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M-TEST OF TWO PARALLEL REGRESSION LINES UNDER UNCERTAIN PRIOR INFORMATION

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ABSTRACT

This paper considers the problem of testing the intercepts of two simple linear models following a pre-test on the suspected equality of slopes. The unrestricted test (UT), restricted test (RT) and pre-test test (PTT) are proposed from the M-tests using the M-estimation methodology. The asymptotic power functions of the UT, RT and PTT are given. The computational comparisons of power function of the three tests are provided. The PTT achieves a reasonable dominance over the other two tests asymptotically.

Keywords: Two parallel regression lines, pre-test, asymptotic power and size, M-estimation, local alternative hypothesis, bivariate non-central chi-square.

1. INTRODUCTION

A researcher may model independent data sets from two random samples for two separate groups of respondents. Often, the researcher may wish to know whether the regression lines for the two groups are parallel (i.e. slopes of the two regression lines are equal) or whether the lines have the same intercept on vertical-axis. An interesting situation would be if the researcher decides to test the equality of the intercepts when the equality of slopes is suspected, but not sure.

Data for this problem can be represented by the following two simple linear regression equations

$$\boldsymbol{X}_{1_{n_1}} = \theta_1 \mathbf{1}_{n_1} + \beta_1 \boldsymbol{c}_1 + \boldsymbol{\varepsilon}_1 \text{ and } \boldsymbol{X}_{2_{n_2}} = \theta_2 \mathbf{1}_{n_2} + \beta_2 \boldsymbol{c}_2 + \boldsymbol{\varepsilon}_2. \tag{1.1}$$

For the first data set, $\boldsymbol{X}_{1_{n_1}} = (X_{1_1}, \dots, X_{1_{n_1}})'$ is a vector of n_1 observable response random variables, $\mathbf{1}_{n_1} = (1, 1, \dots, 1)'$, is an n_1 tuple of 1's, $\boldsymbol{c}_1 = (c_{1_1}, \dots, c_{1_{n_1}})'$ is a vector of n_1 independent variables, θ_1 and β_1 are the unknown intercept and slope parameters respectively. For the second data set, $\boldsymbol{X}_{2_{n_2}} = (X_{2_1}, \dots, X_{2_{n_2}})'$ is a vector of n_2 observable response random variables, $\mathbf{1}_{n_2} = (1, 1, \dots, 1)'$, is an n_2 tuple of 1's, $\boldsymbol{c}_2 = (c_{2_1}, \dots, c_{2_{n_2}})'$ is a vector of

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n_2 independent variables, θ_2 and β_2 are the unknown intercept and slope parameters respectively. Assume the error $\varepsilon_{j_i} = X_{j_i} - \theta_j - \beta_j c_{j_i}$, for $i = 1, \ldots, n_j$ and $j = 1, 2$ are mutually independent and identically distributed with cdf F.

The researcher may wish to test the intercept vector $\boldsymbol{\theta} = (\theta_1, \theta_2)'$ of the two regression lines equal to a fixed vector $\boldsymbol{\theta}_0 = (\theta_{01}, \theta_{02})'$ while it not sure if the two slope parameters are equal. In this situation, three different scenarios associated with the value of the slopes are considered: the value of the slopes would either be (i) completely unspecified, (ii) equal at an arbitrary constant, β_0, or (iii) suspected to be equal at an arbitrary constant, β_0. The unrestricted test (UT), the restricted test (RT) and the pre-test test (PTT) are defined respectively for the three scenarios of the slope parameters. Thus, the UT is for testing $H_0^{(1)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_A^{(1)} : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ when $\boldsymbol{\beta} = (\beta_1, \beta_2)'$ is unspecified, the RT is for testing $H_0^{(1)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_A^{(1)} : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ when $\boldsymbol{\beta} = \beta_0 \mathbf{1}_2$ (fixed vector) and the PTT is for testing $H_0^{(1)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_A^{(1)} : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ after pre-testing $H_0^\star : \boldsymbol{\beta} = \beta_0 \mathbf{1}_2$ against $H_A^\star : \boldsymbol{\beta} > \beta_0 \mathbf{1}_2$ (to remove the uncertainty). The PTT is a choice between the UT and the RT. If the null hypothesis $H_0^\star$ is rejected in the pre-test (PT), then the UT is used, otherwise the RT is used.

