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SYMMETRY RESTORATION IN COLLISIONS OF SOLITONS IN FRACTIONAL COUPLERS

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Abstract

Recently, we analysed spontaneous symmetry breaking (SSB) of solitons in linearly coupled dual-core waveguides with fractional diffraction and cubic nonlinearity. In a practical context, the system can serve as a model for optical waveguides with the fractional diffraction or Bose–Einstein condensate of particles with Lévy index $\alpha < 2$. In an earlier study, the SSB in the fractional coupler was identified as the bifurcation of subcritical type, becoming extremely subcritical in the limit of $\alpha \to 1$. There, the moving solitons and collisions between them at low speeds were also explored. In the present paper, we present new numerical results for fast solitons demonstrating restoration of symmetry in post-collision dynamics.

2020 *Mathematics subject classification*: primary 35Q55; secondary 35Q51, 35Q60, 35S05, 35S16, 60G65.

Keywords and phrases: fractional diffraction, nonlinear Schrödinger equation, coupled waveguides, fractional derivatives, Lévy index, spontaneous symmetry breaking, subcritical bifurcation, Bose–Einstein condensates.

1. Introduction

For a particle moving by way of Lévy flights, Laskin derived the Schrödinger equation with the kinetic energy operator represented by a fractional derivative [7]. It was assumed that stochastic motion of the respective classical particle in one dimension is characterized by mean distance from the initial position growing with time as $|x| \sim t^{1/\alpha}$,

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where $\alpha < 2$ is the Lévy index (LI) [13]. After rescaling, the Schrödinger equation has the form

$$i\frac{\partial\psi}{\partial t} = \frac{1}{2}\Big(-\frac{\partial^2}{\partial x^2}\Big)^{\alpha/2}\psi, \tag{1.1}$$

where the fractional operator represents the Riesz derivative [1],

$$\Big(-\frac{\partial^2}{\partial x^2}\Big)^{\alpha/2}\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dp|p|^{\alpha}\int_{-\infty}^{+\infty} dx'\, e^{ip(x-x')}\psi(x'). \tag{1.2}$$

The fractional operator is constructed via the direct and inverse Fourier transforms incorporating fractional differentiation. In the limit of $\alpha = 2$, the operator (1.2) reduces to the usual second derivative.

The fractional quantum mechanics description in terms of wave equations was also applied to Lévy crystals [17] and polariton condensates [16]. There were recent reports of experimental realization of the fractional group-velocity dispersion in a fibre-laser cavity [9]. Theoretical models involving the fractional diffraction were extended to situations with external potentials (in particular, parity-time ($\mathcal{PT}$) symmetric ones [23]) and the Airy waves in two dimensions. In optical waveguide models, the fractional diffraction/dispersion is included in conjunction with the self-focusing Kerr nonlinearity leading to the fractional nonlinear Schrödinger equations (FNLSEs) [12].

The analyses of FNLSEs revealed the modulational instability of continuous waves [22], critical or supercritical collapse [2] and other effects. Spontaneous symmetry breaking (SSB) in double-well potentials produced by self-trapping nonlinearity [11] was studied in [3] and experimentally in [4] in various physical contexts. A recent focus of fractional diffraction was in the study of nonlinear systems with symmetric potentials [8].

It is well known that SSB might arise in dual waveguides in the form of two-component solitons [10]. Such a setting is relevant for modelling double-core optical fibres. In fibres, the SSB bifurcation of symmetric solitons into asymmetric two-component solitons was studied theoretically in a number of works including [19], and it was demonstrated in an experiment [14]. In a recent work [21], families of symmetric and asymmetric solitons were found in the double-core system with fractional diffraction.

Recently, Strunin and Malomed [18] identified the SSB bifurcation of two- component solitons in the fractional dual-core waveguide. This was done both analytically, by means of the variational approximation (VA), and by systematic numerical computations. The comparison demonstrated that a relatively simple VA, based on the straightforward sech ansatz, produces quite accurate results. An essential finding was that *deeper* system's fractionality (that is, smaller LI α in (1.2)) enhances the subcritical character [5] of the soliton bifurcation, which is its characteristic feature in the usual (nonfractional) double-core system [10]. Thus, the fractionality makes the SSB of two-component solitons a more strongly pronounced phase transition of the first kind.

