

THE ROLE OF RHEOLOGY IN MODELLING ELASTIC WAVES WITH GAS BUBBLES IN GRANULAR FLUID-SATURATED MEDIA

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Elastic waves in fluid-saturated granular media depend on the rheology which includes elements representing the fluid and, if necessary, gas bubbles. We investigated the effect of different rheological schemes, including and excluding the bubbles, on the linear Frenkel–Biot waves of P1 type. For the wave with the bubbles the scheme consists of three segments representing the solid continuum, fluid continuum, and a bubble surrounded by the fluid. We derived the Nikolaevskiy-type equations describing the velocity of the solid matrix in the moving reference system. The equations are linearized to yield the decay rate λ as a function of the wave number k . We compared the $\lambda(k)$ -dependence for the cases with and without the bubbles, using typical values of the input mechanical parameters. For the both cases, the $\lambda(k)$ -curve lies entirely below zero, which is in line with the notion of the elastic wave being an essentially passive system. We found that the increase of the radius of the bubbles leads to faster decay, while the increase in the number of the bubbles leads to slower decay of the elastic wave.

A list of symbols can be found on page 116.

1. Introduction

The problem of wave propagation in porous media is of interest in various fields of science and engineering. Over the recent years, researchers studied diverse phenomena of this type in large-scale earthquakes, soil mechanics, acoustics, earthquake engineering, and many other disciplines. The fundamentals of the theory of the wave propagation in porous elastic solids can be found in [Biot 1956a; 1956b] or in a more recent review [Frenkel 2005]. Biot [1956a; 1956b] generalized the first principles of linear elasticity and today, most studies in acoustics, geophysical, and geological mechanics rely on his theory. Biot [1962a; 1962b] also deduced the dynamical equations for the wave propagation in poroelastic media. According to Frenkel–Biot’s theory, there are two types of longitudinal waves propagating in a saturated porous medium. The wave of the first type is fast and weakly damped (P1-wave), whereas the wave of the second type is slow and strongly damped (P2-wave). Yang et al. [2014] showed that the dispersion of velocity and attenuation of the P1-wave are greatly affected by the viscoelasticity of the medium.

Liu et al. [1976] demonstrated that rheology based on the scheme often referred to as Generalized Standard Linear Solid (GSLs) helps to better describe measured characteristics of seismic waves in earth continua. The importance of complex multicomponent GSLs was acknowledged by Bohlen [2002] who employed the rheology with many Maxwell bodies connected in parallel. Nikolaevskiy [1989] used complex stress-strain relations in a fluid-saturated grain, where the solid matrix and fluid are in contact.

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1 This resulted, in the final analysis, in the nonlinear higher-order partial differential equation of the form

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = \sum_{p=1}^5 \varepsilon^{p-1} A_{p+1} \frac{\partial^{p+1} v}{\partial x^{p+1}}, \quad (1)$$

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6 where v is the particle velocity of the solid matrix, ε is the small parameter reflecting slow evolution
7 of the wave (this is discussed below), and A_{p+1} are the coefficients linked to mechanical parameters of
8 the system. From the standpoint of wave dynamics, the even derivatives in (1) are responsible for the
9 dissipation and odd derivatives for the dispersion effects. Equation (1) assumes the form of the Korteweg–
10 de Vries–Burgers equation if the index p goes from just 1 and 2. But with the range of p going further
11 as shown, the equation manifests an extension of this classical equation to include high-order spatial
12 derivatives. As explained below, this extension results from the complex rheology of the system.

13 Experimental evidence indicates that the presence of gas bubbles changes the characteristics of the
14 wave [Dunin and Nikolaevskiy 2005; Van Wijngaarden 1968; Anderson 1980]. Typically, in rocks satu-
15 rated with fluids, the P1-wave is the only observable wave [Nikolaevskiy 2008]. However, the presence
16 of gas, even in small proportion, can affect the wave type [Nikolaevskiy and Strunin 2012], so that the
17 P2-wave may also be visible. Dunin et al. [2006] studied the effect of gas bubbles on linear P1- and P2-
18 waves, and derived the dispersion relation connecting the frequency and wave number. Of special interest
19 was the transformation of the wave type due to the bubbles. They found that the transformation was due
20 to the change in the motion of the liquid in the porous space. Instead of the overflow between the pores
21 incurring large Darcy friction, which is characteristic of the P2-waves, the liquid may be displaced into
22 the volume released when a bubble is compressed. In this case the oscillations of the porous matrix and
23 of the bubbles occur in phase and, as a result, the decay of the P2-wave diminishes due to the reduction
24 in the Darcy friction. As far as the rheology is concerned, Dunin et al. [2006] used a rather simple stress-
25 strain relation, $\sigma = Ee$, in standard notations. Various aspects of the wave propagation in multifluid and
26 bubbly flows were studied in [Papageorgiou and Chapman 2015; Tisato et al. 2015; Brunner and Spetzler
27 2001; Collier et al. 2006]. For example, Collier et al. [2006] explored the influence of the gas bubbles
28 on attenuation in volcanic magma, where the bubbles grow not only due to gas expansion, but also due
29 to the exsolution of volatiles, such as water, from the melt into the bubbles. In our present study we do
30 not consider such kind of thermodynamic disequilibrium conditions.

31 The rheological scheme used in [Nikolaevskiy 1989], despite containing several Maxwell bodies, did
32 not include an element representing gas bubble. Nikolaevskiy and Strunin [2012] pointed out the place
33 in this scheme which the bubble element should occupy; see Figure 1. In the present work we aim to
34 include the bubble element into the rheological scheme and then derive and analyze the equation of the
35 type (1), where the coefficients A_p too depend on the bubble-related parameters. The resulting equation
36 will describe the decay (or attenuation) of the freely propagating seismic wave. We will investigate the
37 influence of the bubble-related parameters, including their radius and concentration, on the decay rate.

38 During its propagation the seismic wave decays due to the viscous friction both within individual
39 phases, e.g., fluid, and between the phases. The decay may be described in terms of the decay rate in
40 time as in [Nikolaevskiy 1989], or decay rate in space via the attenuation factor as in [Dunin et al. 2006].
41 These descriptions are closely connected and just correspond to different realizations of the wave. To
42 illustrate this, let us represent Fourier modes of the linear wave as $\exp(i\xi_1)$, where $\xi_1 = \omega t + kx$, ω is

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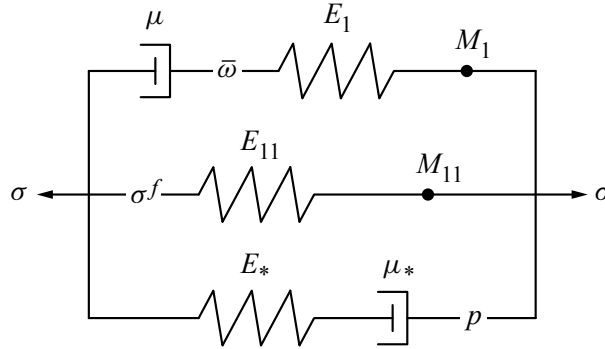


Figure 1. Rheological scheme with the branch ϖ corresponding to a bubble [Nikolaevskiy and Strunin 2012].

the frequency, and k the wave number connected with each other via the dispersion relation. Writing this relation in the form $w = w(k)$ with k being real-valued, we can find the corresponding complex-valued w . Its imaginary part determines how fast the wave decays in time. Physically this situation corresponds to the wave in an unbounded medium, which decays as time goes. Alternatively, one may write down the dispersion relation as $k = k(w)$ and consider the real-valued frequency w as the argument, whereas the wave number k becomes complex-valued. Imaginary part of k governs the decay of the wave in space. From physical standpoint this realization can be associated with the wave which propagates, say, from the surface into underground. The decay of such a wave against the distance is characterized by the attenuation factor.

Importantly, the dynamics of the fluid in porous media may exhibit boundary layers. They form when the frequency of the seismic wave is relatively high. This contrasts the low-frequency waves where the viscous forces dominate throughout the fluid volume so that inertial effects may be neglected. However, at high frequencies the inertial effects dominate in the bulk of the fluid, while the viscous friction concentrates within narrow boundary layers near solid walls due to the no-slip conditions. Allowing for the boundary layers in the analysis significantly affects the frequency dependence of the attenuation of the wave. Namely the low- and high-frequency branches of the attenuation curve become asymmetric. Masson et al. [2006] confirmed this effect by numerical computations of the governing mechanics equations. The model that we use in our present paper is one-dimensional; therefore, it disregards the boundary layers.

In our study of the wave decay we choose to analyze the decay in time, that is, the $w = w(k)$ form of the dispersion relation. We will execute a procedure similar to [Nikolaevskiy 2008], where one-dimensional (x -dependent) dynamics are considered, and all the functions of interest are decomposed into series in small parameter ε characterizing slow evolution of the wave in space and time. Let us denote the phase velocity of the wave by c . Introducing the running variable ξ and using ε to scale the distance x and time t as

$$\xi = \varepsilon(x - ct), \quad \tau = \left(\frac{1}{2}\varepsilon^2\right)t, \tag{2}$$

we seek the velocity of the solid matrix in the form

$$v = \varepsilon v_1 + \varepsilon^2 v_2 + \dots$$

1 We will show, in line with Nikolaevskiy [2008], that the complex rheology generates higher-order time
 2 derivatives [Nikolaevskiy 1989]. They, in turn, translate into high-order derivatives in ξ in the resulting
 3 equation (1) because, using (2),

$$\frac{\partial}{\partial t} = \left(\frac{1}{2}\varepsilon^2\right) \frac{\partial}{\partial \tau} - \varepsilon c \frac{\partial}{\partial \xi} \approx -\varepsilon c \frac{\partial}{\partial \xi},$$

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 7 where the quadratic term in the small parameter ε is neglected in comparison with the linear term. Once
 8 derived, (1) gives the dispersion relation, which is the main point of interest in this paper. Our focus is
 9 on the dissipation controlled by the even x -derivatives. Therefore, we will study a truncated form of (1):

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \xi} = A_2 \frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 A_4 \frac{\partial^4 v}{\partial \xi^4} + \varepsilon^4 A_6 \frac{\partial^6 v}{\partial \xi^6}. \quad (3)$$

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 13 Equation (3) can be complemented by spatially periodic boundary conditions, leading to a spatially
 14 periodic solution. Substituting the Fourier modes of the solution, $v \sim \exp(\lambda t + ikx)$, into (3) gives

$$\lambda = -A_2 k^2 + \varepsilon^2 A_4 k^4 - \varepsilon^4 A_6 k^6, \quad (4)$$

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 17 with k being the wave number associated with the scaled length ξ . When analyzing (4) we remember
 18 that k is not allowed to be too large, otherwise the assumption of the slow variation of the wave in space
 19 will be violated. As we noted, the slowness is facilitated by the smallness of ε . Therefore, in (4) the term
 20 $\varepsilon^2 A_4 k^4$ should be treated just as a correction to the leading term $A_2 k^2$, and the following term $\varepsilon^4 A_6 k^6$
 21 as a correction to the term $\varepsilon^2 A_4 k^4$. Thus, the value of λ remains negative at all plausible values of the
 22 mechanical parameters of the system (such as elastic moduli and viscosities). This reflects the essentially
 23 dissipative nature of the seismic wave, or, in other words, the impossibility of self-excitation of motion.
 24 In view of the crucial presence of the small parameter ε in (3) and (4) we revise our earlier attempt
 25 [Strunin 2014] to guarantee this important property of the freely propagating seismic wave in the model.
 26 In [Strunin 2014] a popular form of (3) was considered where the small parameter ε was omitted. It was
 27 reasoned that the mechanical parameters, of which A_4 and A_6 are composed, should therefore assume
 28 special limited values, in order to guarantee that $\lambda < 0$. However, negativity of λ is simply ensured by
 29 the smallness of ε , which is the essential part of (3) as explicitly shown.