The inclusion of non-sample prior information (NSPI) in the parameter estimation usually improve the quality of an estimator. In many cases, the prior information is available as a suspected value of the parameter interest. However, such a prior value is likely to be uncertain. This has led to the suggestion of pre-testing the suspected value to remove the uncertainty. The idea of pre-testing by Bancroft (1964) arouses a number of studies in literature. Akritas et al. (1984), Lambert et al. (1985a) and Khan (2003) are among authors who considered the problem of estimating the intercepts parameters when it is *apriori* suspected that the regression lines are parallel.

In literature, the effects of pre-testing on the performance of the ultimate test are studied for some parametric cases by Bechhofer (1951), Bozivich et al. (1956) and Mead et al. (1973). For nonparametric cases, Tamura (1965) investigated the performance of the PTT for one sample and two sample problem while Saleh and Sen (1982, 1983) developed the PTT for the simple linear model and multivariate simple model using nonparametric rank tests. Lambert et al. (1985b) used least-squares (LS) based tests to propose the UT, RT and PTT for the parallelism model. However, LS estimates are non robust with respect to deviation from the assumed (normal) distribution (c.f. Jurečková and Sen, 1996, p.21), so, it is suspected that the UT, RT and PTT defined using the LS based tests are also non robust. In this paper, the M-test which is originally proposed by Sen (1982) to test the significance of the slope is used to define the UT, RT and PTT. Recently, Yunus and Khan (2010) used M-tests to define the UT, RT and PTT for the simple linear regression model.

The comparison of the power of the UT, RT and PTT are studied by Lambert et al. (1985b) analytically. The cdf of the bivariate noncentral chi-square distribution is required to compute the power of the PTT. The bivariate noncentral chi-square distribution function used in their paper is complicated and not practical for computation, so there is no graphical representation on the comparison of the power of the tests provided in their paper. In this paper, Yunus and Khan (2009) is referred for the computation of the cdf of the bivariate noncentral chi-square distribution.

Along with some preliminary notions, the UT, RT and PTT are proposed in Sections 2. In Section 3, the asymptotic power functions for UT, RT and PTT are given. The graphical representation is given in Section 4. The final Section contains comments and conclusion.

2. THE UT, RT AND PTT

2.1 THE UNRESTRICTED TEST (UT)

If $\boldsymbol{\beta}$ is unspecified, ϕ_n^{UT} is the test function of $H_0^{(1)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_A^{(1)} : \boldsymbol{\theta} > \boldsymbol{\theta}_0$. We consider the test statistic

$$\left[T_n^{UT} = n^{-1} \frac{\boldsymbol{M}_{n_1}(\boldsymbol{\theta}_0, \tilde{\boldsymbol{\beta}})' \boldsymbol{\Lambda}_0^{\star -1} \boldsymbol{M}_{n_1}(\boldsymbol{\theta}_0, \tilde{\boldsymbol{\beta}})}{{S_n^{(1)}}^2}, \right]$$

where $\tilde{\boldsymbol{\beta}} = \frac{1}{2} [\sup\{\boldsymbol{b} : \boldsymbol{M}_{n_2}(\boldsymbol{\theta}_0, \boldsymbol{b}) > 0\} + \inf\{\boldsymbol{b} : \boldsymbol{M}_{n_2}(\boldsymbol{\theta}_0, \boldsymbol{b}) < 0\}]$ is a constrained M-estimator of $\boldsymbol{\beta}$ under $H_0^{(1)}$. For $\boldsymbol{a} = (a_1, a_2)'$ and $\boldsymbol{b} = (b_1, b_2)'$, vectors of real numbers a_j and b_j, $j = 1,\ 2$, $\boldsymbol{M}_{n_1}(\boldsymbol{a}, \boldsymbol{b}) = \left(M_{n_1}^{(1)}(a_1, b_1), M_{n_1}^{(2)}(a_2, b_2) \right)'$ and $\boldsymbol{M}_{n_2}(\boldsymbol{a}, \boldsymbol{b}) = \left(M_{n_2}^{(1)}(a_1, b_1), M_{n_2}^{(2)}(a_2, b_2) \right)'$ where

$$\begin{aligned} M_{n_1}^{(j)}(a_j, b_j) &= \sum_{i=1}^{n_j} \psi \left(\frac{X_{j_i} - a_j - b_j c_{j_i}}{S_n} \right) \quad \text{and} \\ M_{n_2}^{(j)}(a_j, b_j) &= \sum_{i=1}^{n_j} c_{j_i} \psi \left(\frac{X_{j_i} - a_j - b_j c_{j_i}}{S_n} \right). \end{aligned}$$

