 1,2,\ldots,K, l \neq i) - \sum_j \alpha_j e_{ii} y_i^j,$$

where $i = 1,2,\ldots,K$.

By convexity of $Q_i(x, y_l, l = 1,2,\ldots,K, l \neq i)$, we have

$$0 \leq \sum_j \alpha_j (Q_i(x^j, y_l^j, l = 1,2,\ldots,K, l \neq i) - e_{ii} y_i^j),$$

where $i = 1,2,\ldots,K$. But by a definition,

$$Q_i(x^j, y_l^j, l = 1,2,\ldots,K, l \neq i) = \min_{y_i \in S(x^j)} e_{ii} y_i \leq e_{ii} y_i^j, \ i = 1,2,\ldots,K.$$

Therefore, $Q_i(x^j, y_l^j, l = 1,2,\ldots,K, l \neq i) - e_{ii} y_i^j \leq 0, j = 1,2,\ldots,\bar{r}$, $i = 1,2,\ldots,K$. Noting that $\alpha_j \geq 0, j = 1,2,\ldots,\bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q_i(x^j, y_l^j, l = 1,2,\ldots,K, l \neq i) - e_{ii} y_i^j = 0$, $j = 1,2,\ldots,\bar{r}$, $i = 1,2,\ldots,K$. This implies that $(x^j, y_1^j, \ldots, y_K^j) \in IR$, $j = 1,2,\ldots,\bar{r}$ and $(x^*, y_1^*, \ldots, y_K^*)$ can be written as a convex combination of points in IR. Because $(x^*, y_1^*, \ldots, y_K^*)$ is a vertex of IR, a contradiction results unless $\bar{r} = 1$. The proof is completed.

Corollary 1 If x is an extreme point of IR, it is an extreme point of S.

Proof: Let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^r, y_1^r, \ldots, y_K^r)$ be the distinct vertices of S. Since any point in S can be written a convex combination of these vertices, let $(x^*, y_1^*, \ldots, y_K^*) = \sum_{j=1}^r \alpha_j (x^j, y_1^j, \ldots, y_K^j)$, where $\sum_{j=1}^r \alpha_j = 1, \alpha_j \geq 0, j = 1,2,\ldots,\bar{r}$ and

$\bar{r} \le r$. It must be shown that $\bar{r} = 1$. To see this let us write the constraints to (3) at $(x^*, y_1^*, \dots, y_K^*)$ in their piecewise linear form (7).

$$0 = Q_i(x, y_l^*, l = 1,2,\dots,K, l \ne i) - e_{ii} y_i^*,\ i = 1,2,\dots,K\ .$$

Rewrite the above formulation as follows

$$0 = Q_i(\sum_j \alpha_j (x^j, y_l^j, l = 1,2,\dots,K, l \ne i)) - e_{ii}(\sum_j \alpha_j y_i^j)$$

$$\le \sum_j \alpha_i Q_i(x^j, y_l^j, l = 1,2,\dots,K, l \ne i) - \sum_j \alpha_j e_{ii} y_i^j\ ,$$

where $i = 1,2,\dots,K$.

By convexity of $Q_i(x, y_l, l = 1,2,\dots,K, l \ne i)$, we have

$$0 \le \sum_j \alpha_j (Q_i(x^j, y_l^j, l = 1,2,\dots,K, l \ne i) - e_{ii} y_i^j)\ ,$$

where $i = 1,2,\dots,K$. But by a definition,

$$Q_i(x^j, y_l^j, l = 1,2,\dots,K, l \ne i) = \min_{y_i \in S(x^j)} e_{ii} y_i \le e_{ii} y_i^j,\ i = 1,2,\dots,K\ .$$

Therefore, $Q_i(x^j, y_l^j, l = 1,2,\dots,K, l \ne i) - e_{ii} y_i^j \le 0,\ j = 1,2,\dots,\bar{r}$, $i = 1,2,\dots,K$. Noting that $\alpha_j \ge 0,\ j = 1,2,\dots,\bar{r}$, the equality in the preceding expression must hold or else a contradiction would result in the sequence above. Consequently, $Q_i(x^j, y_l^j, l = 1,2,\dots,K, l \ne i) - e_{ii} y_i^j = 0$, $j = 1,2,\dots,\bar{r}$, $i = 1,2,\dots,K$. This implies that $(x^j, y_1^j, \dots, y_K^j) \in IR$, $j = 1,2,\dots,\bar{r}$ and $(x^*, y_1^*, \dots, y_K^*)$ can be written as a convex combination of points in IR. Because $(x^*, y_1^*, \dots, y_K^*)$ is a vertex of IR, a contradiction results unless $\bar{r} = 1$. This means that $(x^*, y_1^*, \dots, y_K^*)$ is an extreme point of S . The proof is completed.