31 2. Basic equations of one-dimensional dynamics

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 33 **2.1. Conservation of mass and momentum.** For a one-dimensional case the momentum and mass bal-
 34 ance equations are [Nikolaevskiy 1990]

$$\begin{aligned} \frac{\partial}{\partial t} (1-m)\rho^{(s)}v + \frac{\partial}{\partial x} (1-m)\rho^{(s)}vv &= \frac{\partial}{\partial x} \sigma^{(ef)} - (1-m) \frac{\partial p}{\partial x} - I, \\ \frac{\partial}{\partial t} m\rho^{(f)}u + \frac{\partial}{\partial x} m\rho^{(f)}uu &= -m \frac{\partial p}{\partial x} + I, \\ \frac{\partial}{\partial t} (1-m)\rho^{(s)} + \frac{\partial}{\partial x} (1-m)\rho^{(s)}v &= 0, \\ \frac{\partial}{\partial t} m\rho^{(f)} + \frac{\partial}{\partial x} m\rho^{(f)}u &= 0, \end{aligned} \quad (5)$$

1 where the subscripts s and f label the solid and gas-liquid mixture respectively, ρ , v , and u are the
 2 corresponding densities and mass velocities, m is the porosity, $\sigma^{(ef)}$ is the effective Terzaghi stress, p is
 3 the pore pressure, and I is the interfacial viscous force approximated by

$$I = \delta m(v - u), \quad \delta = \frac{\mu^{(f)} m}{\ell},$$

4 where $\mu^{(f)}$ is the gas-liquid mixture viscosity and ℓ is the intrinsic permeability.

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 8 **2.2. Dynamics of bubbles.** The equation of the dynamics of a bubble [Dunin et al. 2006] has the form

$$R \frac{\partial^2}{\partial t^2} R + \frac{3}{2} \left(\frac{\partial}{\partial t} R \right)^2 + \frac{4\mu}{\rho^{(L)}} \left(\frac{1}{R} + \frac{m}{4\ell} R \right) \frac{\partial}{\partial t} R = (p_g - p) / \rho^{(L)}, \quad (6)$$

9 where R is the bubble radius, p is the pressure in the liquid, $p_g = p_0(R_0/R)^\chi$ is the gas pressure inside
 10 the bubble (here $\chi = 3\zeta$, and ζ is the adiabatic exponent), $\rho^{(L)}$ is the density of the liquid without the
 11 bubbles, and μ is the viscosity of the liquid without the bubbles. The density equations for the solid and
 12 liquid without gas are

$$\rho^{(s)} = \rho_0^{(s)} (1 - \beta^{(s)} \sigma) = \rho_0^{(s)} \left[1 + \beta^{(s)} p - \frac{\beta^{(s)} \sigma^{(ef)}}{1-m} \right] \approx \rho_0^{(s)} [1 + \beta^{(s)} p - \beta^{(s)} \sigma^{(ef)}], \quad (7)$$

$$\rho^{(L)} = \rho_0^{(L)} (1 + \beta^{(L)} p). \quad (8)$$

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 20 The mean density of the gas-liquid mixture is

$$\rho^{(f)} = (1 - \phi) \rho^{(L)} + \phi \rho^{(g)}, \quad (9)$$

21 where

$$\phi = \frac{4\pi}{3} R^3 n_0.$$

22 Here σ is the true stress, ϕ is the volume gas content and n_0 is the number density of the bubbles per
 23 unit volume. In (9) we can neglect the density of the gas $\rho^{(g)}$ due to the low gas content. The change in
 24 ϕ is due to the change in the bubble radius R . Then (9) becomes

$$\rho^{(f)} = \rho_0^{(L)} (1 + \beta^{(L)} p) \left(1 - \frac{4\pi}{3} R_0^3 n_0 \right). \quad (10)$$

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 30 Similarly to [Dunin et al. 2006] we also assume that the pore pressure p is equal to the pressure in the
 31 liquid far from the bubble.

32 **2.3. Stress-strain relation.** In this section we derive the stress-strain relation for the viscoelastic medium
 33 based on the rheological Maxwell–Voigt model, which includes the gas bubble. The model includes two
 34 friction elements with viscosities μ_1 and μ_2 , three elastic springs with the elastic moduli E_1 , E_2 , and
 35 E_3 , and three oscillating masses M_1 , M_2 , and M_3 . The total stress is denoted σ . We also denote the
 36 displacements of the elements of the model by e with respective subscripts as shown in Figure 2. Now
 37 we write the second Newton's law for the elements and the kinematic relations:

$$\begin{aligned} M_1 \frac{d^2 e_1}{dt^2} + M_2 \frac{d^2 e_2}{dt^2} &= \sigma - E_1 e_1 - E_2 e_2, & e &= e_2 = e_1 + e_3 + e_4 + e_5, \\ M_3 \frac{d^2 e_3}{dt^2} &= E_1 e_1 - E_3 e_3, & E_3 e_3 &= \mu_2 \frac{de_4}{dt} = \mu_1 \frac{de_5}{dt}. \end{aligned} \quad (11)$$

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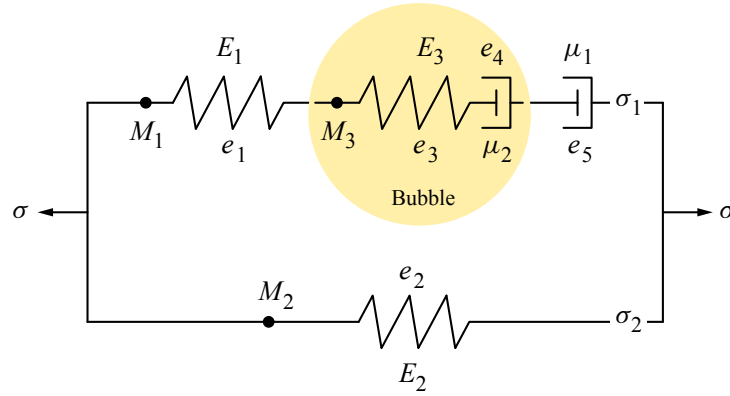


Figure 2. Rheological scheme including a gas bubble.

Equations (11) generate the following relation between the stress and strain:

$$\begin{aligned} & \left[E_1 E_3 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] \sigma + (E_3 + E_1) \frac{d\sigma}{dt} + M_3 \frac{d^3 \sigma}{dt^3} \\ &= \left[E_1 E_2 E_3 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] e + [(E_2 + E_1) E_3 + E_1 E_2] \frac{de}{dt} + \left[E_1 E_3 M_2 \left(\frac{1}{\mu_1} + \frac{1}{\mu_2} \right) \right] \frac{d^2 e}{dt^2} \\ & \quad + [((E_2 + E_1) M_3 + (E_3 + E_1) M_2) + E_3 M_1] \frac{d^3 e}{dt^3} + [(M_2 + M_1) M_3] \frac{d^5 e}{dt^5}. \end{aligned} \quad (12)$$

Generalizing (12) using a similar approach to [Nikolaevskiy 2008], we get

$$\sigma^{(ef)} + \eta \sum_{q=1,3} b_q \frac{D^q \sigma^{(ef)}}{Dt^q} = E_2 e + \beta^{(s)} k_b p + \eta \sum_{q=1,2,3,5} a_q \frac{D^q e}{Dt^q}, \quad (13)$$

where $\sigma^{(ef)}$ is the effective stress, $\eta = [E_1 E_3 (1/\mu_1 + 1/\mu_2)]^{-1}$, k_b is the bulk elastic module of the porous matrix, and the coefficients a_q and b_q are expressed as

$$\begin{aligned} a_1 &= [(E_2 + E_1) E_3 + E_1 E_2], & a_2 &= M_2, & a_3 &= [(E_2 + E_1) M_3 + (E_3 + E_1) M_2 + E_3 M_1], \\ a_5 &= [(M_2 + M_1) M_3], & b_1 &= (E_3 + E_1), & b_3 &= M_3. \end{aligned}$$

Finally, we add the closing relation between the deformation e and the velocity v of the solid:

$$\frac{De}{Dt} \equiv \frac{\partial e}{\partial t} + v \frac{\partial e}{\partial x} = \frac{\partial v}{\partial x}. \quad (14)$$

3. Elastic waves in saturated media including gas bubbles

Following the approach of Nikolaevskiy [2008], we consider the P1-wave in a porous media under the full saturation. Accordingly we assume that the mass velocities v and u have the same sign:

$$v = u + O(\varepsilon v), \quad (15)$$

1 where ε is the small parameter. The Darcy force has the order as shown:

$$I = \varepsilon^\gamma \delta m(v - u) = \varepsilon^\gamma \delta m v, \quad \delta = m\mu/k = O(1). \quad (16)$$

2 Describing a weakly nonlinear wave we use the running coordinate system with simultaneous scale
3 change:

$$\xi = \varepsilon(x - ct), \quad \tau = \frac{1}{2}\varepsilon^2 t, \quad \frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial t} = \varepsilon \left(\frac{1}{2}\varepsilon \frac{\partial}{\partial \tau} - c \frac{\partial}{\partial \xi} \right). \quad (17)$$