Here, S_n is an appropriate scale statistic for some functional $S = S(F) > 0$ and ψ is the score function in the M-estimation methodology. Note, $\boldsymbol{\Lambda}_0^\star = \text{Diag}\left(\frac{\lambda_1 {C_1^\star}^2}{{C_1^\star}^2 + \bar{c}_1^2}, \frac{\lambda_2 {C_2^\star}^2}{{C_2^\star}^2 + \bar{c}_2^2} \right)$, where $\lambda_j = \lim_{n\to\infty} \frac{n_j}{n}$ $(0 < \lambda_j < 1)$ with $n = n_1 + n_2$. Also, $\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n_j} c_{j_i} = \lambda_j \bar{c}_j$ $(|\bar{c}_j| < \infty)$ and $\lim_{n\to\infty} n^{-1} {C_{n_j}^\star}^2 = \lambda_j {C_j^\star}^2$, where ${C_{n_j}^\star}^2 = \sum_{i=1}^{n_j} c_{j_i}^2 - n_j \bar{c}_{n_j}^2$ and $\bar{c}_{n_j} = n_j^{-1} \sum_{i=1}^{n_j} c_{j_i}$. Let ${S_n^{(1)}}^2 = \frac{1}{n} \sum_{j=1}^2 \sum_{i=1}^{n_j} \psi^2 \left(\frac{X_{i_j} - \theta_{0j} - \tilde{\beta}_j c_{j_i}}{S_n} \right)$. The asymptotic results in Yunus and Khan (2010) is adapted for the parallelism model. Thus, we find T_n^{UT} is χ_2^2 (chi-square distribution with 2 degrees of freedom).

Let ℓ_{n,α_1}^{UT} be the critical value of T_n^{UT} at the α_1 level of significance. So, for the test function $\phi_n^{UT} = I(T_n^{UT} > \ell_{n,\alpha_1}^{UT})$, the power function of the UT becomes $\Pi_n^{UT}(\boldsymbol{\theta}) = E(\phi_n^{UT} | \boldsymbol{\theta}) = P(T_n^{UT} > \ell_{n,\alpha_1}^{UT} | \boldsymbol{\theta})$, where $I(A)$ is an indicator function of the set A. It takes value 1 if A occurs, otherwise it is 0.

2.2 THE RESTRICTED TEST (RT)

If $\boldsymbol{\beta} = \beta_0 \mathbf{1}_2$, the test function for testing $H_0^{(1)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_A^{(1)} : \boldsymbol{\theta} > \boldsymbol{\theta}_0$ is ϕ_n^{RT}. The proposed test statistic is

$$\left[T_n^{RT} = n^{-1} \frac{\boldsymbol{M}_{n_1}(\boldsymbol{\theta}_0, \beta_0 \mathbf{1}_2)' \boldsymbol{\Lambda}_0^{-1} \boldsymbol{M}_{n_1}(\boldsymbol{\theta}_0, \beta_0 \mathbf{1}_2)}{{S_n^{(2)}}^2}, \right]$$

where ${S_n^{(2)}}^2 = \frac{1}{n} \sum_{j=1}^{2} \sum_{i=1}^{n_j} \psi^2 \left(\frac{X_{i_j} - \theta_{0j} - \beta_0 c_{j_i}}{S_n} \right)$ and $\boldsymbol{\Lambda}_0 = \text{Diag}(\lambda_1, \lambda_2)$. Again, using the asymptotic results in Yunus and Khan (2010) and adapt them for use in the parallelism model, we obtain for large n, T_n^{RT} is χ_2^2 under $H_0^{(2)} : \boldsymbol{\theta} = \boldsymbol{\theta}_0, \boldsymbol{\beta} = \beta_0 \mathbf{1}_2$. Again, let ℓ_{n,α_2}^{RT} be the critical value of T_n^{RT} at the α_2 level of significance. So, for the test function $\phi_n^{RT} = I(T_n^{RT} > \ell_{n,\alpha_2}^{RT})$, the power function of the RT becomes $\Pi_n^{RT}(\boldsymbol{\theta}) = E(\phi_n^{RT} | \boldsymbol{\theta}) = P(T_n^{RT} > \ell_{n,\alpha_2}^{RT} | \boldsymbol{\theta})$.