4. An Extended *K*th-best Algorithm for Linear Bilevel Multifollower Programming with Partial Shared Variables among Followers

Theorem and Corollary have provided theoretical foundation for our new algorithm. It means that by searching extreme points on the constraint region S, we can efficiently find an optimal solution for a linear BLMFP problem. The basic idea of our algorithm is that according to the objective function of the upper level, we arrange all the extreme points in S in descending order, and select the first extreme point to check if it is on the inducible region IR. If yes, the current extreme point is the optimal solution. Otherwise, the next one will be selected and checked.

More specifically, let $(x^1, y_1^1, \ldots, y_K^1), \ldots, (x^N, y_1^N, \ldots, y_K^N)$, denote the N ordered extreme points to the linear BLMFP problem

$$\min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1, \ldots, y_K) \in S\}, \tag{8}$$

such that $cx^j + \sum_{s=1}^{K} d_s y_s^j \le cx^{j+1} + \sum_{s=1}^{K} d_s y_s^{j+1}$, $j = 1,2,\ldots, N-1$. Let $(\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_K)$ denote the optimal solution to the following problem.

$$\min(f_i(x^j, y_1, \ldots, y_K) : y_i \in S_i(x^j), i = 1,2,\ldots, K). \tag{9}$$

We only need to find the smallest j, $j = 1,2,\ldots, N$ under which $y_i^j = \tilde{y}_i$, $i = 1,2,\ldots, K$. Let us write (9) as follows

$$\min f_i(x, y_1, \ldots, y_K)$$

$$\text{subject to } y_i \in S(x)$$

$$x = x^j,$$

where $i = 1,2,\ldots,K$. We only need to find the smallest j under which $y_i^j = \tilde{y}_i$, $i = 1,2,\ldots,K$. From Definition 2(b), we have

$$\min f_i(x, y_1, \ldots, y_K) = c_i x + \sum_{s=1}^{K} e_{is} y_s \tag{10a}$$

$$\text{subject to } Ax + \sum_{s=1}^{K} B_s y_s \le b \tag{10b}$$

$$A_l x + \sum_{s=1}^{K} C_{ls} y_s \le b_l,\ l = 1,2,\ldots,K \tag{10c}$$

$$x = x^j \tag{10d}$$

$$y_1 \ge 0, y_2 \ge 0, \ldots, y_K \ge 0, \tag{10f}$$

where $i = 1,2,\ldots,K$.

The solving is equivalent to select one ordered extreme point $(x^j, y_1^j, \ldots, y_K^j)$, then solve (10) to obtain the optimal solution $\tilde{y}_i$. If for all i, $y_i^j = \tilde{y}_i$, then $(x^j, y_1^j, \ldots, y_K^j)$ is the global optimum to (3). Otherwise, check the next extreme point. It can be accomplished with the following procedure.

Step 1: Put $j \leftarrow 1$. Solve (8) with the simplex method to obtain the optimal solution $(x^1, y_1^1, \ldots, y_K^1)$. Let $W = (x^1, y_1^1, \ldots, y_K^1)$ and $T = \phi$. Go to Step 2.

Step 2: Solve (10) with the bounded simplex method. Let $\tilde{y}_i$ denote the optimal solution to (10). If $y_i^j = \tilde{y}_i$ for all i, $i = 1,\ldots,K$, stop; $(x^j, y_1^j, \ldots, y_K^j)$ is the global optimum to (3). Otherwise, go to Step 3.

Step 3: Let $W_{[j]}$ denote the set of adjacent extreme points of $(x^j, y_1^j, \ldots, y_K^j)$ such that $(x, y_1, \ldots, y_K) \in W_{[j]}$ implies $cx + \sum_{s=1}^{K} d_s y_s \le cx^j + \sum_{s=1}^{K} d_s y_s^j$. Let $T = T \cup \{(x^j, y_1^j, \ldots, y_K^j)\}$ and $W = (W \cup W_{[j]}) \setminus T$. Go to Step 4.