4 Thus, the constitutive law (13) transforms into the form

$$\begin{aligned} \sigma^{(ef)} + \eta \sum_{q=1,3} b_q \varepsilon^q \left(\frac{1}{2}\varepsilon \frac{\partial}{\partial \tau} + (v - c) \frac{\partial}{\partial \xi} \right)^q \sigma^{(ef)} \\ = E_2 e + \beta^{(s)} k_b p + \eta \sum_{q=1,2,3,5} a_q \varepsilon^q \left(\frac{1}{2}\varepsilon \frac{\partial}{\partial \tau} + (v - c) \frac{\partial}{\partial \xi} \right)^q e. \end{aligned} \quad (18)$$

5 Now, we seek the unknown functions as power series:

$$\begin{aligned} v &= \varepsilon v_1 + \varepsilon^2 v_2 + \dots, & u &= \varepsilon u_1 + \varepsilon^2 u_2 + \dots, \\ \sigma^{ef} &= \sigma_0^{(ef)} + \varepsilon \sigma_1^{(ef)} + \varepsilon^2 \sigma_2^{(ef)} + \dots, & p &= p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots, \\ m &= m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots, & e &= e_0 + \varepsilon e_1 + \varepsilon^2 e_2 + \dots, \\ \phi &= \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \dots, & R &= R_0(1 + \varepsilon R_1 + \varepsilon^2 R_2 + \dots). \end{aligned} \quad (19)$$

6 **3.1. The first approximation.** Using series (19) in the mass and momentum equations (5), (6), and the
7 stress-strain relation (13) and collecting linear terms $\sim \varepsilon$, we eventually arrive at the system

$$(1 - \beta^{(s)} E_2) v_1 + (B - k_b \beta^{(s)} \beta^{(s)}) p_0 \chi R_1 c = 0, \quad (20)$$

$$(E_2 - \rho_0 c^2) v_1 - (A - k_b \beta^{(s)} \beta^{(s)}) p_0 \chi R_1 c = 0. \quad (21)$$

8 We address the reader to [Appendix A](#) for details of derivation of (20) and (21).

9 Equations (20) and (21) must coincide, therefore,

$$\begin{vmatrix} (1 - \beta^{(s)} E_2) & (B - k_b \beta^{(s)} \beta^{(s)}) p_0 \chi \\ (E_2 - \rho_0 c^2) & -(A - k_b \beta^{(s)} \beta^{(s)}) p_0 \chi \end{vmatrix} = 0. \quad (22)$$

10 Equation (22) gives the velocity of the wave,

$$c^2 = \frac{(A - k_b \beta^{(s)} \beta^{(s)}) Z_1 + E_2}{\rho_0}, \quad (23)$$

11 where

$$Z_1 = \frac{1 - \beta^{(s)} E_2}{B - k_b \beta^{(s)} \beta^{(s)}}.$$

Thus, all the variables are expressed through any one selected variable; for example, the velocity v_1 :

$$\begin{aligned} e_1 &= -\frac{v_1}{c}, \quad \sigma_1^{(ef)} = -(E_2 - k_b \beta^{(s)} Z_1) \frac{v_1}{c}, \quad p_1 = Z_1 \frac{v_1}{c}, \quad R_1 = -\frac{Z_1}{p_0 \chi} \frac{v_1}{c}, \\ m_1 &= [(1 - m_0) - k_b \beta^{(s)}] \beta^{(s)} Z_1 + \beta^{(s)} E_2 - (1 - m_0) \frac{v_1}{c}, \\ \rho_1^{(f)} &= \rho_0^{(L)} Z_1 \left(\beta^{(L)} \kappa_1 + \frac{\kappa_2 4\pi n_0 R_0^3}{p_0 \chi} \right) \frac{v_1}{c}, \\ \rho_1^{(s)} &= \rho_0^{(s)} \beta^{(s)} [Z_1 (1 - k_b \beta^{(s)}) + E_2] \frac{v_1}{c}. \end{aligned}$$

3.2. The second approximation. In the second approximation, we collect quadratic terms $\sim \varepsilon^2$ in the mass and momentum equations (5), (6), and stress-strain relation (13) and come to the following system:

$$\begin{aligned} \frac{\partial}{\partial \xi} [(E_2 - \rho_0 c^2) v_2 - (1 - \beta^{(s)} k_b) p_0 \chi R_2 c] &= E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi} - c(1 - \beta^{(s)} k_b) \frac{\partial \Gamma}{\partial \xi}, \quad (24) \\ \frac{\partial}{\partial \xi} \left[(1 - E_2 \beta^{(s)}) v_2 - \left(\omega_1 p_0 \chi - \frac{4\pi m_0 n_0 R_0^3}{\kappa_1} \right) R_2 c \right] &= \Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi} - c \omega_1 \frac{\partial \Gamma}{\partial \xi} + c \omega_2 \frac{\partial R_1^2}{\partial \xi}. \quad (25) \end{aligned}$$

The details of the derivation of (24) and (25) are provided in Appendix B. The determinant of the left-hand side of the system of equations (24) and (25) coincides with the determinant of (22), which equals zero. A nonzero solution for v_2 exists only if the following compatibility condition takes place:

$$\begin{vmatrix} (E_2 - \rho_0 c^2) & \frac{\partial}{\partial \xi} \left(E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi} - (1 - \beta^{(s)} k_b) c \frac{\partial \Gamma}{\partial \xi} \right) \\ (1 - E_2 \beta^{(s)}) & \frac{\partial}{\partial \xi} \left[\Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi} - c \omega_1 \frac{\partial \Gamma}{\partial \xi} + c \omega_2 \frac{\partial R_1^2}{\partial \xi} \right] \end{vmatrix} = 0 \quad (26)$$

(see Appendix C). This is the evolution equation with respect to $v \cong v_1$:

$$cM \frac{\partial \Gamma}{\partial \xi} - cN \frac{\partial T}{\partial \xi} + c\omega_2 \psi \frac{\partial R_1^2}{\partial \xi} + \Lambda \psi + c\Sigma(1 - E_2 \beta^{(s)}) - E_2 F N = 0, \quad (27)$$

where

$$\psi = (E_2 - \rho_0 c^2), \quad M = (1 - \beta^{(s)} k_b)(1 - E_2 \beta^{(s)}) - \omega_1 \psi, \quad N = (1 - \beta^{(s)} \rho_0 c^2).$$

Now, we rewrite (27) in terms of v and regroup:

$$\begin{aligned} & \frac{1}{2} \left[Y_1 + \psi \left((1 - \kappa_1 \kappa_2) \bar{m}_1 - E_2 \beta^{(s)} - \left(Y_2 + \frac{4\pi n_0 R_0^3}{p_0 \chi} \right) Z_1 \right) \right] \frac{\partial v}{\partial \tau} \\ & - \left[N \eta c^2 (a_1 - b_1 (E_2 - k_b \beta^{(s)} Z_1)) + \frac{M Z_1 \mu c^2}{p_0 \chi} \left(4 + \frac{m_0 R_0^2}{\ell} \right) \right] \frac{\partial^2 v}{\partial \xi^2} + \varepsilon N \eta c^3 a_2 \frac{\partial^3 v}{\partial \xi^3} \\ & - \varepsilon^2 N \eta c^4 [a_3 - b_3 (E_2 - k_b \beta^{(s)} Z_1)] \frac{\partial^4 v}{\partial \xi^4} - \varepsilon^4 N \eta a_5 c^6 \frac{\partial^6 v}{\partial \xi^6} - [\zeta_1 + \zeta_2] \frac{\partial v v}{\partial \xi} = 0, \quad (28) \end{aligned}$$

1 where

$$\begin{aligned}
 \bar{m}_1 &= ((1-m_0)-k_b\beta^{(s)})\beta^{(s)}Z_1+\beta^{(s)}E_2-(1-m_0), \\
 Y_1 &= (E_2N+c^2\rho_0(1-E_2\beta^{(s)})), \quad Y_2 = m_0(\kappa_1\beta^{(L)}-\beta^{(s)})+\beta^{(s)}(1-\beta^{(s)}k_b), \\
 \zeta_1 &= \psi \left(\bar{m}_1-(1-m_0)\beta^{(s)}Z_1+\beta^{(s)}(E_2-k_b\beta^{(s)}Z_1)(1-\bar{m}_1)-\beta^{(s)}Z_1\bar{m}_1 \right. \\
 &\quad \left. +\kappa_1\beta^{(L)}Z_1\bar{m}_1-\kappa_1\kappa_2+4\pi n_0\kappa_2R_0^3\frac{Z_1}{p_0\chi}(\bar{m}_1-m_0)-m_0\kappa_1\beta^{(L)}Z_1+\omega_2\frac{Z_1^2}{(p_0\chi)^2} \right), \\
 \zeta_2 &= c^2(1-E_2\beta^{(s)}) \left(\rho_0-\rho_0\kappa_1\beta Z_1-\rho_0^{(s)}\beta^{(s)}(E_2-k_b\beta^{(s)}Z_1)-m_0\kappa_2\rho^{(L)}\frac{Z_1}{p_0\chi}+\bar{m}_1(\rho^{(s)}-\kappa_1\kappa_2\rho^{(L)}) \right) \\
 &\quad +\frac{M(\chi+1)}{2p_0\chi}Z_1^2+E_2N.
 \end{aligned}$$

14 In short, the evolution equation (28) can be written as

$$C_1\frac{\partial v}{\partial \tau}-C_2\frac{\partial^2 v}{\partial \xi^2}+\varepsilon C_3\frac{\partial^3 v}{\partial \xi^3}-\varepsilon^2 C_4\frac{\partial^4 v}{\partial \xi^4}-\varepsilon^4 C_6\frac{\partial^6 v}{\partial \xi^6}-\zeta\frac{\partial vv}{\partial \xi}=0, \quad (29)$$

18 where

$$\begin{aligned}
 C_1 &= \frac{1}{2}\left[Y_1+\psi\left((1-\kappa_1\kappa_2)\bar{m}_1-E_2\beta^{(s)}-\left(Y_2+\frac{4\pi n_0R_0^3}{p_0\chi} \right)Z_1 \right) \right], \\
 C_2 &= \left[N\eta c^2(a_1-b_1(E_2-k_b\beta^{(s)}Z_1))+\frac{MZ_1\mu c^2}{p_0\chi}\left(4+\frac{m_0R_0^2}{\ell} \right) \right], \quad C_3 = N\eta c^3 a_2, \\
 C_4 &= N\eta c^4(a_3-b_3(E_2-k_b\beta^{(s)}Z_1)), \quad C_6 = N\eta a_5 c^6, \quad \zeta = \zeta_1 + \zeta_2.
 \end{aligned}$$

4. Elastic waves in saturated media without gas bubbles

28 Our goal is to study the effect of inclusion of gas bubbles into the rheological scheme on the elastic wave
 29 decay. For this purpose we will remove the bubble-representing segment from Figure 2 and rederive the
 30 wave equation (note that our rheological scheme consists of only two branches: one for the solid and the
 31 other for the bubble-fluid mixture). This differs from the original Nikolaevskiy scheme, which includes
 32 three parallel branches [Nikolaevskiy 1989; 2008].