In this paper, we address the linearly coupled system of FNLSEs with the cubic self-focusing nonlinearity and the same fractional diffraction as in (1.1),

$$
\begin{aligned}
i\frac{\partial u_1}{\partial t} &= \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u_1 - |u_1|^2 u_1 - u_2,\\
i\frac{\partial u_2}{\partial t} &= \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} u_2 - |u_2|^2 u_2 - u_1,
\end{aligned}
\tag{1.3}
$$

where the coupling coefficient in front of terms $(-u_2)$ and $(-u_1)$ in the first and second equations is set to be 1 by means of scaling. The only irreducible control parameter of the normalized system (1.3) is LI α, and intrinsic parameters of soliton solutions will be the propagation constant k and velocity (tilt) c (see (2.1) and (2.4) below). Throughout the paper, we call solutions with equal components, $u_1 = u_2$, symmetric. Solutions with unequal components are called asymmetric.

The paper is organized as follows: the framework for the construction of soliton solutions, and analysis of their stability and dynamics are presented in Section 2; numerical results, for quiescent and moving solitons, are summarized in Section 3; the conclusion is given in Section 4.

2. Soliton solutions

Stationary solutions to (1.3) with propagation constant k are looked for in the form

$$
u_{1,2}(x,t) = U_{1,2}(x)e^{ikt} \tag{2.1}
$$

with real functions $u_{1,2}(x)$. These solutions must satisfy

$$
\begin{aligned}
kU_1 + \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} U_1 - U_1^3 - U_2 &= 0,\\
kU_2 + \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} U_2 - U_2^3 - U_1 &= 0.
\end{aligned}
\tag{2.2}
$$

As is well established, the single one-dimensional FNLSE generates stable solitons for

$$
1 < \alpha \le 2, \tag{2.3}
$$

while at $\alpha \le 1$, the solitons are unstable because of the collapse [12], and hence we consider values of LI belonging to interval (2.3) as was done in [18].

In the case of the usual diffraction, $\alpha = 2$, a simple solution of (2.2) in the form of symmetric solitons is

$$
U_1 = U_2 = \sqrt{2(k-1)}\,\mathrm{sech}\left(\sqrt{2(k-1)}\,x\right).
$$

The norm (power) of this solution is

$$
N = \int_{-\infty}^{+\infty} [(U_1(x))^2 + (U_2(x))^2]\,dx = 4\sqrt{2(k-1)}.
$$

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With the increase of N, the symmetric states become unstable through SSB, and stable asymmetric solitons appear. The SSB is caused by the lowest energy level achieved through asymmetric solution, which manifests itself as a new ground state of the system. While there are no exact solutions for the asymmetric solitons, the SSB point, at which they emerge, can be found exactly for $\alpha = 2$ [19]:

$$(N_{\mathrm{SSB}})_{\mathrm{exact}}(\alpha = 2) = 8/\sqrt{3}.$$

Strunin and Malomed [18] confirmed this criterion to a high accuracy.

Solutions of (1.3) for solitons moving with speed c are sought for as

$$u_{1,2} = u_{1,2}(\xi \equiv x - ct, t). \tag{2.4}$$

In terms of (ξ, t), (1.3) take the form

$$\begin{aligned} i\frac{\partial u_1}{\partial t} - ic\frac{\partial u_1}{\partial \xi} &= \frac{1}{2}\Big(-\frac{\partial^2}{\partial \xi^2}\Big)^{\alpha/2} u_1 - |u_1|^2 u_1 - u_2, \\ i\frac{\partial u_2}{\partial t} - ic\frac{\partial u_2}{\partial \xi} &= \frac{1}{2}\Big(-\frac{\partial^2}{\partial \xi^2}\Big)^{\alpha/2} u_2 - |u_2|^2 u_2 - u_1. \end{aligned} \tag{2.5}$$

Solutions to (2.5) are further looked for as $u_{1,2}(\xi, t) = U_{1,2}(\xi)e^{ikt}$ [see (2.1)], with complex functions $U_{1,2}(\xi)$ satisfying the system of stationary equations

$$\begin{aligned} kU_1 + ic\frac{dU_1}{d\xi} + \frac{1}{2}\Big(-\frac{d^2}{d\xi^2}\Big)^{\alpha/2} U_1 - |U_1|^2 U_1 - U_2 = 0, \\ kU_2 + ic\frac{dU_2}{d\xi} + \frac{1}{2}\Big(-\frac{d^2}{d\xi^2}\Big)^{\alpha/2} U_2 - |U_2|^2 U_2 - U_1 = 0. \end{aligned} \tag{2.6}$$

