

# Solution of a modified fractional diffusion equation

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## Abstract

Recently, a modified fractional diffusion equation has been proposed [I. Sokolov, J. Klafter, From diffusion to anomalous diffusion: a century after Einstein's brownian motion, *Chaos* 15 (2005) 026103; A.V. Chechkin, R. Gorenflo, I.M. Sokolov, V.Yu. Gonchar, Distributed order time fractional diffusion equation, *Frac. Calc. Appl. Anal.* 6 (3) (2003) 259–279; I.M. Sokolov, A.V. Chechkin, J. Klafter, Distributed-order time fractional kinetics, *Acta. Phys. Pol. B* 35 (2004) 1323.] for describing processes that become less anomalous as time progresses by the inclusion of a second fractional time derivative acting on the diffusion term. In this letter we give the solution of the modified equation on an infinite domain. In contrast to the solution of the traditional fractional diffusion equation, the solution of the modified equation requires a summation of Fox functions instead of a single Fox function.

*Key words:* Modified Fractional diffusion equation, Anomalous diffusion, Fox function

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## 1 Introduction

In recent years numerous physical and biological systems have been reported in which the diffusion rates of species cannot be characterized by the single parameter of the diffusion constant [1]. Instead, the (anomalous) diffusion is characterized by a scaling parameter  $\gamma$  as well as a diffusion constant  $D$  and the mean square displacement of diffusing species  $\langle r^2(t) \rangle$  scales as a nonlinear power law in time, i.e.,  $\langle r^2(t) \rangle \sim t^\gamma$ . As examples, single particle

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tracking experiments and photo-bleaching recovery experiments have revealed sub-diffusion ( $0 < \gamma < 1$ ) of proteins and lipids in a variety of cell membranes [2–8]. Anomalous subdiffusion has also been observed in neural cell adhesion molecules [9]. Indeed anomalous sub-diffusion (the case with  $0 < \gamma < 1$ ) is generic in media with obstacles [10,11] or binding sites [12]. Recently Reynolds [13] have shown that the solution of fractional diffusion equation fits well the diffusion of proteins within plasma membranes.

There are numerous approaches to modelling anomalous diffusive behaviour such as, Continuous Time Random Walks (CTRW), Monte Carlo simulations [11], Langevin equations and fractional diffusion equations [14]. The fractional diffusion equation is characterised by the presence of either a fractional temporal derivative or fractional spatial derivative or both (time-fractional diffusion equations were introduced by Zaslavsky [15], see Refs. [14,16] for a recent review). Other fractional variants are the fractional Fokker-Planck equation [17,18] for anomalous diffusion due to an externally force and fractional reaction-diffusion equations [19–23] for reactions where the products and reactants diffuse anomalously. Recently Henry et al. [24,25] have shown that the inclusion of a fractional temporal derivative greatly affects the solution behaviour of Turing-instability induced patterns compared with the solution of the standard non-fractional reaction-diffusion equations. Note these equations involve only a single temporal fractional derivative acting on the diffusion term.

However recently, Sokolov and Klafter [1,26,27] proposed a model for describing processes that become less anomalous as time progresses by the inclusion of a secondary fractional time derivative acting on a linear operator,  $\mathcal{L}_x$ ,

$$\frac{\partial P(x, t)}{\partial t} = \left( A \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) \mathcal{L}_x P(x, t), \quad (1)$$

where  $0 < \alpha < \beta \leq 1$  and  $A$  and  $B$  are positive dimensionless constants. A possible application of this equation is in econophysics where there is an increasing interest in modelling using CTRWs [28–34]. In particular the crossover between more and less anomalous behaviour has been observed in the volatility of some share prices [35–37].

Note in Eq. (1) the operator

$$\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}}, \quad (2)$$

is the Riemann-Liouville fractional derivative operator defined by

$$\frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} Y(t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{Y(t')}{(t-t')^{1-\alpha}} dt'. \quad (3)$$

In this article we find the Green solution of this model in the case of where the linear operator is taken as the diffusion operator (though it can be replaced by the Fokker-Planck operator)

$$\mathcal{L}_x = K \frac{\partial^2}{\partial x^2}, \quad (4)$$

with diffusion coefficient,  $K$ .

In dimensionless variables this equation reads

$$\frac{\partial P}{\partial t} = \left( A \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} + B \frac{\partial^{1-\beta}}{\partial t^{1-\beta}} \right) \frac{\partial^2 P}{\partial x^2}, \quad (5)$$

where  $K$  has been absorbed into  $A$  and  $B$ .