2.3 THE PRE-TEST (PT)

For the pre-test on the slope, the test function of $H_0^\star : \boldsymbol{\beta} = \beta_0 \mathbf{1}_2$ against $H_A^\star : \boldsymbol{\beta} > \beta_0 \mathbf{1}_2$ is ϕ_n^{PT}. The proposed test statistic is

$$\left[T_n^{PT} = n^{-1} \frac{\boldsymbol{M}_{n_2}(\tilde{\boldsymbol{\theta}}, \beta_0 \mathbf{1}_2)' \boldsymbol{\Lambda}_2^{\star -1} \boldsymbol{M}_{n_2}(\tilde{\boldsymbol{\theta}}, \beta_0 \mathbf{1}_2)}{{S_n^{(3)}}^2}, \right]$$

where $\tilde{\boldsymbol{\theta}} = \frac{1}{2} [\sup\{\boldsymbol{a} : \boldsymbol{M}_{n_1}(\boldsymbol{a}, \beta_0 \mathbf{1}_2) > 0\} + \inf\{\boldsymbol{a} : \boldsymbol{M}_{n_1}(\boldsymbol{a}, \beta_0 \mathbf{1}_2) < 0\}]$ is a constrained M-estimator of $\boldsymbol{\theta}$ and ${S_n^{(3)}}^2 = \frac{1}{n} \sum_{j=1}^{2} \sum_{i=1}^{n_j} \psi^2 \left(\frac{X_{i_j} - \tilde{\theta}_j - \beta_0 c_{j_i}}{S_n} \right)$ and $\boldsymbol{\Lambda}_2^\star = \text{Diag}\,(\lambda_1 C_1^{\star 2}, \lambda_2 C_2^{\star 2})$. It follows that as $n \to \infty$, $T_n^{PT} \xrightarrow{d} \chi_2^2$ under $H_0^\star$.

2.4 THE PRE-TEST-TEST (PTT)

We are now in the position to formulate ϕ_n^{PTT} for testing $H_0^{(1)}$ following a pre-test on $\boldsymbol{\beta}$. Since the PTT is a choice between RT and UT, define,

$$\phi_n^{PTT} = I[(T_n^{PT} < \ell_{n,\alpha_3}^{PT}, T_n^{RT} > \ell_{n,\alpha_2}^{RT}) \text{ or } (T_n^{PT} > \ell_{n,\alpha_3}^{PT}, T_n^{UT} > \ell_{n,\alpha_1}^{RT})], \tag{2.1}$$

where ℓ_{n,α_3}^{PT} is the critical value of T_n^{PT} at the α_3 level of significance. The power function of the PTT is given by

$$\Pi_n^{PTT}(\boldsymbol{\theta}) = E(\phi_n^{PTT} | \boldsymbol{\theta}) \tag{2.2}$$

and the size of the PTT is obtained by substituting $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ in equation (2.2).

3. ASYMPTOTIC POWER FUNCTIONS

Let $\{K_n^\star\}$ be a sequence of alternative hypotheses, where

$$K_n^\star : (\boldsymbol{\theta}, \boldsymbol{\beta}) = (\boldsymbol{\theta}_0 + n^{-\frac{1}{2}}\boldsymbol{\delta}_1, \beta_0 \mathbf{1}_2 + n^{-\frac{1}{2}}\boldsymbol{\delta}_2), \tag{3.1}$$

with $\boldsymbol{\delta}_1 = n^{\frac{1}{2}}(\boldsymbol{\theta} - \boldsymbol{\theta}_0) > \mathbf{0}$ and $\boldsymbol{\delta}_2 = n^{\frac{1}{2}}(\boldsymbol{\beta} - \beta_0\mathbf{1}_2) > \mathbf{0}$. Here, $\boldsymbol{\delta}_1 = (\delta_{1_1}, \delta_{1_2})'$, $\boldsymbol{\delta}_2 = (\delta_{2_1}, \delta_{2_2})'$ are vectors of fixed real numbers.