The stability of solitons is analysed by considering solutions including small perturbations $a_{1,2}$ and $b_{1,2}$,

$$u_{1,2}(x, t) = [U_{1,2}(x) + a_{1,2}(x)e^{\lambda t} + b^*_{1,2}(x)e^{\lambda^* t}]e^{ikt},$$

where λ is the eigenvalue with real part $\mathrm{Re}(\lambda)$ responsible for the growth/decay rate, and $*$ stands for the complex conjugate. The linearization of (1.3) for the perturbations leads to the system of Bogoliubov–de Gennes (BdG) equations:

$$\begin{aligned} &\Big[-(k - i\lambda) - \frac{1}{2}\Big(-\frac{d^2}{d\xi^2}\Big)^{\alpha/2} + 2|U_{1,2}|^2\Big]a_{1,2} + U^2_{1,2}b_{1,2} + a_{2,1} = 0, \\ &\Big[-(k + i\lambda) - \frac{1}{2}\Big(-\frac{d^2}{d\xi^2}\Big)^{\alpha/2} + 2|U_{1,2}|^2\Big]b_{1,2} + (U^*_{1,2})^2 a_{1,2} + b_{2,1} = 0. \end{aligned} \tag{2.7}$$

The BdG equations can be straightforwardly updated for the moving soliton case ($c > 0$). The solitons are (neutrally) stable, if solutions of (2.7) produce only eigenvalues with $\mathrm{Re}(\lambda) = 0$ [20]. We support the stability outcome derived from the BdG equations, by direct simulations of the evolution of perturbed solitons.

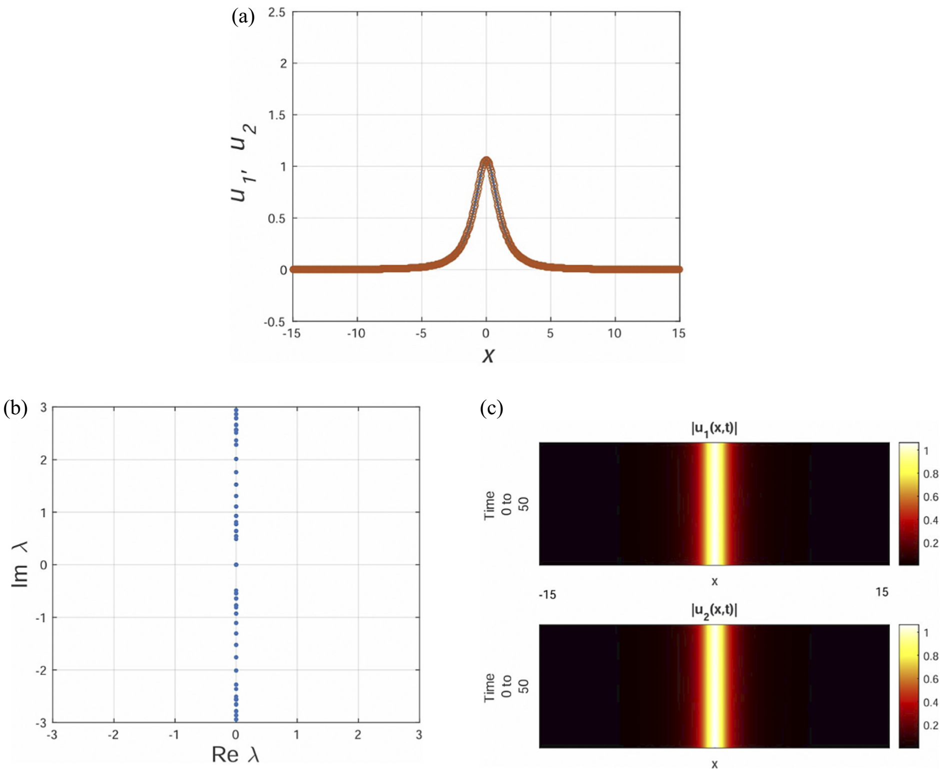

FIGURE 1. (a) The stationary profile of a stable symmetric soliton, (b) the respective spectrum of perturbation eigenvalues and (c) perturbed evolution of the soliton, for $\alpha = 1.6$, $k = 1.5$ and $N = 3.721$. The amplitudes $U_1(x)$ (blue line) and $U_2(x)$ (red circles) are shown in panel (a); the functions $|u_1(x, t)|$ and $|u_2(x, t)|$ are shown in panel (c). Similar labels are used in Figures 2 and 3.