At this point we note that Henry and Wearne [19] find in their derivation of fractional reaction-diffusion equations an additional term

$$A\mathcal{L}^{-1} \left[ \frac{\partial^{-\alpha}}{\partial t^{-\alpha}} \frac{\partial^2 P}{\partial x^2} \right]_{t=0} + B\mathcal{L}^{-1} \left[ \frac{\partial^{-\beta}}{\partial t^{-\beta}} \frac{\partial^2 P}{\partial x^2} \right]_{t=0}, \quad (6)$$

on the right of Eq. (5). The value of this term is unclear as it necessitates the behaviour of the term to be known near  $t = 0$ . However it can be shown from the solution,  $P(x, t)$ , that these terms are zero and can be neglected.

In the next section we find the solution to Eq. (5) in an infinite domain in terms of Fox functions [14]. In the final section we compare this solution with the traditional fractional diffusion equation and give some concluding remarks.

## 2 Infinite solution

In this section we find the Green's solution for Eq. (5) in an infinite domain. We now take a spatial Fourier Transform and a temporal Laplace Transform noting that the Laplace transform of the Riemann-Liouville fractional derivative is given by

$$\mathcal{L} \left\{ \frac{\partial^{1-\alpha}}{\partial t^{1-\alpha}} Y(t) \right\} (s) = s^{1-\alpha} \mathcal{L} \{ Y \} (s) - \left[ \frac{\partial^{-\alpha}}{\partial t^{-\alpha}} Y(t) \right]_{t=0}, \quad (7)$$

for  $0 < \alpha \leq 1$ . The expression for the solution in Fourier-Laplace space is

$$\hat{\tilde{P}}(q, s) = \frac{\tilde{f}(q)}{s + Aq^2 s^{1-\alpha} + Bq^2 s^{1-\beta}}, \quad (8)$$

where  $q$  and  $s$  are the Fourier and Laplace variables and the tilde and hat denote Fourier and Laplace transformed functions, respectively. The function  $\tilde{f}(q)$  is defined in terms of the initial conditions to the problem and is given by

$$\tilde{f}(q) = \tilde{P}(q, 0) + Aq^2 \left[ \frac{\partial^{-\alpha}}{\partial t^{-\alpha}} \tilde{P}(q, t) \right]_{t=0} + Bq^2 \left[ \frac{\partial^{-\beta}}{\partial t^{-\beta}} \tilde{P}(q, t) \right]_{t=0}. \quad (9)$$

Note if we include the additional terms in Eq. (6) then the last two terms for  $\tilde{f}(q)$  cancel. However if we do not include the additional terms then it can be shown using the solution of  $P(x, t)$  that the two fractional integrals are nevertheless zero.

In terms of finding the Green's function we need only to concentrate on the function

$$\hat{\tilde{G}}(q, s) = \frac{1}{s + Aq^2 s^{1-\alpha} + Bq^2 s^{1-\beta}}, \quad (10)$$

and the final solution can then be found through a convolution.

We can show by inverting the Fourier transform in Eq. (10)

$$\hat{G}(x, s) = \frac{1}{2\sqrt{s(A s^{1-\alpha} + B s^{1-\beta})}} \exp\left(-|x| \sqrt{\frac{s}{A s^{1-\alpha} + B s^{1-\beta}}}\right), \quad (11)$$

that the Green's function is an even function of  $x$ . Unfortunately, inverting the Laplace transform from this equation is problematic.

However we can invert the Laplace transform in Eq. (10) by first rewriting  $\hat{\tilde{G}}(q, s)$  in the form

$$\hat{\tilde{G}}(q, s) = \frac{s^{\alpha-1}}{s^\alpha + Aq^2} \frac{1}{1 + \frac{Bq^2 s^{\alpha-\beta}}{s^\alpha + Aq^2}}. \quad (12)$$

Now expanding the second fraction and simplifying we have

$$\hat{\tilde{G}}(q, s) = \sum_{k=0}^{\infty} \frac{(-Bq^2)^k k! s^{\alpha-(1+(\beta-\alpha)k)}}{k! (s^\alpha + Aq^2)^{k+1}}. \quad (13)$$

From Podlubny [38] we have the following Laplace transform involving the derivative of the Mittag-Leffler function,

$$\mathcal{L} \left\{ t^{\alpha k + \beta - 1} E_{\alpha, \beta}^{(k)}(-at^\alpha) \right\} (s) = \frac{k! s^{\alpha-\beta}}{(s^\alpha + a)^{k+1}}, \quad (14)$$

where the derivative of the Mittag-Leffler function is given by

$$E_{\alpha, \beta}^{(k)}(y) = \frac{d^k E_{\alpha, \beta}(y)}{dy^k} = \sum_{j=0}^{\infty} \frac{(j+k)! y^j}{j! \Gamma(\alpha(j+k) + \beta)}. \quad (15)$$