Under $\{K_n^\star\}$, for large sample, asymptotically (T_n^{RT}, T_n^{PT}) are independently distributed as bivariate non-central chi-square distribution with 2 degrees of freedom and (T_n^{UT}, T_n^{PT}) are distributed as correlated bivariate non-central chi-square distribution with 2 degrees of freedom and non-centrality parameters,

$$\begin{aligned}
\theta^{UT} &= (\gamma \boldsymbol{\Lambda}_0^\star \boldsymbol{\delta}_1)' \, \boldsymbol{\Lambda}_0^{\star -1} (\gamma \boldsymbol{\Lambda}_0^\star \boldsymbol{\delta}_1) / \sigma_0^2 && (3.2)\\
\theta^{RT} &= [\gamma(\boldsymbol{\Lambda}_0 \boldsymbol{\delta}_1 + \boldsymbol{\Lambda}_{12}\boldsymbol{\delta}_2)]' \, \boldsymbol{\Lambda}_0^{-1} [\gamma(\boldsymbol{\Lambda}_0 \boldsymbol{\delta}_1 + \boldsymbol{\Lambda}_{12}\boldsymbol{\delta}_2)] / \sigma_0^2, && (3.3)\\
\theta^{PT} &= (\gamma \boldsymbol{\Lambda}_2^\star \boldsymbol{\delta}_2)' \, \boldsymbol{\Lambda}_2^{\star -1} (\gamma \boldsymbol{\Lambda}_2^\star \boldsymbol{\delta}_2) / \sigma_0^2, && (3.4)
\end{aligned}$$

where $\boldsymbol{\Lambda}_{12} = \mathrm{Diag}(\lambda_1 \bar{c}_1, \lambda_2 \bar{c}_2)$, $\sigma_0^2 = \int_{-\infty}^{\infty} \psi^2(z/S) dF(z)$ and $\gamma = \int_{-\infty}^{\infty} \frac{1}{S}\psi'(z/S) dF(z)$.

Thus, under $\{K_n^\star\}$, the asymptotic power functions for the UT, RT and PT which are denoted by $\Pi^h(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)$ for h any of the UT, RT and PT, are defined as

$$\Pi^h(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \lim_{n\to\infty} \Pi_n^h(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2) = \lim_{n\to\infty} P(T_n^h > \ell_{n,\alpha_\nu}^h | K_n^\star) = 1 - G_2(\chi^2_{2,\alpha_\nu}; \theta^h), \tag{3.5}$$

where is $G_2(\chi^2_{2,\alpha_\nu}, \theta^h)$ is the cdf of a non-central chi-square distribution with 2 degre es of freedom and non-centrality parameter θ^h. The level of significance, $\alpha_\nu,\ \nu = 1, 2, 3$ are chosen together with the critical values ℓ_{n,α_ν}^h for the UT, RT and PT. Here, $\chi^2_{2,\alpha}$ is the upper $100\alpha\%$ critical value of a central chi-square distribution and $\ell_{n,\alpha_1}^{UT} \to \chi^2_{2,\alpha_1}$ under $H_0^{(1)}$, $\ell_{n,\alpha_2}^{RT} \to \chi^2_{2,\alpha_2}$ under $H_0^{(2)}$ and $\ell_{n,\alpha_3}^{PT} \to \chi^2_{2,\alpha_3}$under $H_0^\star$.

For testing $H_0^{(1)}$ following a pre-test on $\boldsymbol{\beta}$, using equation (2.1), the asymptotic power function for the PTT under $\{K_n^\star\}$ is given as

$$\begin{aligned}
&\Pi^{PTT}(\boldsymbol{\delta}_1, \boldsymbol{\delta}_2)\\
=\ & \lim_{n\to\infty} P(T_n^{PT} \le \ell_{n,\alpha_3}^{PT}, T_n^{RT} > \ell_{n,\alpha_2}^{RT} | K_n^\star) + \lim_{n\to\infty} P(T_n^{PT} > \ell_{n,\alpha_3}^{PT}, T_n^{UT} > \ell_{n,\alpha_1}^{UT} | K_n^\star)\\
=\ & G_2(\chi^2_{2,\alpha_3}; \theta^{PT})\{1 - G_2(\chi^2_{2,\alpha_2}; \theta^{RT})\} + \int_{\chi^2_{2,\alpha_1}} \int_{\chi^2_{2,\alpha_3}} \phi^\star(w_1, w_2) dw_1 dw_2,
\end{aligned} \tag{3.6}$$

where $\phi^\star(\cdot)$ is the density function of a bivariate non-central chi-square distribution with 2 degrees of freedom, non-centrality parameters, θ^{UT} and θ^{PT} and correlation coefficient