3. Numerical simulations

First, we present some typical results for the quiescent soliton solutions and their stability. Soliton solutions of (2.2) are obtained by the squared-operator iteration method [20]. The stability eigenvalues λ are computed by solving (2.7) using the Fourier collocation method. Both algorithms were realized in Matlab [20]. When implementing the Fourier collocation method, we represented the functions of interest as Fourier series [18], which was convenient for our purposes, as the action of fractional derivative (1.2) in the Fourier space amounts to multiplication by factor $|p|^{\alpha}$. Direct simulations of (1.3) are performed by means of the pseudospectral method [20], based on the discrete Fourier transform and the Runge–Kutta time-stepping method.

Strunin and Malomed [18] presented an example of unstable asymmetric solitons. In Figures 1, 2 and 3, we show typical examples of stable and unstable symmetric solitons and stable asymmetric solitons. The latter lead to particularly interesting collision results as described further below. Each figure displays profiles of two components

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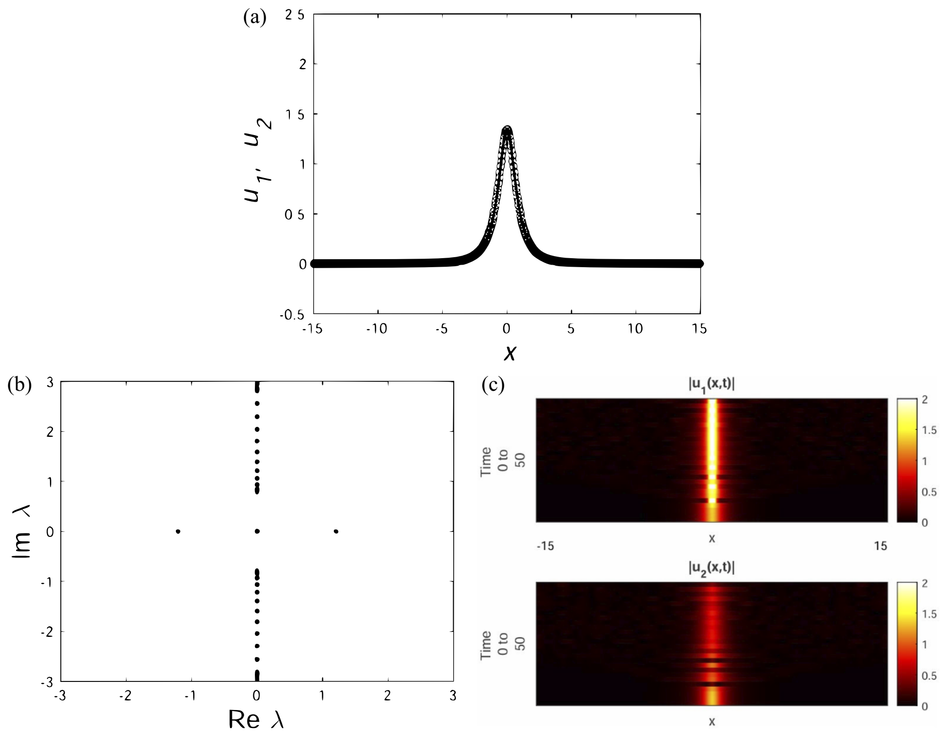

FIGURE 2. The same as in Figure 1, but for an unstable symmetric soliton, with $\alpha = 1.6$, $k = 1.8$ and $N = 4.438$.

of the stationary solution, the spectrum of its stability eigenvalues (recall the stability implies that all eigenvalues must have zero real parts) and direct simulations of the perturbed evolution of the solitons.

From Figure 2, we observe that the instability of the symmetric soliton spontaneously turns it into an asymmetric one, with residual internal oscillations. Shortly after the start of the dynamic, one of the components, namely u_1 in this case, increases due to the positive eigenvalue, while u_2 decreases due to the negative eigenvalue.