Setting  $\alpha = \alpha$ ,  $a = q^2$ , and  $\beta = 1 + (\beta - \alpha)k$  in Eq. (14) we can then invert the Laplace transform in Eq. (13) to give

$$\tilde{G}(q, t) = \sum_{k=0}^{\infty} \frac{(-Bt^\beta)^k}{k!} q^{2k} E_{\alpha, 1+(\beta-\alpha)k}^{(k)}(-Aq^2 t^\alpha). \quad (16)$$

Previously, [39] has shown that the Fourier inverse of the derivative of the Mittag-Leffler function in Eq. (15) can be achieved by first rewriting the derivative in terms a Fox function [14]

$$E_{\alpha, \beta}^{(k)}(y) = H_{1,2}^{1,1} \left[ -y \left| \begin{matrix} (-k, 1) \\ (0, 1) \end{matrix} \right. \begin{matrix} (1 - (\alpha k + \beta), \alpha) \end{matrix} \right]. \quad (17)$$

So to invert the transform in Eq. (16) we need only to invert, for each  $k$ , the term

$$\tilde{h}_k(q, t) = q^{2k} H_{1,2}^{1,1} \left[ Aq^2 t^\alpha \left| \begin{matrix} (-k, 1) \\ (0, 1) \end{matrix} \right. \begin{matrix} (-k\beta, \alpha) \end{matrix} \right]. \quad (18)$$

To invert the Fourier transform we first recall the Fourier transform of an even function can be written in terms of a Fourier Cosine transform

$$\mathcal{F}\{f(x)\}(q) = \sqrt{\frac{\pi}{2}} \mathcal{F}_c\{f(x) + f(-x)\}(q), \quad (19)$$

and that the Mellin transform of a Fourier Cosine transform is given by [40]

$$\mathcal{M}\{\mathcal{F}_c[\phi(x)](q)\}(z) = \sqrt{\frac{2}{\pi}} \Gamma(z) \cos\left(\frac{\pi z}{2}\right) \mathcal{M}\{\phi(x)\}(1-z), \quad (20)$$

where  $z$  is the Mellin transform variable.

Now since we know from Eq. (11) the Green's function is an even function then we can assume  $f(x)$  is also even and we then have the Mellin transform of the Fourier transform of  $f(x)$  is

$$\begin{aligned} \mathcal{M}\{\mathcal{F}[f(x)](q)\}(z) &= 2\sqrt{\frac{\pi}{2}} \mathcal{M}\{\mathcal{F}_c[f(x)](q)\}(z) \\ &= 2\Gamma(z) \cos\left(\frac{\pi z}{2}\right) \mathcal{M}\{f(x)\}(1-z). \end{aligned} \quad (21)$$

So to invert the Fourier transform,  $\tilde{h}_k(q, t)$ , in Eq. (18) we first evaluate its Mellin transform (the left hand side of Eq. (21)) to find the Mellin transform of  $h_k(x, t)$  (the right hand side of Eq. (21)). The Mellin transform then need only be inverted to find the Fourier inverse,  $h_k(x, t)$ .

To find the Mellin transform of Eq. (18) we note the Mellin transform of a Fox function is given by [41]

$$\mathcal{M} \left\{ H_{p,q}^{m,n} \left[ ax \left| \begin{matrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{matrix} \right. \right] \right\} (z) = a^{-z} \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j z) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j z)}{\prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j z) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j z)}, \quad (22)$$

when the following conditions are met

$$\delta = \sum_{j=1}^q \beta_j - \sum_{j=1}^p \alpha_j > 0, \quad (23)$$

$$A = \sum_{j=1}^n \alpha_j - \sum_{j=n+1}^p \alpha_j + \sum_{j=1}^m \beta_j - \sum_{j=m+1}^q \beta_j > 0, \quad (24)$$

$$|\arg(a)| < \frac{1}{2} A\pi, \quad (25)$$

and

$$-\min_{1 \leq j \leq m} \left[ \Re \left( \frac{b_j}{\beta_j} \right) \right] < \Re(z) < \min_{1 \leq j \leq n} \left[ \Re \left( \frac{1 - a_j}{\alpha_j} \right) \right]. \quad (26)$$

Another useful identity is [40]

$$\mathcal{M} \{ x^\nu \phi(ax^p) \} (z) = \frac{1}{p} a^{-\frac{z+\nu}{p}} \mathcal{M} \{ \phi(x) \} \left( \frac{z+\nu}{p} \right), \quad (27)$$

for  $p > 0$  and  $a > 0$ . We now invert Eq. (18) by using Eqs. (22) and (27) along with Eq. (21) to first derive the Mellin transform of  $h_k(x, t)$

$$\begin{aligned} \mathcal{M} \{ h_k(x, t) \} (z) &= \frac{1}{2} \frac{1}{\sqrt{4\pi At^\alpha}} \left( \frac{1}{At^\alpha} \right)^k \left( \frac{1}{\sqrt{4At^\alpha}} \right)^{-z} \\ &\quad \times \frac{\Gamma\left(\frac{z}{2}\right) \Gamma\left(\frac{1}{2} + k - \frac{z}{2}\right) \Gamma\left(\frac{1}{2} + \frac{z}{2}\right)}{\Gamma\left(1 - \frac{\alpha}{2} + (\beta - \alpha)k + \frac{\alpha z}{2}\right) \Gamma\left(\frac{1}{2} - \frac{z}{2}\right)}. \end{aligned} \quad (28)$$