34 **4.1. Stress-strain relation.** By removing the bubble segment from the rheological scheme, we get Figure 3.

35 The second Newton's law for the scheme and the kinematic relation are

$$M_1\frac{d^2 e_1}{dt^2}+M_2\frac{d^2 e}{dt^2}=\sigma-E_1 e_1-E_2 e, \quad e=e_1+e_3, \quad E_1 e_1=\mu_1\frac{de_3}{dt}. \quad (30)$$

39 Equations (30) lead to the well-known stress-strain relation [Nikolaevskiy 1985]

$$\sigma+\theta\frac{d\sigma}{dt}=E_2 e+(E_1+E_2)\theta\frac{de}{dt}+M_2\frac{d^2 e}{dt^2}+(M_1+M_2)\theta\frac{d^3 e}{dt^3}, \quad (31)$$

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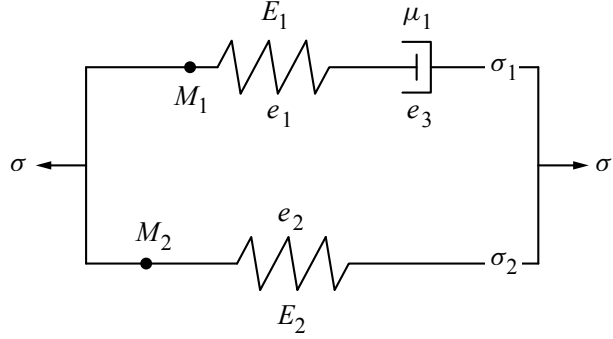


Figure 3. Rheological scheme without gas bubble.

where $\theta = \mu_1/E_1$. Hence, the constitutive law (31) will be written as

$$\sigma^{(ef)} + b_1 \varepsilon \left(\frac{1}{2} \varepsilon \frac{\partial}{\partial \tau} + (v - c) \frac{\partial}{\partial \xi} \right) \sigma^{(ef)} = E_2 e + \beta^{(s)} k_b p + \sum_{q=1}^3 a_q \varepsilon^q \left(\frac{1}{2} \varepsilon \frac{\partial}{\partial \tau} + (v - c) \frac{\partial}{\partial \xi} \right)^q e, \quad (32)$$

where

$$a_1 = (E_1 + E_2)\theta, \quad a_2 = M_2, \quad a_3 = (M_1 + M_2)\theta, \quad b_1 = \theta.$$

4.2. First approximation of the system without gas bubbles. Following the approach of Section 3.1, in the first approximation for the system without bubbles we arrive at the system

$$(1 - \beta^{(s)} E_2) v_1 = (\beta - \beta^{(s)} \beta^{(s)} k_b) c p_1, \quad (33)$$

$$(\rho_0 c^2 - E_2) v_1 = (1 - \beta^{(s)} k_b) c p_1. \quad (34)$$

Appendix D provides the derivation details of these equations. The determinant of the system of equations (33) and (34) gives the wave velocity c :

$$\begin{vmatrix} (1 - \beta^{(s)} E_2) & -(\beta - \beta^{(s)} \beta^{(s)} k_b) \\ (\rho_0 c^2 - E_2) & -(1 - \beta^{(s)} k_b) \end{vmatrix} = 0. \quad (35)$$

Thus,

$$c^2 = \frac{E_2 + Z_2(1 - \beta^{(s)} k_b)}{\rho_0}, \quad (36)$$

where

$$Z_2 = \frac{(1 - \beta^{(s)} E_2)}{(\beta - \beta^{(s)} \beta^{(s)} k_b)}.$$

Again we can express all the variables through the velocity v_1 :

$$\begin{aligned} e_1 &= -\frac{v_1}{c}, & p_1 &= Z_2 \frac{v_1}{c}, & \rho_1^{(s)} &= \rho^{(s)} \beta^{(s)} [Z_2(1 - \beta^{(s)} k_b) + E_2] \frac{v_1}{c}, \\ \rho_1^{(L)} &= \rho^{(L)} \beta^{(L)} Z_2 \frac{v_1}{c}, & m_1 &= [(1 - m_0) \beta^{(s)} Z_2 + \beta^{(s)} (E_2 - \beta^{(s)} k_b Z_2) - (1 - m_0)] \frac{v_1}{c}, \\ \sigma_1^{(ef)} &= -(E_2 - \beta^{(s)} k_b Z_2) \frac{v_1}{c}. \end{aligned}$$

1 **4.3. Second approximation for the system without gas bubbles.** Following the approach of Section 3.2,
 2 in the second approximation we arrive at the system

$$3 \frac{\partial}{\partial \xi} [(E_2 - \rho_0 c^2)v_2 + (1 - \beta^{(s)}k_b)cp_2] = E_2 F - c\Sigma + c \frac{\partial T}{\partial \xi}, \quad (37)$$

$$5 \frac{\partial}{\partial \xi} [(1 - E_2\beta^{(s)})v_2 - (\beta - \beta^{(s)}\beta^{(s)})cp_2] = \Lambda - \beta^{(s)}E_2 F - c\beta^{(s)} \frac{\partial T}{\partial \xi}. \quad (38)$$

7 See Appendix E for the derivation details. In analogy to (26), the compatibility condition for the system
 8 of equations (37), (38) has the form

$$9 \left| \begin{array}{c} (E_2 - \rho_0 c^2) \frac{\partial}{\partial \xi} (E_2 F - c\Sigma + c \frac{\partial T}{\partial \xi}) \\ (1 - E_2\beta^{(s)}) \frac{\partial}{\partial \xi} (\Lambda - \beta^{(s)}E_2 F - c\beta^{(s)} \frac{\partial T}{\partial \xi}) \end{array} \right| = 0. \quad (39)$$

13 Then the evolution equation for $v \cong v_1$ is

$$15 \Lambda \psi - cN \frac{\partial T}{\partial \xi} + c\Sigma(1 - E_2\beta^{(s)}) - E_2 FN = 0. \quad (40)$$

17 Rearranging, we arrive at

$$18 c^2 \rho_0 (1 - E_2\beta^{(s)}) \frac{\partial v}{\partial \tau} - Nc^2 (a_1 - b_1(E_2 - \beta^{(s)}k_b Z_2)) \frac{\partial^2 v}{\partial \xi^2} \\ 20 + \varepsilon N a_2 c^3 \frac{\partial^3 v}{\partial \xi^3} - \varepsilon^2 N a_3 c^4 \frac{\partial^4 v}{\partial \xi^4} + [G_1 + G_2] \frac{\partial v v}{\partial \xi} = 0, \quad (41)$$

23 where

$$24 \hat{m}_1 = (1 - m_0)\beta^{(s)} Z_2 + \beta^{(s)}(E_2 - \beta^{(s)}k_b Z_2) - (1 - m_0),$$

$$25 G_1 = \psi \left(-((1 - m_0) + \hat{m}_1)\beta^{(s)} Z_2 - \beta^{(s)}(E_2 - \beta^{(s)}k_b Z_2) \left(1 + \frac{\hat{m}_1}{(1 - m_0)} \right) + \beta^{(L)} Z_2 \hat{m}_1 - m_0 \beta^{(L)} Z_2 \right),$$

$$27 G_2 = c^2 (1 - E_2\beta^{(s)}) (\rho_0 - \rho_0 \beta^{(s)} Z_2 - \rho^{(s)} \beta^{(s)} (E_2 - \beta^{(s)}k_b Z_2) + \hat{m}_1 (\rho^{(s)} - \rho^{(L)})) + E_2 (1 - \beta^{(s)} \rho_0 c^2).$$

29 Finally, we rewrite the evolution equation (41) as

$$30 D_1 \frac{\partial v}{\partial \tau} - D_2 \frac{\partial^2 v}{\partial \xi^2} + \varepsilon D_3 \frac{\partial^3 v}{\partial \xi^3} - \varepsilon^2 D_4 \frac{\partial^4 v}{\partial \xi^4} + G \frac{\partial v v}{\partial \xi} = 0, \quad (42)$$

32 where

$$33 D_1 = c^2 \rho_0 (1 - E_2\beta^{(s)}), \quad D_2 = Nc^2 (a_1 - b_1(E_2 - \beta^{(s)}k_b Z_2)), \\ 34 D_3 = N a_2 c^3, \quad D_4 = N a_3 c^4, \quad G = G_1 + G_2.$$

36 We remark that for the wave propagating to the left, that is, with $\xi = \varepsilon(x + ct)$, one obtains (as we
 37 checked) the same equation (42).
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39 5. Linearized model

41 In this section we investigate the linearized versions of the model with and without the bubbles, that is,
 42 (29) and (42). Our particular interest is in its dissipative part responsible for decay of the wave.