Step 4: Set $j \leftarrow j+1$ and choose $(x^j, y_1^j, \ldots, y_K^j)$ so that

$$cx^j + \sum_{s=1}^{K} d_s y_s^j = \min\{cx + \sum_{s=1}^{K} d_s y_s : (x, y_1 \ldots, y_K) \in W\}.$$

Go to Step 2.

4. A Numeric Example

Let us give a following example to show how the *K*th-best approach works.

Example 1

Consider a following linear BLMFP problem with $x \in R^1$, $y_1, y_2 \in R^1$, $z \in R^1$ and $X = \{x \ge 0\}$, $Y = \{y_1 \ge 0, y_2 \ge 0\}$, $Z = \{z \ge 0\}$

$$\min_{x \in X} F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$

subject to $x \le 1$

$$\min_{y_1 \in Y, z \in Z} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to $y_1 \le 1$

$$\min_{y_2 \in Y, z \in Z} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to $y_2 \le 1$

$$z \le 1 .$$

The followers share the variable z. According to the way of model transformation, (1), (2) and (3), we have as follows:

$$\min_{x \in X} F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$

subject to $x \leq 1$

$$\min_{y_1 \in Y} f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$

subject to $y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$.

According to the extended *K*th-best approach, the transferred form of Example can be rewritten as follow in the format of (8),

$$\min F(x, y_1, y_2, z) = -8x + y_1 + 2y_2 - z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x \geq 0, y_1 \geq 0, y_2 \geq 0, z \geq 0$.

Step 1, set $j = 1$, and solve the above problem with the simplex method to obtain the optimal solution $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,0,0,1)$. Let $W = \{(1,0,0,1)\}$ and $T = \phi$. Go to Step 2.

Loop 1:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$. Because of $\tilde{y}_{1j} \neq y_{1[j]}$, we go to Step 3. We have: $W_{[j]} = \{(1,0,1,1), (1,1,0,1), (1,0,0,0), (0,0,0,1)\}$, $T = \{(1,0,0,1)\}$ and $W = \{(1,0,1,1), (1,1,0,1), (1,0,0,0), (0,0,0,1)\}$, then go to Step 4. Update $j = 2$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,1,0,1)$, then go to Step 2.

Loop 2:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \le 1$

$x = 1$

$y_1 \ge 0$

$y_2 \ge 0$

$z \ge 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$ and $\tilde{y}_{1j} = y_{1[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to $x \le 1$

$y_1 \le 1$

$y_2 \le 1$

$z \le 1$

$x = 1$

$y_1 \ge 0$

$y_2 \ge 0$

$z \ge 0$.

Using the bounded simplex method, we have $\tilde{y}_{2j} = 1$. Because of $\tilde{y}_{2j} \ne \tilde{y}_{2[j]}$, we go to Step 3. We have $W_{[j]} = \{(0,1,0,1), (1,1,1,1), (1,0,0,0), (1,1,0,0)\}$, $T = \{(1,0,0,1), (1,1,0,1)\}$ and

$W = \{(0,0,0,1), (1,0,0,0), (1,0,1,1), (0,1,0,1), (1,1,1,1), (1,1,0,0)\}$, then go to Step 4. Update $j = j+1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,0,0,0)$,

Loop 3:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

$$\text{subject to } \quad x \le 1$$
$$y_1 \le 1$$
$$y_2 \le 1$$
$$z \le 1$$
$$x = 1$$
$$y_1 \ge 0$$
$$y_2 \ge 0$$
$$z \ge 0 .$$

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$. Because of $\tilde{y}_{1j} \ne y_{1[j]}$, we go to Step 3. We have: $W_{[j]} = \{(1,0,1,0),(0,0,0,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0)\}$ and $W = \{(0,0,0,1),(1,0,1,1),(0,1,0,1),(1,1,1,1),(1,1,0,0),(1,0,1,0),(0,0,0,0)\}$, then go to Step 4. Update $j = j + 1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,0,1,1)$, then go to Step 2.