Once moving stable solitons are produced, they can be used to explore collisions of soliton pairs. For this purpose, two solitons were numerically constructed as solutions of (2.6), $u^{\pm}_{1,2}$, with velocities $\pm c$. Then, direct simulations of (1.3) are run, with the input in the form of the pair of solitons $u^{\pm}_{1,2}(x)$ placed respectively at $x < 0$ and $x > 0$, with a sufficiently large distance between them.

In this paper, we focus on the collisions between mutually symmetric solitons, with equal values of the propagation constant, k: (1) two stable symmetric solitons; (2) two stable asymmetric solitons, with the same k, in the *flipped* configuration, where soliton $u^{+}_{1,2}$ has a larger component u_1 and a smaller one u_2, and vice versa for $u^{-}_{1,2}$ (see [15]).

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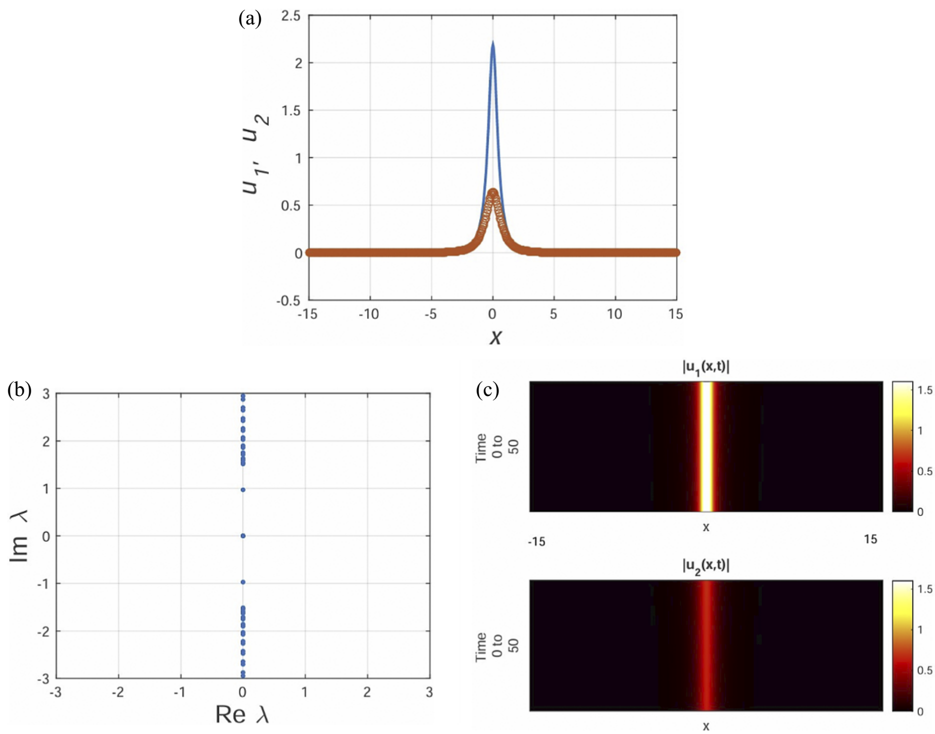

FIGURE 3. The same as in Figure 1, but for a stable asymmetric soliton, with $\alpha = 1.6$, $k = 2.5$ and $N = 3.726$.

In [18], the outcomes of collisions between stable symmetric solitons, at gradually increasing speeds $\pm c$, were presented. In all cases, the colliding solitons bounced back – naturally, remaining far separated for smallest speeds and approaching closer to each other for larger c. Up to $c = 0.06$, the entire picture remains fully symmetric, with respect to both the two components in each soliton and the two colliding solitons as well. Next, starting from $c = 0.08$, the simulations demonstrated the onset of collision-induced SSB, which became obvious at $c = 0.10$. In the latter case, the collision broke the symmetry between the components, as well as between the colliding solitons. It is worth noting that the post-collision amplitude of component u_2 in the left soliton is much larger than before the collision. The collision-induced SSB effect was explained by the instability of the transient state formed by the colliding solitons when they are separated by a relatively small distance.

In the present paper, our main interest is the collisions at significantly higher speeds [6]. We observe an interesting effect of restoration of symmetry as illustrated in Figure 4. The restoration is observed starting from $c = 0.34$. At the highest speed presented, $c = 0.40$, the collision is seen to be fully elastic, with the post-collision solitons being identical to their counterparts before the collision.