5.1. Evaluation of the parameters and the wave velocity. From [Dunin et al. 2006; Nikolaevskiy 1985; 2016; Sutton and Biblarz 2016; Carcione 1998; Mikhailov 2010; Smeulders 2005], the values of the parameters are: the density $\rho_0^{(L)} = 1000 \text{ kg/m}^3$ for water, $\rho^{(g)} = 2 \text{ kg/m}^3$ for gas, $\rho_0^{(s)} = 2500 \text{ kg/m}^3$ for solid; porosity $m_0 = 0.25$; bulk modulus $k_b = 1.7 \times 10^9 \text{ Pa}$ for the matrix, $k_b = 30 \times 10^9 \text{ Pa}$ for the solid; compressibility $\beta^{(L)} = 2 \times 10^{-9} \text{ Pa}^{-1}$ for water, $\beta^{(L)} = 2.4 \times 10^{-6} \text{ Pa}^{-1}$ for gas, $\beta^{(s)} = 2 \times 10^{-10} \text{ Pa}^{-1}$ for solid; steady pressure $p_0 = 10^5 \text{ Pa}$; bubble radius $R_0 = 5 \times 10^{-5} \text{ m}$; volume gas content $\phi_0 = 10^{-3}$; viscosity $\mu_1 = 10^{-3} \text{ Pa}\cdot\text{s}$ for water, $\mu_2 = 2 \times 10^{-5} \text{ Pa}\cdot\text{s}$ for gas; adiabatic exponent $\zeta = 1.4$, and permeability $\ell = 2 \times 10^{-11} \text{ m}^2$. Using the data from [Nikolaevskiy and Strunin 2012; Nikolaevskiy 1985; 2016; Nikolaevskiy and Stepanova 2005], the values of the parameters of the rheological scheme in Figure 2 are

$$M_1 = \rho^{(L)} L_s^2 = 10^{-2} \text{ kg/m}, \quad M_2 = \rho^{(s)} L_s^2 = 0.02 \text{ kg/m}, \quad M_3 = \rho^{(g)} L_s^2 = 2 \times 10^{-6} \text{ kg/m},$$

and

$$(a) \quad E_1 = 1/\beta^{(L)} = 4 \times 10^5 \text{ Pa}, \quad E_2 = c^2 \rho_0 = 2 \times 10^7 \text{ Pa}, \quad E_3 = 3\chi p_0 = 4 \times 10^7 \text{ Pa},$$

where we used, just for the purpose of evaluating of E_i and M_i , the typical velocity $c \sim 100 \text{ m/s}$ and the linear size of the oscillator $L_s = 0.3 \text{ cm}$ from [Nikolaevskiy 1985; Vilchinska et al. 1985]. Note that the above values of E_i are known only approximately. With this in mind, in the present study we also explore other the values of E_i that are considerably different from variant (a):

$$(b) \quad E_1 = 5 \times 10^5 \text{ Pa}, \quad E_2 = 5 \times 10^8 \text{ Pa}, \quad E_3 = 5 \times 10^4 \text{ Pa},$$

$$(c) \quad E_1 = 6 \times 10^5 \text{ Pa}, \quad E_2 = 2 \times 10^9 \text{ Pa}, \quad E_3 = 5 \times 10^3 \text{ Pa}.$$

The reason for this choice is that the two different rheological schemes that we use (for the wave with and without the bubbles) give close values of λ when we put $R_0 = 0$ and $n_0 = 0$.

Now we apply the formulas for the wave velocity (23) and (36) to show that they give reasonable orders of magnitude. For variant (a) formula (23) for the wave with the bubbles gives $c \approx 577 \text{ m/s}$, and formula (36) for the wave without the bubbles gives $c \approx 2100 \text{ m/s}$. For variant (b) the wave with the bubbles has the velocity $c \approx 726 \text{ m/s}$ and the wave without the bubbles the velocity $c \approx 2000 \text{ m/s}$. For variant (c) the wave with the bubbles has $c \approx 1100 \text{ m/s}$, and the wave without the bubbles $c \approx 1800 \text{ m/s}$. This illustrates, in line with the previous studies, that the bubbles may result in a considerable change of the wave velocity. However, our main interest in this study is the dissipation rate of the wave, which we explore in the next section.

5.2. Dispersion (dissipation) relation. Analyzing the linearized model, we are interested in the influence of the bubbles on the wave dissipation. This effect is controlled by the even derivatives, so we truncate the linearized equations (29) to the form

$$\frac{\partial v}{\partial \tau} = \frac{C_2}{C_1} \frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 \frac{C_4}{C_1} \frac{\partial^4 v}{\partial \xi^4} + \varepsilon^4 \frac{C_6}{C_1} \frac{\partial^6 v}{\partial \xi^6}. \quad (43)$$

For the Fourier modes $v \sim \exp(\lambda t + ikx)$, we get the dispersion (or dissipation) relation

$$\lambda(k) = -\frac{C_2}{C_1} k^2 + \varepsilon^2 \frac{C_4}{C_1} k^4 - \varepsilon^4 \frac{C_6}{C_1} k^6, \quad (44)$$

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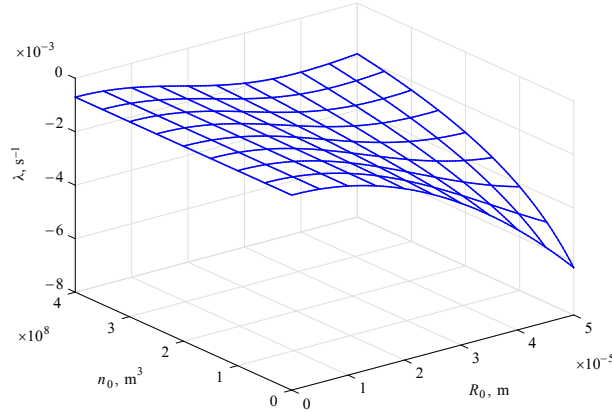


Figure 4. The decay rate by formula (44) for variant (a), $k_* = 0.25 \text{ m}^{-1}$.

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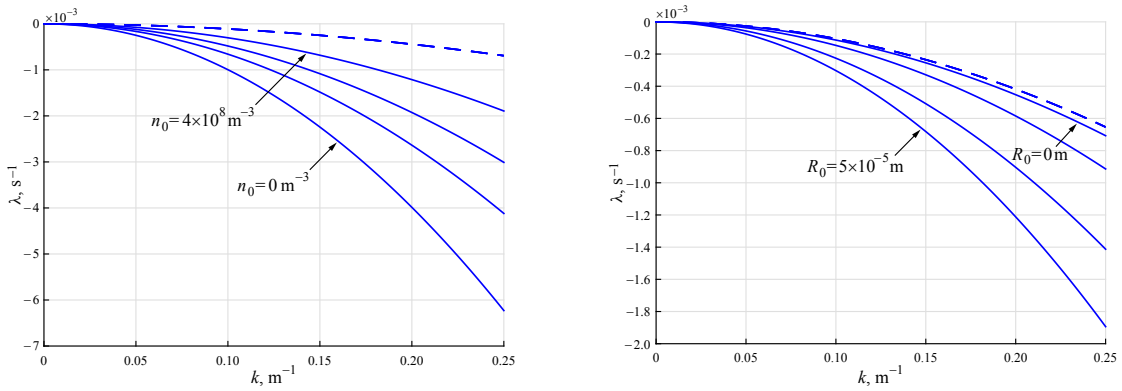


Figure 5. The decay rate by formulas (44) and (46) for variant (a). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

25 where λ is the decay rate and k is the wave number. For the model without the bubbles, the linearized
26 form of (42) is

$$\frac{\partial v}{\partial \tau} = \frac{D_2}{D_1} \frac{\partial^2 v}{\partial \xi^2} + \varepsilon^2 \frac{D_4}{D_1} \frac{\partial^4 v}{\partial \xi^4} \quad (45)$$

27 (we again consider only even derivatives). Accordingly, the dispersion relation is

$$\lambda(k) = -\frac{D_2}{D_1} k^2 + \varepsilon^2 \frac{D_4}{D_1} k^4. \quad (46)$$

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Figure 4 shows the decay rate by formula (44) at fixed $k_* = 0.25 \text{ m}^{-1}$ [Nikolaevskiy 1989] against R_0 and n_0 . See that the increase in R_0 significantly affects the decay rate and makes its absolute value larger due to the bubbles increasing their role through the pressure $p_1 = -p_0 \chi R_1$. As for n_0 , one should disregard the region of small n_0 in Figure 4 since the equations of continuum mechanics in the form adopted in the model become invalid when there are too few bubbles. This is because one can no longer assume that every fluid particle contains its own bubble (as suggested by (6)) because this would imply that the fluid particles are no longer small and, hence, the continuum mechanics description fails.

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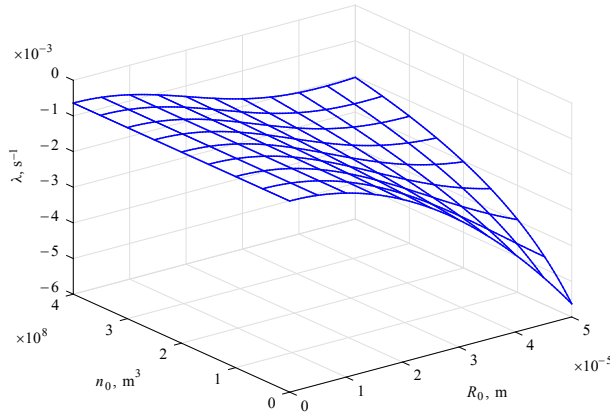


Figure 6. The decay rate by formula (44) for variant (b), $k_* = 0.25 \text{ m}^{-1}$.

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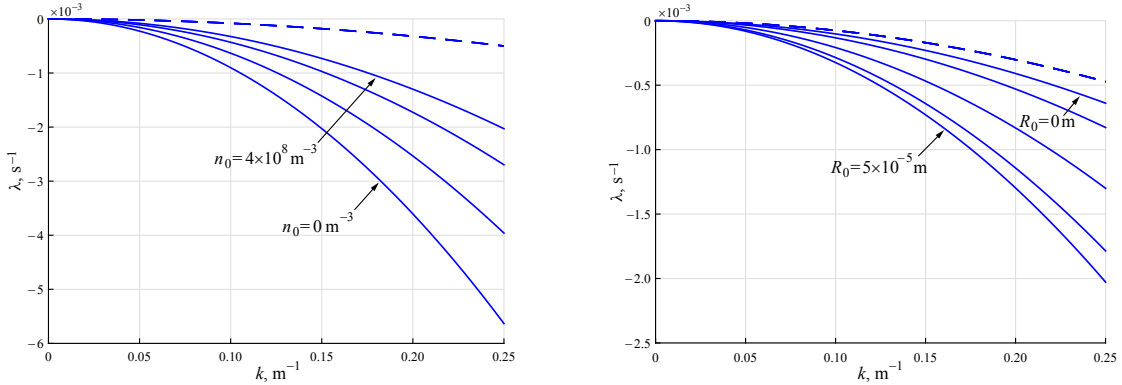


Figure 7. The decay rate by formulas (44) and (46) for variant (b). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

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Figure 5 compares the dispersion curves of the wave with the bubbles and the wave without the bubbles. The dashed line describes the case without the bubbles and the solid lines correspond to the wave with the bubbles. The figure on the left is for varying n_0 and fixed R_0 ; the figure on the right is for varying R_0 and fixed n_0 . The decay rate depends on the number and radius of the bubbles. We note that this result agrees with the conception discussed in [Strunin 2014; Strunin and Ali 2016] about the passive nature of the freely propagating elastic wave. Similar results are obtained for variants (b) and (c) as shown in Figures 6–9.

For a different $k_* = 0.52 \text{ m}^{-1}$ [Beresnev and Nikolaevskiy 1993], the results are similar; see Figures 10 and 11.