Loop 4:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

$$\text{subject to } \quad x \le 1$$
$$y_1 \le 1$$
$$y_2 \le 1$$
$$z \le 1$$
$$x = 1$$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$. Because of $\tilde{y}_{1j} \neq y_{1[j]}$, we go to Step 3. We have: $W_{[j]} = \{(0,0,1,1)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1)\}$ and $W = \{(0,0,0,1),(0,1,0,1),(1,1,1,1),(1,1,0,0),(1,0,1,0),(0,0,0,0),(0,0,1,1\}$, then go to Step 4. Update $j = j+1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,1,0,0)$, then go to Step 2.

Loop 5:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$ and $\tilde{y}_{1j} = y_{1[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to $x \le 1$

$y_1 \le 1$

$y_2 \le 1$

$z \le 1$

$x = 1$

$y_1 \ge 0$

$y_2 \ge 0$

$z \ge 0$.

Using the bounded simplex method, we have $\tilde{y}_{2j} = 1$. Because of $\tilde{y}_{2j} \ne \tilde{y}_{2[j]}$, we go to Step 3. We have

$W_{[j]} = \{(0,1,0,0),(1,1,1,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1),(1,1,0,0)\}$ and $W = \{(0,0,0,1),(0,1,0,1),(1,1,1,1),(1,0,1,0),(0,0,0,0),(0,0,1,1\},(0,1,0,0),(1,1,1,0)\}$, then go to Step 4. Update $j = j+1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,1,1,1)$, then go to Step 2.

Loop 6:

Setting $i \leftarrow 1$ and by (10), we have

min $f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$

subject to $x \le 1$

$y_1 \le 1$

$y_2 \le 1$

$z \le 1$

$x = 1$

$y_1 \ge 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$ and $\tilde{y}_{1j} = y_{1[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{2j} = 1$ and $\tilde{y}_{2j} = \tilde{y}_{2[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{z}_j = 0$. Because of $\tilde{z}_j \neq z_{[j]}$, we go to Step 3. We have

$W_{[j]} = \{(0,1,1,1)\}$, $T = \{(1,0,0,1), (1,1,0,1), (1,0,0,0), (1,0,1,1), (1,1,0,0), (1,1,1,1)\}$ and $W = \{(0,0,0,1), (0,1,0,1), (1,0,1,0), (0,0,0,0), (0,0,1,1\}, (0,1,0,0), (1,1,1,0), (0,1,1,1)\}$, then go to Step 4. Update $j = j+1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,0,1,0)$, then go to Step 2.

Loop 7:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$. Because of $\tilde{y}_{1j} \neq y_{1[j]}$, we go to Step 3. We have:

$W_{[j]} = \{(0,0,1,0)\}$, $T = \{(1,0,0,1),(1,1,0,1),(1,0,0,0),(1,0,1,1),(1,1,0,0),(1,1,1,1),(1,0,1,0)\}$ and $W = \{(0,0,0,1),(0,1,0,1),(0,0,0,0),(0,0,1,1\},(0,1,0,0),(1,1,1,0),(0,1,1,1),(0,0,1,0)\}$, then go to Step 4. Update $j = j+1$, and choose $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,1,1,0)$, then go to Step 2.

Loop 8:

Setting $i \leftarrow 1$ and by (10), we have

$$\min f_1(x, y_1, y_2, z) = x - 2y_1 + y_2 + z$$

$$\begin{aligned} \text{subject to} \quad & x \le 1 \\ & y_1 \le 1 \\ & y_2 \le 1 \\ & z \le 1 \\ & x = 1 \\ & y_1 \ge 0 \\ & y_2 \ge 0 \\ & z \ge 0\,. \end{aligned}$$

Using the bounded simplex method, we have $\tilde{y}_{1j} = 1$ and $\tilde{y}_{1j} = y_{1[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{y_2 \in Y} f_2(x, y_1, y_2, z) = x + y_1 - 2y_2 + z$$

$$\begin{aligned} \text{subject to} \quad & x \le 1 \\ & y_1 \le 1 \\ & y_2 \le 1 \\ & z \le 1 \end{aligned}$$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{y}_{2j} = 1$ and $\tilde{y}_{2j} = \tilde{y}_{2[j]}$. Setting $i \leftarrow i+1$ and by (10), we have

$$\min_{z \in Z} f_3(x, y_1, y_2, z) = 2x - y_1 - y_2 + 2z$$

subject to $x \leq 1$

$y_1 \leq 1$

$y_2 \leq 1$

$z \leq 1$

$x = 1$

$y_1 \geq 0$

$y_2 \geq 0$

$z \geq 0$.