$-1 < \rho < 1$. The probability integral in (3.6) is given by

$$
\begin{aligned}
& \int_{\chi^2_{2,\alpha_1}} \int_{\chi^2_{2,\alpha_3}} \phi^\star(w_1, w_2) dw_1 dw_2 \\
= & \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa_1=0}^{\infty}\sum_{\kappa_2=0}^{\infty} (1-\rho^2)^p \frac{\Gamma(1+j)}{\Gamma(1)j!}\frac{\Gamma(1+k)}{\Gamma(1)k!}\rho^{2(j+k)} \\
& \times \left[1-\gamma^\star\left(1+j+\kappa_1, \frac{\chi^2_{2,\alpha_1}}{2(1-\rho^2)}\right)\right]\left[1-\gamma^\star\left(1+k+\kappa_2, \frac{\chi^2_{2,\alpha_3}}{2(1-\rho^2)}\right)\right] \\
& \times \frac{e^{-\theta^{UT}/2}(\theta^{UT}/2)^{\kappa_1}}{\kappa_1!}\frac{e^{-\theta^{PT}/2}(\theta^{PT}/2)^{\kappa_2}}{\kappa_2!}. \qquad (3.7)
\end{aligned}
$$

Let $\rho^2 = \sum_{j=1}^{2} \frac{1}{2}\rho_j^2$ be the mean correlation, where $\rho_j = -c_j/\sqrt{C_j^{\star 2} + \bar{c}_j^2}$ is the correlation coefficient between $\left(M_{n_1}^{(j)}(\theta_{0_j}, \tilde{\beta}_j), M_{n_2}^{(j)}(\tilde{\theta}_j, \beta_0)\right)$. Here, $\gamma^\star(v, d) = \int_0^d x^{v-1}\, e^{-x}/\Gamma(v)dx$ is the incomplete gamma function. For details on the evaluation of the bivariate integral, see Yunus and Khan (2009). The density function of the bivariate noncentral chi-square distribution given above is a mixture of the bivariate central chi-square distribution of two central chi-square random variables (see Gunst and Webster, 1973, Wright and Kennedy, 2002), with the probabilities from the Poisson distribution. Krishnaiah et al., 1963, Gunst and Webster, 1973 and Wright and Kennedy, 2002)

4. ILLUSTRATIVE EXAMPLE

The power functions given in equations (3.5) and (3.6) are computed for graphical view. The non-centrality parameters for UT, RT and PT respectively are

$$
\begin{aligned}
\theta^{UT} &= \begin{bmatrix} \xi_{1_1}\lambda_1 \frac{C_1^{\star 2}}{C_1^{\star 2}+\bar{c}_1^2} \\ \xi_{1_2}\lambda_2 \frac{C_2^{\star 2}}{C_2^{\star 2}+\bar{c}_2^2} \end{bmatrix}' \begin{bmatrix} \lambda_1 \frac{C_1^{\star 2}}{C_1^{\star 2}+\bar{c}_1^2} & 0 \\ 0 & \lambda_2 \frac{C_2^{\star 2}}{C_2^{\star 2}+\bar{c}_2^2} \end{bmatrix}^{-1} \begin{bmatrix} \xi_{1_1}\lambda_1 \frac{C_1^{\star 2}}{C_1^{\star 2}+\bar{c}_1^2} \\ \xi_{1_2}\lambda_2 \frac{C_2^{\star 2}}{C_2^{\star 2}+\bar{c}_2^2} \end{bmatrix}, \\
\theta^{RT} &= \begin{bmatrix} \xi_{1_1}\lambda_1 + \xi_{2_1}\lambda_1\bar{c}_1 \\ \xi_{1_2}\lambda_2 + \xi_{2_2}\lambda_2\bar{c}_1 \end{bmatrix}' \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{-1} \begin{bmatrix} \xi_{1_1}\lambda_1 + \xi_{2_1}\lambda_1\bar{c}_1 \\ \xi_{1_2}\lambda_2 + \xi_{2_2}\lambda_2\bar{c}_1 \end{bmatrix} \text{ and} \\
\theta^{PT} &= \begin{bmatrix} \xi_{2_1}\lambda_1 C_1^{\star 2} \\ \xi_{2_2}\lambda_2 C_2^{\star 2} \end{bmatrix}' \begin{bmatrix} \lambda_1 C_1^{\star 2} & 0 \\ 0 & \lambda_2 C_2^{\star 2} \end{bmatrix}^{-1} \begin{bmatrix} \xi_{2_1}\lambda_1 C_1^{\star 2} \\ \xi_{2_2}\lambda_2 C_2^{\star 2} \end{bmatrix},
\end{aligned}
$$

where $\xi_{k_l} = \delta_{k_l}\gamma/\sigma_0$ for $k, l = 1, 2$ and $\delta_{1_l} = \sqrt{n}(\theta_l - \theta_{0_l})$ and $\delta_{2_l} = \sqrt{n}(\beta_l - \beta_{0_l})$.