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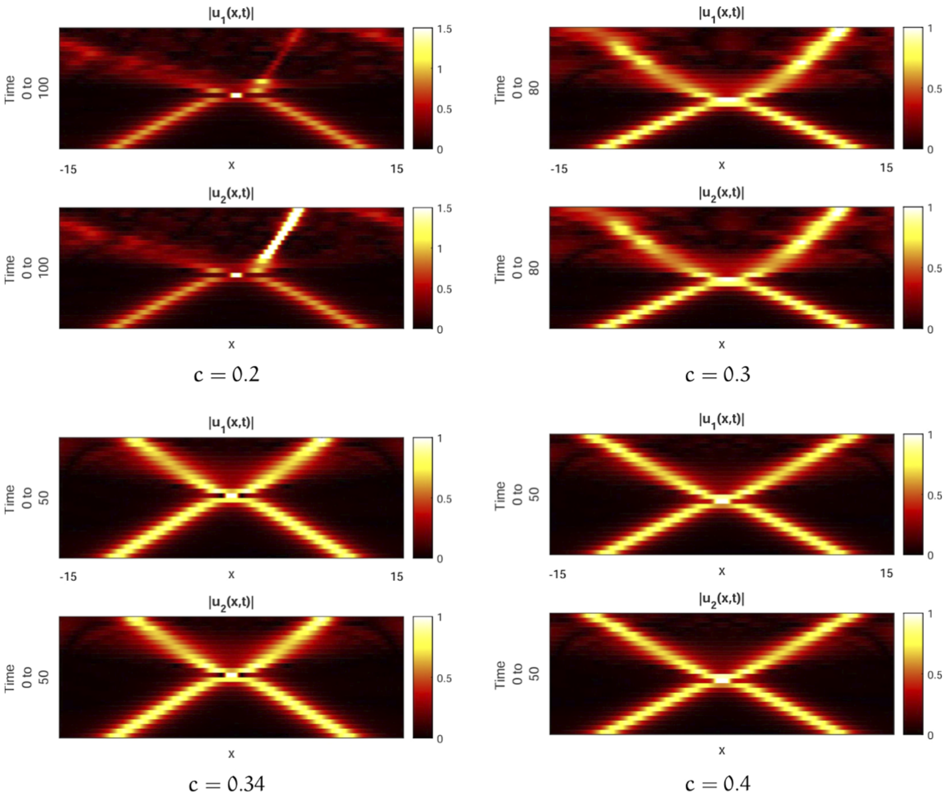

FIGURE 4. Gradual restoration of the symmetry at high speeds in collisions of symmetric solitons for $\alpha = 1.6$, $k = 1.4$. The norm of each soliton is $N = 3.303$ ($c = 0.2$), $N = 3.152$ ($c = 0.3$), $N = 3.075$ ($c = 0.34$), $N = 2.939$ ($c = 0.4$).

Next, in Figure 5, we demonstrate results of collisions between stable asymmetric solitons in the flipped state, as defined above. The general picture is similar to that outlined above for the collisions between symmetric solitons. Namely, at low speeds, $c \leq 0.04$, the solitons bounce back, without breaking the symmetry between the colliding ones. In fact, in this case, each soliton switches from the intrinsic asymmetric shape into a nearly symmetric one, as concerns the relation between its two components. Then, starting from $c = 0.06$, the collision-induced SSB effect sets in, leading to strong symmetry breaking at $c = 0.1$, with a dominant u_1 component of the left soliton in the post-collision state. Then, Figure 5 demonstrates that strong SSB persists, with the increase of the speed, up to $c = 0.40$. Starting from $c = 0.60$, the further increase of the speed gradually leads to the restoration of the symmetry between the colliding solitons and at the speed $c = 0.80$, the collision appears to be quasi-elastic.