6. Conclusions

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41 We studied the effect of rheology with and without gas bubbles and of the bubble dynamics on the
42 dissipation of elastic waves in porous solids. The Frenkel–Biot waves of P1 type are analyzed in the

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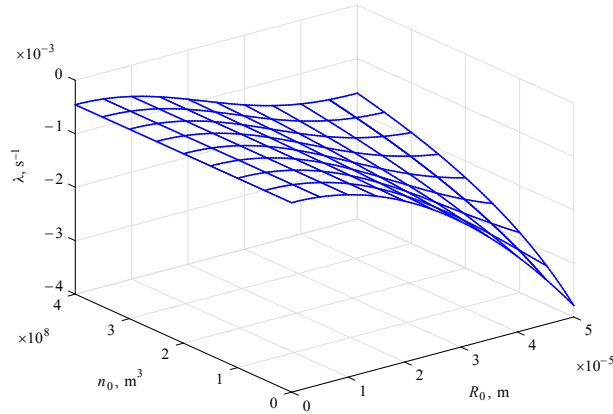


Figure 8. The decay rate by formula (44) for variant (c), $k_* = 0.25 m^{-1}$.

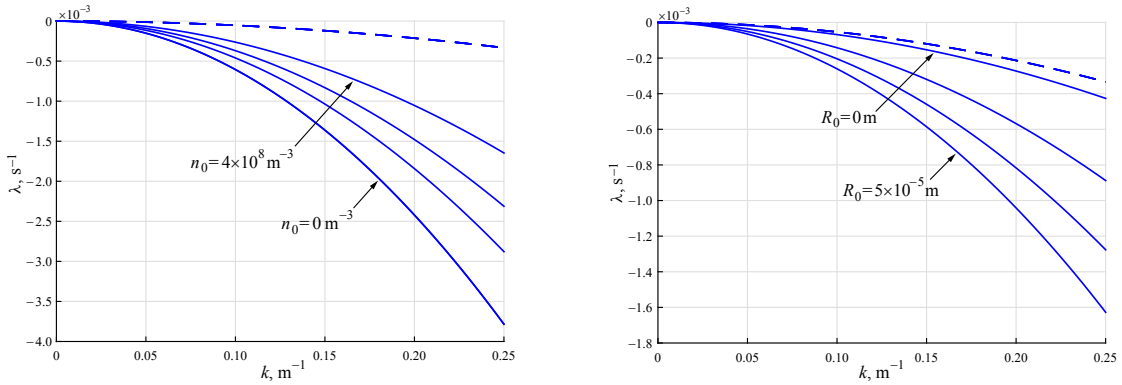


Figure 9. The decay rate by formulas (44) and (46) for variant (c). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

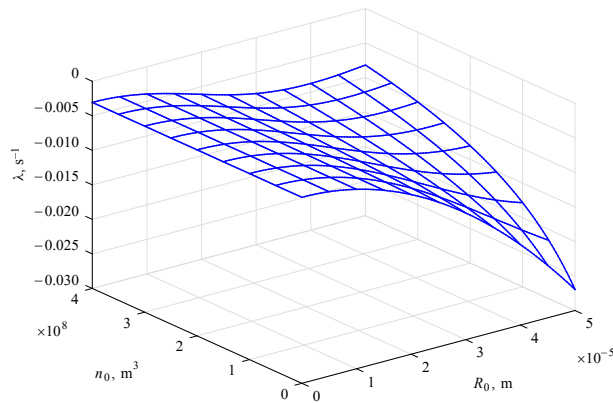


Figure 10. The decay rate by formula (44) for variant (a), $k_* = 0.52 m^{-1}$.

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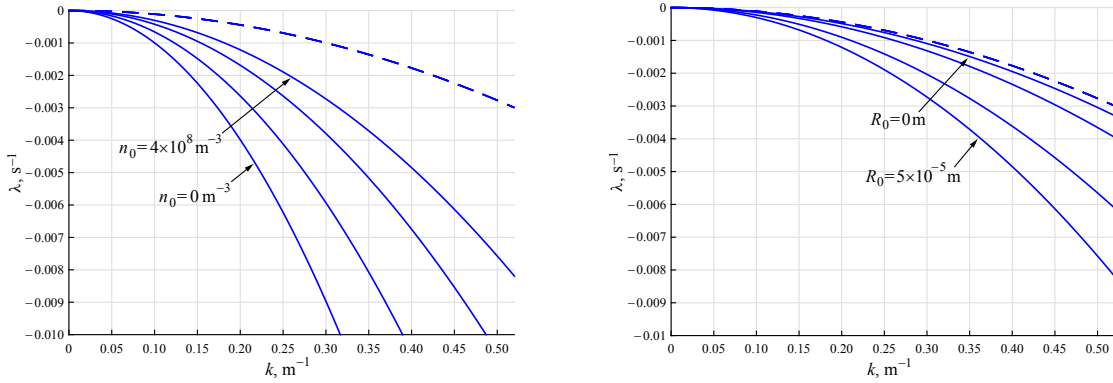


Figure 11. The decay rate by formulas (44) and (46) for variant (a). Left: n_0 varies, $R_0 = 5 \times 10^{-5}$. Right: R_0 varies, $n_0 = 4 \times 10^8$.

fluid-saturated environment. Using the three-segment rheological scheme (with the bubbles) and two-segment scheme (without the bubbles), we derived the Nikolaevskiy-type equations for the velocity of the solid matrix. The linearized versions of the equations are compared in terms of the decay rate $\lambda(k)$ of the Fourier modes. For the both cases — with and without the bubbles — the $\lambda(k)$ -curve lies entirely below the zero. We found out that $|\lambda(k)|$ increases with the increase of the radius of the bubbles but decreases with the increase of the number of the bubbles.

List of symbols

$\beta^{(s)}$: compressibility of solid (Pa ⁻¹)	M_i : masses (kg/m)
$\beta^{(L)}$: compressibility of water and gas (Pa ⁻¹)	k_b : bulk modulus (Pa)
$\rho_0^{(s)}$: density of solid (kg/m ³)	p : pressure (Pa)
$\rho_0^{(L)}$: density of water (kg/m ³)	$\sigma^{(ef)}$: effective stress (Pa)
$\rho_0^{(g)}$: density of gas (kg/m ³)	ϕ : volume gas content
R_0 : bubble radius (m)	m : porosity
ε : small parameter	ζ : adiabatic exponent
n_0 : number of bubbles (m ⁻³)	k : wave number (m ⁻¹)
μ : viscosity (Pa·s)	c : wave velocity (m/s)
ℓ : permeability (m ²)	v : particle velocity (m/s)
E_i : elastic moduli (Pa)	λ : decay rate (s ⁻¹)

Appendix A

Using equations (19), we collect the linear terms $\sim \varepsilon$ in system (5):

$$-(1 - m_0)\rho_0^{(s)}c \frac{\partial v_1}{\partial \xi} = \frac{\partial \sigma_1^{(ef)}}{\partial \xi} - (1 - m_0) \frac{\partial p_1}{\partial \xi},$$

$$-m_0\rho_0^{(f)}c \frac{\partial u_1}{\partial \xi} = -m_0 \frac{\partial p_1}{\partial \xi},$$

$$\begin{aligned}
 1^{1/2} \frac{1}{2} \quad & \rho_0^{(s)} c \frac{\partial m_1}{\partial \xi} - (1 - m_0) c \frac{\partial \rho_1^{(s)}}{\partial \xi} + (1 - m_0) \rho_0^{(s)} \frac{\partial v_1}{\partial \xi} = -\frac{1}{2} (1 - m_0) \frac{\partial \rho_0^{(s)}}{\partial \tau}, \\
 3 \quad & \\
 4 \quad & -m_0 c \frac{\partial \rho_1^{(f)}}{\partial \xi} - \rho_0^{(f)} c \frac{\partial m_1}{\partial \xi} + m_0 \rho_0^{(f)} \frac{\partial u_1}{\partial \xi} = -\frac{1}{2} m_0 \frac{\partial \rho_0^{(f)}}{\partial \tau}. \quad (47) \\
 5 \quad &
 \end{aligned}$$

6 The system (47) gives the integrals

$$\begin{aligned}
 8 \quad & (1 - m_0) \rho_0^{(s)} c v_1 = -\sigma_1^{(ef)} + (1 - m_0) p_1, \quad m_0 \rho_0^{(f)} c u_1 = m_0 p_1, \\
 9 \quad & (1 - m_0) \rho_0^{(s)} v_1 = ((1 - m_0) \rho_1^{(s)} - \rho_0^{(s)} m_1) c, \quad m_0 \rho_0^{(f)} u_1 = (\rho_0^{(f)} m_1 + m_0 \rho_1^{(f)}) c. \quad (48) \\
 10 \quad &
 \end{aligned}$$

11 According to (7) and (10) the terms $\sim \varepsilon$ in the density series are

$$13 \quad \rho_1^{(s)} = \rho_0^{(s)} \left(\beta^{(s)} p_1 - \frac{\beta^{(s)} \sigma_1^{(ef)}}{(1 - m_0)} \right), \quad \rho_1^{(f)} = \rho_0^{(L)} (\beta^{(L)} \kappa_1 p_1 - 4\pi n_0 \kappa_2 R_0^3 R_1), \quad (49)$$

15 and also

$$16 \quad \rho_0^{(f)} = \kappa_1 \kappa_2 \rho_0^{(L)}, \quad (50)$$

18 where

$$19 \quad \kappa_1 = 1 - \frac{4\pi}{3} R_0^3 n_0, \quad \kappa_2 = 1 + \beta^{(L)} p.$$

20^{1/2} Inserting (49) and (50) into the last two equations in (48) (mass equations), we get

$$22 \quad (1 - m_0) v_1 = [(1 - m_0) \beta^{(s)} p_1 - \beta^{(s)} \sigma_1^{(ef)} - m_1] c, \quad (51)$$

$$24 \quad m_0 u_1 = \left[m_1 + \frac{m_0 \beta^{(L)} p_1}{\kappa_2} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c. \quad (52)$$

26 The combination of (51) and (52) gives

$$28 \quad (1 - m_0) v_1 + m_0 u_1 = \left[\frac{(\beta + (1 - m_0) \beta^{(s)} \beta^{(L)} p_0) p_1}{\kappa_2} - \beta^{(s)} \sigma_1^{(ef)} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c, \quad (53)$$

31 where $\beta = (1 - m_0) \beta^{(s)} + m_0 \beta^{(L)}$.