A special case of the two sample problem (Saleh, 2006, p.67) is considered with $n_j = n_{j_1} + n_{j_2}$ for $j = 1, 2$, $n_{j_1}/n_j \to 1 - P$, $c_{j_1} = \ldots = c_{j_{n_1}} = 0$ and $c_{j_{n_1+1}} = \ldots = c_{j_n} = 1$. So $\bar{c}_j = 1 - P$ and $C_j^{\star 2} = P(1 - P)$. In this example, let $P = 0.5$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.05$.

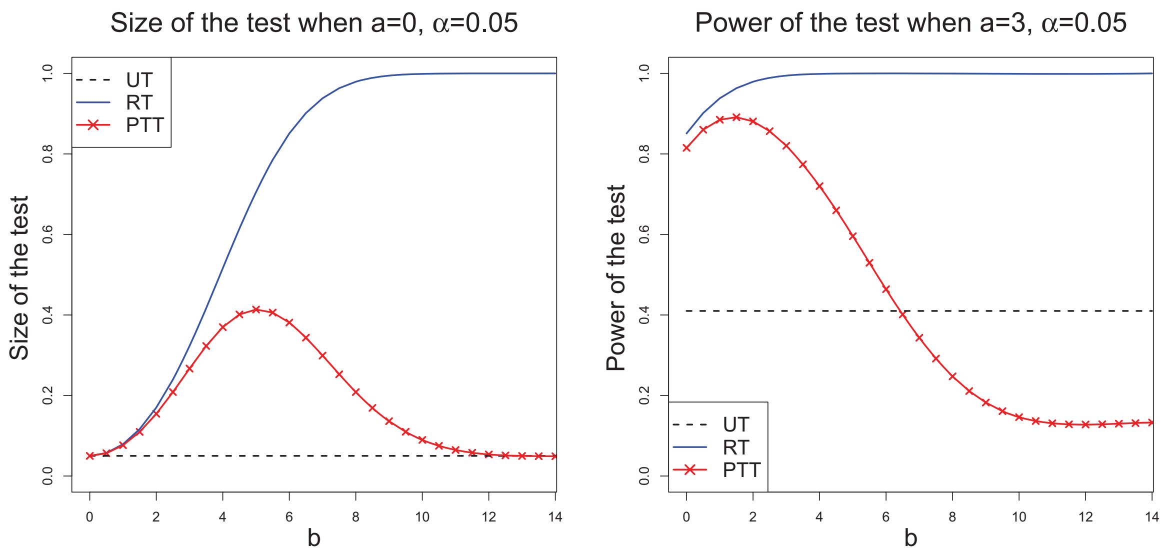

Figure 1: Graphs of power functions as a function of (b) $((= \xi_{2_1} = \xi_{2_2}))$ for selected values of (ξ_{1_1}) and (ξ_{1_2}) with $(\bar{c} > 0)$ and $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.05)$. Here, $(\xi_{k_l} = \delta_{k_l}\gamma/\sigma_0, k, l = 1, 2.)$

Also, let $n_1, n_2 = 50$ so $n = n_1 + n_2 = 100$. As a result, the correlation coefficient, $\rho_j, j = 1, 2$ for both regression lines are the same since $\bar{c}_1^2 = \bar{c}_2^2 = \bar{c}^2$ for both samples, (X_{n_1}, c_1) and (X_{n_2}, c_2) of the two regression lines. Note, in plotting the power functions for the PTT, a bivariate non-central chi-square distribution is used.