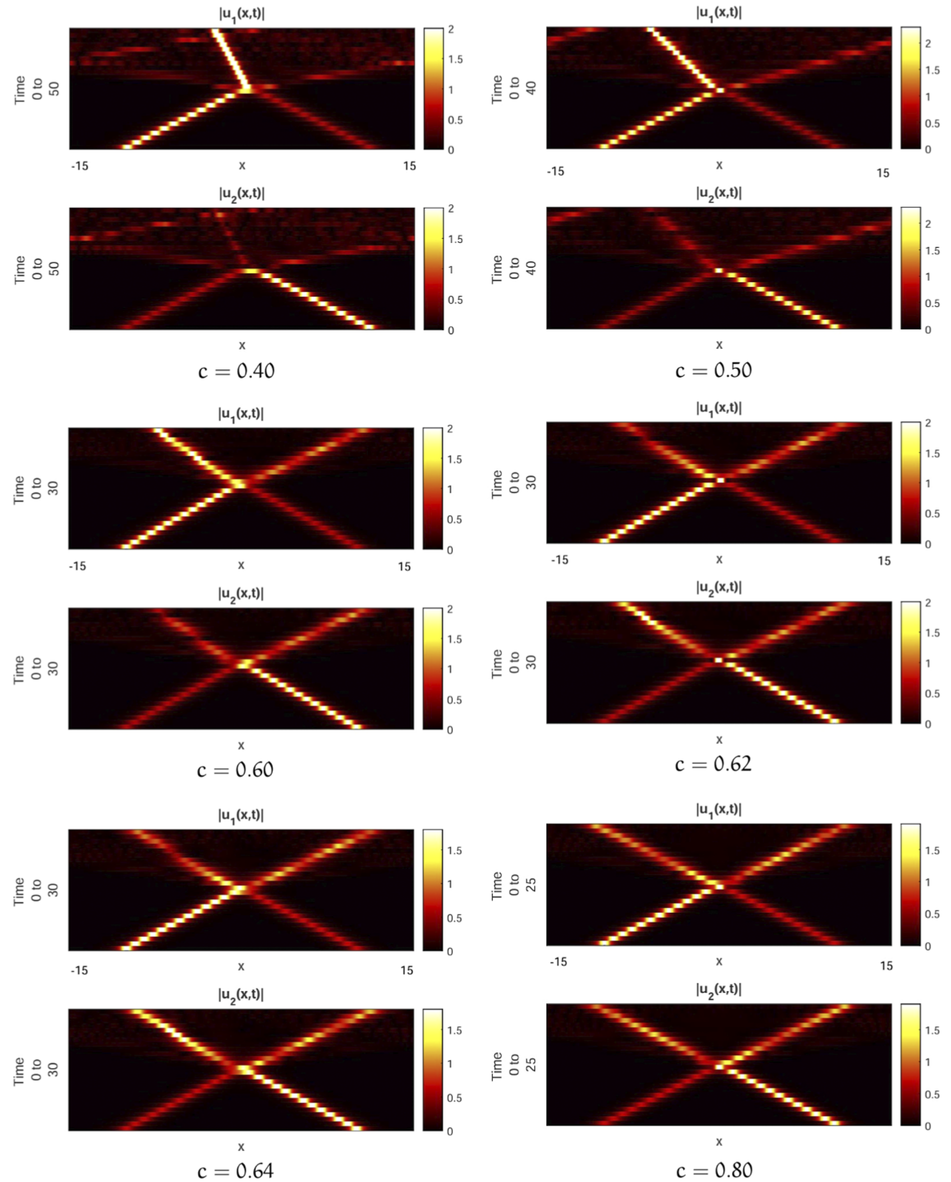

FIGURE 5. Gradual restoration of the symmetry at high speeds in collisions of the mutually flipped asymmetric solitons for $\alpha = 1.6$, $k = 2.6$. The norm of each soliton is $N = 3.638$ ($c = 0.4$), $N = 3.576$ ($c = 0.5$), $N = 3.500$ ($c = 0.6$), $N = 3.483$ ($c = 0.62$), $N = 3.466$ ($c = 0.64$), $N = 3.312$ ($c = 0.80$).

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4. Conclusion

We analysed solitons in systems with fractional diffraction, especially the SSB (spontaneous symmetry breaking) in the one-dimensional dual-core configuration. The systems involve the Riesz fractional derivative, cubic self-focusing acting in the cores and inter-core linear coupling. The corresponding system of FNLSEs (fractional nonlinear Schrödinger equations) models tunnel-coupled planar optical waveguides with the fractional diffraction, as well as coupled waveguides with the fractional group-velocity dispersion in the temporal domain [9]. Using numerical methods, we obtained static and moving solitons of symmetric and asymmetric shapes. Collisions between the moving solitons are explored, demonstrating restoration of symmetry at high soliton speeds.

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