32 The condition (15) means $v_1 = u_1$, therefore (53) becomes

$$34 \quad v_1 = \left[\frac{(\beta + (1 - m_0) \beta^{(s)} \beta^{(L)} p_0) p_1}{\kappa_2} - \beta^{(s)} \sigma_1^{(ef)} - \frac{4\pi n_0 m_0 R_0^3 R_1}{\kappa_1} \right] c. \quad (54)$$

36 Due to the conditions $v_1 = u_1$, $\rho_0 = (1 - m_0) \rho_0^{(s)} + m_0 \rho_0^{(L)}$ and using (50), the first two of the momentum equations (48) give

$$39 \quad \rho_0 c v_1 = -\sigma_1^{(ef)} + A p_1, \quad (55)$$

40 where

$$42 \quad A = (1 - m_0) + \frac{m_0}{\kappa_1 \kappa_2}.$$

1 Now, the linear terms $\sim \varepsilon$ in relations (14) and (18) give

$$1^{1/2} \frac{1}{2} \frac{\partial e_0}{\partial \tau} - c \frac{\partial e_1}{\partial \xi} + v_1 \frac{\partial e_0}{\partial \xi} = \frac{\partial v_1}{\partial \xi}, \quad (56)$$

$$\frac{3}{4} \sigma_1^{(ef)} - E_2 e_1 - \beta^{(s)} k_b p_1 = T, \quad (57)$$

5 where

$$6 \quad T \equiv \eta \left[\sum_{q=1,2,3,5} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_0}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1} \frac{\partial^q \sigma_0^{(ef)}}{\partial \xi^q} \right].$$

10 The linear terms $\sim \varepsilon$ in the bubble (6) give

$$11 \quad -\frac{\mu c}{\rho_0^{(L)} \kappa_2} \left[\frac{4}{R_0} + \frac{m_0 R_0}{\ell} \right] \frac{\partial R_0}{\partial \xi} = -\frac{1}{\rho_0^{(L)} \kappa_2} (p_0 \chi R_1 + p_1). \quad (58)$$

14 Equations (56), (57), and (58) lead to the integrals

$$15 \quad e_1 = -\frac{v_1}{c}, \quad \sigma_1^{(ef)} = E_2 e_1 + \beta^{(s)} k_b p_1, \quad p_1 = -p_0 \chi R_1. \quad (59)$$

18 The effective stress $\sigma_1^{(ef)}$ in (59) can be rewritten as

$$20^{1/2} \sigma_1^{(ef)} = -\left[\frac{E_2 v_1}{c} + p_0 \chi \beta^{(s)} k_b R_1 \right]. \quad (60)$$

22 Substituting (60) and the value of p_1 from (59) into (54) leads to

$$23 \quad (1 - \beta^{(s)} E_2) v_1 + (B - k_b \beta^{(s)} \beta^{(s)}) p_0 \chi R_1 c = 0, \quad (61)$$

25 where

$$26 \quad B = \frac{(\beta + (1 - m_0) \beta^{(s)} \beta^{(L)} p_0)}{\kappa_2} + \frac{4\pi n_0 m_0 R_0^3}{\kappa_1 p_0 \chi}.$$

28 Now, from (55) and using the value of p_1 from (59), we obtain the effective stress as

$$30 \quad \sigma_1^{(ef)} = -(\rho_0 c v_1 + A) p_0 \chi R_1. \quad (62)$$

32 The combination of (60) and (62) results in

$$33 \quad (E_2 - \rho_0 c^2) v_1 - (A - k_b \beta^{(s)}) p_0 \chi R_1 c = 0. \quad (63)$$

35 Appendix B

37 Collecting the quadratic terms $\sim \varepsilon^2$ in (18), we get

$$38 \quad \sigma_2^{(ef)} - E_2 e_2 - \beta^{(s)} k_b p_2 = T, \quad (64)$$

39^{1/2} where

$$40 \quad T \equiv \eta \left[\sum_{q=1,2,3,5} a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_1}{\partial \xi^q} + \sum_{q=1,3} b_q c^q \varepsilon^{q-1} \frac{\partial^q \sigma_1^{(ef)}}{\partial \xi^q} \right].$$

1 Note that here we keep (as Nikolaevskiy [2008] did) the higher powers of ε to represent small corrections
 2 to the leading terms. These corrections will eventually translate into small corrections in the derived
 3 Nikolaevskiy equation further in this paper; they will be the object of our study. Thus,

$$\frac{\partial \sigma_2^{(ef)}}{\partial \xi} - E_2 \frac{\partial e_2}{\partial \xi} - \beta^{(s)} k_b \frac{\partial p_2}{\partial \xi} = \frac{\partial T}{\partial \xi}. \quad (65)$$

4 From (14) in the order $\sim \varepsilon^2$, we get

$$\frac{\partial}{\partial \xi} (c e_2 + v_2) = F, \quad F = -\frac{1}{c} \left(\frac{1}{2} \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1 v_1}{\partial \xi} \right). \quad (66)$$

7 Therefore,

$$\frac{\partial e_2}{\partial \xi} = \frac{F}{c} - \frac{1}{c} \frac{\partial v_2}{\partial \xi}. \quad (67)$$

8 Substituting (67) into (65) we obtain

$$\frac{\partial}{\partial \xi} (c \sigma_2^{(ef)} + E_2 v_2 - c \beta^{(s)} k_b p_2) = E_2 F + c \frac{\partial T}{\partial \xi}. \quad (68)$$

9 From the momentum equations (5) for the solid and liquid, we get

$$(1 - m_0) \rho_0^{(s)} c \frac{\partial v_2}{\partial \xi} + \frac{\partial \sigma_2^{(ef)}}{\partial \xi} - (1 - m_0) \frac{\partial p_2}{\partial \xi} = \Sigma_1, \quad (69)$$

10 where

$$\Sigma_1 = (1 - m_0) \rho_0^{(s)} \left(\frac{1}{2} \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1 v_1}{\partial \xi} \right) - (1 - m_0) \rho_1^{(s)} c \frac{\partial v_1}{\partial \xi} + m_1 \rho_0^{(s)} c \frac{\partial v_1}{\partial \xi} - m_1 \frac{\partial p_1}{\partial \xi} + \varepsilon^{\gamma-1} \delta m_0 v_1$$

11 and

$$m_0 \rho_0^{(f)} c \frac{\partial u_2}{\partial \xi} - m_0 \frac{\partial p_2}{\partial \xi} = \Sigma_2, \quad (70)$$

12 where

$$\Sigma_2 = m_0 \rho_0^{(f)} \left(\frac{1}{2} \frac{\partial u_1}{\partial \tau} + \frac{\partial u_1 u_1}{\partial \xi} \right) - m_0 \rho_1^{(f)} c \frac{\partial u_1}{\partial \xi} - m_1 \rho_0^{(f)} c \frac{\partial u_1}{\partial \xi} + m_1 \frac{\partial p_1}{\partial \xi} - \varepsilon^{\gamma-1} \delta m_0 u_1.$$

13 Due to the condition (15), the combination of (69) with (70) gives

$$\rho_0 c \frac{\partial v_2}{\partial \xi} + \frac{\partial \sigma_2^{(ef)}}{\partial \xi} - \frac{\partial p_2}{\partial \xi} = \Sigma, \quad (71)$$

14 where $\Sigma = \Sigma_1 + \Sigma_2$, so that

$$\begin{aligned} \Sigma = & \rho_0 \left(\frac{1}{2} \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1 v_1}{\partial \xi} \right) - c \left((1 - m_0) \rho_0^{(s)} \beta^{(s)} + m_0 \rho_0^{(L)} \beta^{(L)} \kappa_1 \right) \frac{\partial p_1 v_1}{\partial \xi} \\ & + c \rho_0^{(s)} \beta^{(s)} \frac{\partial \sigma_1^{(ef)} v_1}{\partial \xi} + c \rho_0^{(L)} 4\pi n_0 m_0 \kappa_2 R_0^3 \frac{\partial R_1 v_1}{\partial \xi} + c \left(\rho_0^{(s)} - \kappa_1 \kappa_2 \rho_0^{(L)} \right) \frac{\partial m_1 v_1}{\partial \xi}. \end{aligned}$$

15 Equations (68) and (71) result in

$$\frac{\partial}{\partial \xi} [(E_2 - \rho_0 c^2) v_2 + (1 - \beta^{(s)} k_b) c p_2] = E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi}. \quad (72)$$

1 From the bubble equation (6), in the order $\sim \varepsilon^2$,

$$1^{1/2} \frac{-\mu c}{\rho_0^{(L)} \kappa_2} \left(4 + \frac{m_0 R_0^2}{\ell}\right) \frac{\partial R_1}{\partial \xi} = \frac{1}{\rho_0^{(L)} \kappa_2} \left[\frac{\beta^{(L)} p_0 \chi}{\kappa_2} p_1 R_1 + \frac{\beta^{(L)}}{\kappa_2} p_1^2 + \frac{p_0 \chi (\chi + 1)}{2} R_1^2 - p_0 \chi R_2 - p_2 \right]. \quad (73)$$

5 We rewrite (73) as

$$6 \quad p_2 = \Gamma - p_0 \chi R_2, \quad (74)$$

7 where

$$8 \quad \Gamma = \mu c \left(4 + \frac{m_0 R_0^2}{\ell}\right) \frac{\partial R_1}{\partial \xi} + \frac{\beta^{(L)} p_1}{\kappa_2} (p_0 \chi R_1 + p_1) + \frac{1}{2} p_0 \chi (\chi + 1) R_1^2.$$

11 Now we substitute the value of p_2 from (74) into (72) to get

$$12 \quad \frac{\partial}{\partial \xi} [(E_2 - \rho_0 c^2) v_2 - (1 - \beta^{(s)} k_b) p_0 \chi R_2 c] = E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi} - c(1 - \beta^{(s)} k_b) \frac{\partial \Gamma}{\partial \xi}. \quad (75)$$

14 In the second order, the mass balances (5) for the solid and liquid-gas mixture have the form

$$16 \quad \frac{\partial}{\partial \xi} ((1 - m_0) v_2 - [(1 - m_0) \beta^{(s)} p_2 - \beta^{(s)} \sigma_2^{(ef)} - m_2] c) = \Lambda^{(s)} / \rho_0^{(s)}, \quad (76)$$

$$18 \quad \frac{\partial}{\partial \xi} \left(m_0 u_2 - \left[m_2 + \frac{m_0 \beta^{(L)} p_2}{\kappa_2} - \frac{4\pi m_0 n_0 R_0^3 (R_2 + R_1^2)}{\kappa_1} + \frac{4\pi m_0 n_0 R_0^3 p_0 \chi \beta^{(L)} R_1^2}{\kappa_1 \kappa_2} \right] c \right) = \frac{\Lambda^{(L)}}{\rho_0^{(L)}}, \quad (77)$$