Let $\xi_{1_1} = \xi_{1_2} = a$ and $\xi_{2_1} = \xi_{2_2} = b$. Figure 1 shows the power of the test against b at selected values of ξ_{1_1} and ξ_{1_2}. A test with higher size and lower power is a test which makes small probability of Type I and Type II errors. In Figure 1, the UT has the smallest size and the PTT has smaller size than that of the RT. The RT has the largest power as b grows. The PTT has larger power than that of the UT except for large b. In Figure 2, power of the UT, RT and PTT are plotted against a at selected values of ξ_{2_1} and ξ_{2_2}. As a grows large, power of all tests grow large too. Although the power of the UT and RT are increasing to 1 as a is increasing, the power of the PTT is increasing to a value that is less than 1.

Although the NSPI on the slopes parameters may be uncertain, there is a high possibility that the true values are not too far from the suspected values. Therefore, the study on the behaviour of the three tests when the suspected NSPI values is not too far away from that under the null hypothesis is more realistic.

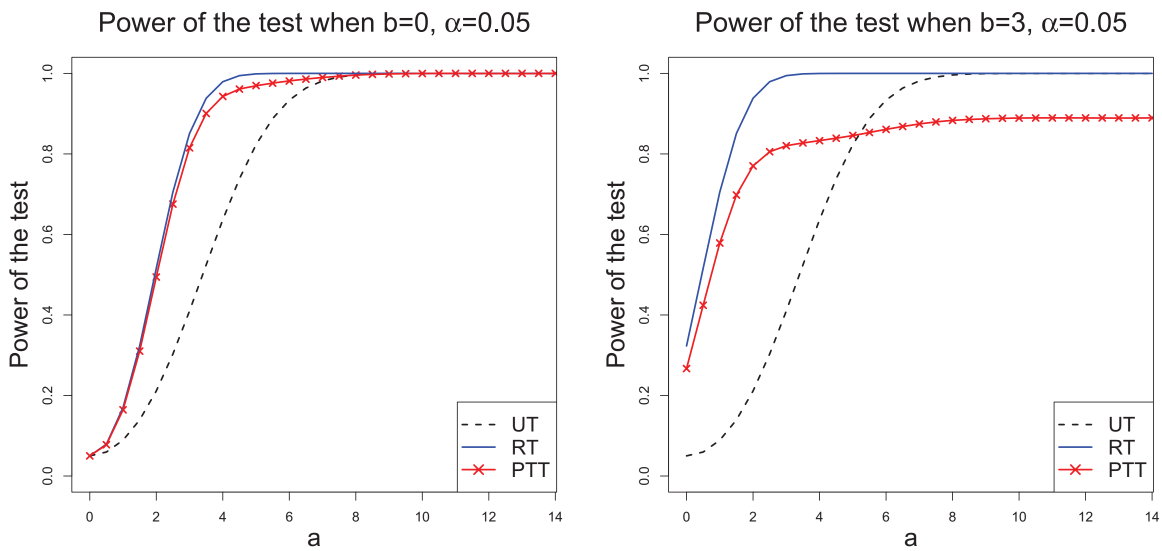

Figure 2: Graphs of power functions as a function of (a) $((= \xi_{1_2} = \xi_{1_2}))$ for selected values of (ξ_{2_1}) and (ξ_{2_2}) with $(\bar{c} > 0)$ and $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha = 0.05)$. Here, $(\xi_{k_l} = \delta_{k_l}\gamma/\sigma_0, k, l = 1, 2.)$

5. COMMENTS AND CONCLUSION

The sampling distributions of the UT, RT and PT follow a univariate noncentral chi-square distribution under the alternative hypothesis when the sample size is large. However, that of the PTT is a bivariate noncentral chi-square distribution as there is a correlation between the UT and PT. Note that there is no such correlation between the RT and PT.

The size of the RT reaches 1 as b (a function of the difference between the true and suspected values of the slopes) increases. This means the RT does not satisfy the asymptotic level constraint, so it is not a valid test. The UT has the smallest constant size; however, it has the smallest power as well, except for very large values of b, that is, when $b > q$, where q is some positive number. Thus, the UT fails to achieve the highest power and lowest size simultaneously. The PTT has a smaller size than the RT and its size does not reach 1 as b increases. It also has higher power than the UT, except for $b > q$.

Therefore, if the prior information is not far away from the true value, that is, b is near 0 (small or moderate), the PTT has a smaller size than the RT and more power than the UT. So, the PTT is a better compromise between the two extremes. Since the prior information comes from previous experience or expert assessment, it is reasonable to expect b should not be too far from 0, although it may not be 0, and hence the PTT achieves a reasonable dominance over the other two tests in a more realistic situation.

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