Using the bounded simplex method, we have $\tilde{z}_j = 1$ and $\tilde{z}_j = z_{[j]}$. Solution $(x_{[j]}, y_{1[j]}, y_{2[j]}, z_{[j]}) = (1,1,1,0)$ is the global solution to Example .

By examining above procedure, we found that the solution occurs at the point $(x^*, y_1^*, y_2^*, z^*) = (1,1,1,0)$ with $F^* = -5$, $f_1^* = 0$ and $f_2^* = 0$ for the Example .

5. Conclusion and further study

This paper proposes the *K*th-best approach for linear bilevel multifollower programming problems with shared variables among followers. A numeric example is given to show how the *K*th-best approach works. The further study of the research is to integrate this method into decision support system (DSS) technology.

References

[1] H. Von Stackelberg, The Theory of the Market Economy. Oxford University Press, New York, Oxford, 1952.

[2] J. Bard, Practical Bilevel Optimization: Algorithms and Applications. Kluwer Academic Publishers, USA 1998.

[3] W. Candler and R. Townsley, A linear two-level programming problem. Computers and Operations Research, vol. 9, pp. 59-76, 1982.

[4] W. Bialas and M. Karwan, Two-level linear programming. Management Science, vol. 30, pp. 1004-1020, 1984.

[5] J. Bard and J. Falk, An explicit solution to the multi-level programming problem. Computers and Operations Research, vol. 9, pp. 77-100 (1982).

[6] W. Bialas and M. Karwan, Multilevel linear programming. Technical Report 78-1, State University of New York at Buffalo, Operations Research Program 1978.

[7] P. Hansen, B. Jaumard, and G. Savard, A new branch-and-bound rules for linear bilevel programming. SIAM Journal on Scientific and Statistical Computing, vol. 13, pp. 1194-1217, 1992.

[8] W. Bialas, M. Karwan, and J. Shaw, A parametric complementary pivot approach for two-level linear programming. Technical Report 80-2, State University of New York at Buffalo, Operations Research Program, 1980.

[9] E. Aiyoshi and K. Shimizu, Hierarchical decentralized systems and its new solution by a barrier method. IEEE Transactions on Systems, Man, and Cybernetics, vol. 11, pp. 444-449, 1981.

[10] D. White and G. Anandalingam, A penalty function approach for solving bi-level linear programs. Journal of Global Optimization, vol. 3, pp. 397-419, 1993.

[11] C. Shi, G. Zhang and J. Lu, On the definition of linear bilevel programming solution. *Applied Mathematics and Computation*, vol.160, pp 169-176, 2005

[12] C. Shi, G. Zhang and J. Lu (2005), An extended *K*th-best approach for linear bilevel programming, *Applied Mathematics and Computation*, 164, 843-855.

[13] C. Shi, J. Lu, and G. Zhang (2005), An extended Kuhn-Tucker approach for linear bilevel programming, *Applied Mathematics and Computation* 162, 51-63.

[14] C. Shi, G. Zhang, J. Lu and H. Zhou (2005), An extended branch and bound algorithm for linear bilevel programming. In press, *Applied Mathematics and Computation*.

[15] J. Lu, C. Shi, and G. Zhang, "Model and algorithm for bilevel multi-follower programming with individual variables and shared information among followers," accepted by *Journal of Global Optimization*, 2005.

[16] C. Shi, G. Zhang and J. Lu (2004), A Kth-best Approach for Linear Bilevel Multi-follower Programming. *Journal of Global Optimization*.

[17] C. Shi, J, Lu and G. Zhang (2004), A branch and bound algorithm for linear bilevel multi-follower programming. In press, *Journal of the Operational Research Society*.

[18] C. Shi, J. Lu, G. Zhang and H, Zhou "An extended Kuhn-Tucker approach for linear bilevel multifollower programming with partial shared among followers," submitted to IEEE SMC'05, The Big Island, Hawaii, USA, 2005.

[19] University of Cambridge, http://thesaurus.maths.org/dictionary/map/word/10037, 2001.