20^{1/2} where

$$22 \quad \Lambda^{(s)} = \rho_0^{(s)} \frac{1}{2} \frac{\partial}{\partial \tau} [(m_1 - (1 - m_0) \beta^{(s)} p_1 + \beta^{(s)} \sigma_1^{(ef)})] \\ 23 \quad + \rho_0^{(s)} \frac{\partial}{\partial \xi} \left[m_1 v_1 - ((1 - m_0) p_1 + \sigma_1^{(ef)}) \beta^{(s)} v_1 - c \beta^{(s)} m_1 \left(p_1 - \frac{\sigma_1^{(ef)}}{(1 - m_0)} \right) \right], \quad (78)$$

$$27 \quad \Lambda^{(L)} = -\rho_0^{(L)} \frac{1}{2} \frac{\partial}{\partial \tau} [\kappa_1 (m_1 \kappa_2 + m_0 \beta^{(L)} p_1) - 4\pi n_0 \kappa_2 R_0^3 R_1] \\ 28 \quad + \rho_0^{(L)} \frac{\partial}{\partial \xi} [(\beta^{(L)} \kappa_1 p_1 - 4\pi n_0 \kappa_2 R_0^3 R_1) (c m_1 - m_0 u_1)] - \kappa_1 \kappa_2 \rho_0^{(L)} \frac{\partial m_1 u_1}{\partial \xi}. \quad (79)$$

30 The combination of (76) and (77) gives

$$32 \quad \frac{\partial}{\partial \xi} \left[v_2 - \left(\frac{(\beta + (1 - m_0) \beta^{(s)} \beta^{(L)} p_0) p_2}{\kappa_2} - \beta^{(s)} \sigma_2^{(ef)} \right. \right. \\ 33 \quad \left. \left. - \frac{4\pi m_0 n_0 R_0^3 (R_2 + R_1^2)}{\kappa_1} + \frac{4\pi m_0 n_0 R_0^3 p_0 \chi \beta^{(L)} R_1^2}{\kappa_1 \kappa_2} \right) c \right] = \Lambda, \quad (80)$$

36 where

$$37 \quad \Lambda \equiv \Lambda^{(s)} / \rho_0^{(s)} + \Lambda^{(L)} / \rho_0^{(L)}.$$

39^{1/2} From (68) we have

$$40 \quad \frac{\partial \sigma_2^{(ef)}}{\partial \xi} = \frac{\partial}{\partial \xi} \left(T + k_b \beta^{(s)} \Gamma - k_b \beta^{(s)} p_0 \chi R_2 - \frac{E_2}{c} v_2 \right) + \frac{1}{c} E_2 F. \quad (81)$$

1 Now we insert (81) and the value of p_2 represented by (74) into (80),

$$\begin{aligned} & \frac{\partial}{\partial \xi} \left[(1 - E_2 \beta^{(s)}) v_2 - \left(\omega_1 p_0 \chi - \frac{4\pi m_0 n_0 R_0^3}{\kappa_1} \right) R_2 c \right] \\ & = \Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi} - c \omega_1 \frac{\partial \Gamma}{\partial \xi} + c \omega_2 \frac{\partial R_1^2}{\partial \xi}, \end{aligned} \quad (82)$$

7 where

$$\omega_1 = k_b \beta^{(s)} \beta^{(s)} + \frac{\beta + (1 - m_0) \beta^{(s)} \beta^{(L)} p_0}{\kappa_2}, \quad \omega_2 = \frac{4\pi m_0 n_0 R_0^3 \beta^{(L)} p_0 \chi}{\kappa_1 \kappa_2} - \frac{4\pi m_0 n_0 R_0^3}{\kappa_1}.$$

Appendix C

13 Equation (26) can be illustrated by the following simple example

$$v_1 + c R_1 = 0, \quad 2v_1 + 4R_1 = 0.$$

16 A nonzero solution of the system exists only if $c = 2$ (the eigenvalue of the problem). Here v_1 and R_1 are
17 analogous to the first approximation from our main text. The second approximation, v_2 and R_2 , satisfies
18 the system

$$v_2 + c R_2 = f[v_1], \quad 2v_2 + 4R_2 = g[v_1],$$

21 which is solvable only if the right-hand sides satisfy the condition $g[v_1] = 2f[v_1]$. This solvability
22 condition is the analogy to the Nikolaevskiy-type equation that we aim to derive.

Appendix D

26 The first approximations for the momentum and mass-balance equations without gas bubbles are the
27 same as for the system (5). As for the density equations, the solid density remains unchanged but for
28 gas-liquid mixture we neglect the volume gas content ϕ in (9):

$$\rho^{(f)} = \rho^{(L)} = \rho_0^{(L)} (1 + \beta^{(L)} p). \quad (83)$$

31 The first approximation of (83) is

$$\rho_1^{(L)} = \rho_0^{(L)} \beta^{(L)} p_1. \quad (84)$$

34 Inserting this into the mass equation for the fluid (48), we get

$$m_0 u_1 = [m_1 + m_0 \beta^{(L)} p_1] c. \quad (85)$$

37 Now, the combination of (51) and (85) yields

$$(1 - m_0) v_1 + n_0 u_1 = [(1 - m_0) \beta^{(s)} p_1 - \beta^{(s)} \sigma_1^{(ef)} + m_0 \beta^{(L)} p_1] c. \quad (86)$$

40 Due to the condition (15), Equation (86) becomes

$$v_1 = [(1 - m_0) \beta^{(s)} p_1 - \beta^{(s)} \sigma_1^{(ef)} + m_0 \beta^{(L)} p_1] c. \quad (87)$$

1 After we apply the conditions

1^{1/2}
2
3

$$v_1 = u_1, \quad \rho_0 = (1 - m_0)\rho_0^{(s)} + m_0\rho_0^{(L)},$$

4 the first two equations in (48) give

5
6
7

$$\rho_0 c v_1 = -\sigma_1^{(ef)} + p_1. \quad (88)$$

8 The first approximation of relation (32) is

9
10

$$\sigma_1^{(ef)} - E_2 e_1 - \beta^{(s)} k_b p_1 = \sum_{q=1}^3 a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_0}{\partial \xi^q} + b_1 c \frac{\partial \sigma_0^{(ef)}}{\partial \xi}. \quad (89)$$

11 Equations (89) and (56) result in the integral

12
13

$$\sigma_1^{ef} = -\frac{E_2 v_1}{c} + \beta^{(s)} k_b p_1. \quad (90)$$

14 Substituting (90) into (87), we get

15
16

$$(1 - \beta^{(s)} E_2) v_1 = c(1 - m_0) \beta^{(s)} p_1 + c m_0 \beta^{(L)} p_1 - \beta^{(s)} \beta^{(s)} k_b c p_1. \quad (91)$$

17 As $\beta = (1 - m_0) \beta^{(s)} + m_0 \beta^{(L)}$, (91) becomes

18
19
20

$$(1 - \beta^{(s)} E_2) v_1 = (\beta - \beta^{(s)} \beta^{(s)} k_b) c p_1. \quad (92)$$

20^{1/2} From (88) we obtain

21
22

$$\sigma_1^{(ef)} = p_1 - \rho_0 c v_1. \quad (93)$$

23 Therefore, the combination of (93) with (90) yields

24
25

$$(\rho_0 c^2 - E_2) v_1 = (1 - \beta^{(s)} k_b) c p_1. \quad (94)$$

Appendix E

28 In the second approximation for the system without the bubbles we again arrive at an equation of the
29 form (72), except the formulas for Σ and T are changed:

30
31

$$\frac{\partial}{\partial \xi} [(E_2 - \rho_0 c^2) v_2 + (1 - \beta^{(s)} k_b) c p_2] = E_2 F - c \Sigma + c \frac{\partial T}{\partial \xi}, \quad (95)$$

32 where

33
34

$$\Sigma = \rho_0 \left(\frac{1}{2} \frac{\partial v_1}{\partial \tau} + \frac{\partial v_1 v_1}{\partial \xi} \right) - c \rho_0 \beta \frac{\partial p_1 v_1}{\partial \xi} + c \rho_0^{(s)} \beta^{(s)} \frac{\partial \sigma_1^{(ef)} v_1}{\partial \xi} + c (\rho_0^{(s)} - \rho_0^{(L)}) \frac{\partial m_1 v_1}{\partial \xi},$$

35
36
37

$$T = \sum_{q=1}^3 a_q (-c)^q \varepsilon^{q-1} \frac{\partial^q e_1}{\partial \xi^q} + b_1 c \frac{\partial \sigma_1^{(ef)}}{\partial \xi}.$$

39^{1/2} The second approximation of the mass balance for the solid is the same as (76), while for the fluid it
40 takes the form

41
42

$$\frac{\partial}{\partial \xi} [m_0 u_2 - (m_2 + m_0 \beta^{(L)} p_2) c] = \Lambda^{(L)} / \rho_0^{(L)}, \quad (96)$$

1 where

$$2 \quad \Lambda^{(L)} = -\frac{1}{2}\rho_0^{(L)} \frac{\partial}{\partial \tau} [m_1 + m_0\beta^{(L)} p_1] + \rho_0^{(L)} \frac{\partial}{\partial \xi} [cm_1\beta^{(L)} p_1 - m_0\beta^{(L)} p_1 u_1 - m_1 u_1].$$

4 The combination of (76) and (96) results in

$$5 \quad \frac{\partial}{\partial \xi} [v_2 - (\beta p_2 - \beta^{(s)} \sigma_2^{(ef)}) c] = \Lambda, \quad (97)$$

7 where

$$8 \quad \Lambda = (\Lambda^{(s)} / \rho_0^{(s)}) + (\Lambda^{(L)} / \rho_0^{(L)}).$$

10 From (68) we find

$$11 \quad \frac{\partial \sigma_2^{(ef)}}{\partial \xi} = \frac{\partial}{\partial \xi} \left(T + k_b \beta^{(s)} p_2 - \frac{E_2}{c} v_2 \right) + \frac{1}{c} E_2 F. \quad (98)$$

13 Substituting (98) into (97) we get

$$14 \quad \frac{\partial}{\partial \xi} [(1 - E_2 \beta^{(s)}) v_2 - (\beta - \beta^{(s)} \beta^{(s)}) c p_2] = \Lambda - \beta^{(s)} E_2 F - c \beta^{(s)} \frac{\partial T}{\partial \xi}. \quad (99)$$

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