Comparing with Eq. (22), we find on inverting the Mellin transform and noting  $x = |x|$  we find

$$h_k(x, t) = \frac{1}{2} \frac{(At^\alpha)^{-k}}{\sqrt{4\pi At^\alpha}} H_{2,3}^{2,1} \left[ \frac{|x|}{\sqrt{4At^\alpha}} \left| \begin{matrix} \left(\frac{1}{2} - k, \frac{1}{2}\right) \left(1 - \frac{\alpha}{2} + (\beta - \alpha)k, \frac{\alpha}{2}\right) \\ \left(0, \frac{1}{2}\right) \left(\frac{1}{2}, \frac{1}{2}\right) \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix} \right. \right]. \quad (29)$$

Using the identity

$$H_{p,q}^{m,n} \left[ x \left| \begin{array}{c} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{array} \right. \right] = c H_{p,q}^{m,n} \left[ x^c \left| \begin{array}{c} (a_p, c\alpha_p) \\ (b_q, c\beta_q) \end{array} \right. \right], \quad (30)$$

with  $c = \frac{1}{2}$  we arrive at the expression for  $h_k(x, t)$

$$h_k(x, t) = \frac{(At^\alpha)^{-k}}{\sqrt{4\pi At^\alpha}} H_{2,3}^{2,1} \left[ \frac{x^2}{4At^\alpha} \left| \begin{array}{c} \left(\frac{1}{2} - k, 1\right) \left(1 - \frac{\alpha}{2} + (\beta - \alpha)k, \alpha\right) \\ (0, 1) \quad \left(\frac{1}{2}, 1\right) \quad \left(\frac{1}{2}, 1\right) \end{array} \right. \right]. \quad (31)$$

The Green's solution of Eq. (5) for an infinite domain is, by Eqs (16) and (31),

$$G(x, t) = \frac{1}{\sqrt{4\pi At^\alpha}} \sum_{k=0}^{\infty} \left(-\frac{B}{A}\right)^k \frac{t^{(\beta-\alpha)k}}{k!} H_{2,3}^{2,1} \left[ \frac{x^2}{4At^\alpha} \left| \begin{array}{c} \left(\frac{1}{2} - k, 1\right) \left(1 - \frac{\alpha}{2} + (\beta - \alpha)k, \alpha\right) \\ (0, 1) \quad \left(\frac{1}{2}, 1\right) \quad \left(\frac{1}{2}, 1\right) \end{array} \right. \right]. \quad (32)$$

If  $\alpha = \beta$  this solution reduces to the solution of the fractional diffusion equation given in [14] which is

$$G(x, t) = \frac{1}{\sqrt{4\pi Dt^\alpha}} H_{1,2}^{2,0} \left[ \frac{x^2}{4Dt^\alpha} \left| \begin{array}{c} \left(1 - \frac{\alpha}{2}, \alpha\right) \\ (0, 1) \quad \left(\frac{1}{2}, 1\right) \end{array} \right. \right] \quad (33)$$

where  $D = A + B$ .

### 3 Results

In Fig. 1 we give an example plot of the solution of the modified fractional diffusion equation in the case  $\alpha = 0.5$  and  $\beta = 0.75$ . For comparison in Figs. 2 and 3 we give the solution of the traditional fractional cable equation in the cases  $\alpha = \beta = 0.5$  and  $\alpha = \beta = 0.75$ , respectively. In each case  $A$  and  $B$  were set to 1.

A comparison of the peak height at  $t = 0.1$  shows the solution of the modified equation ( $\alpha = 0.5$  and  $\beta = 0.75$ ) decays initially faster than the solution of the traditional fractional equation with the larger fractional exponent ( $\alpha = \beta = 0.75$ ) but slower than the traditional solution with the smaller exponent ( $\alpha = \beta = 0.5$ ). This order is reversed for longer times. This demonstrates

the crossover between more and less anomalous behaviour. This is also seen in the expression for the mean squared displacement

$$\langle r^2(t) \rangle = \frac{2A}{\Gamma(1+\alpha)} t^\alpha + \frac{2B}{\Gamma(1+\beta)} t^\beta, \quad (34)$$

which for short times is dominated by the smaller exponent  $\alpha$  (0.5 in Fig. 1) and later by the larger exponent  $\beta$  (0.75 in Fig. 1). Such crossover behaviour has been observed in the volatility (return variance) of share prices [35–37].

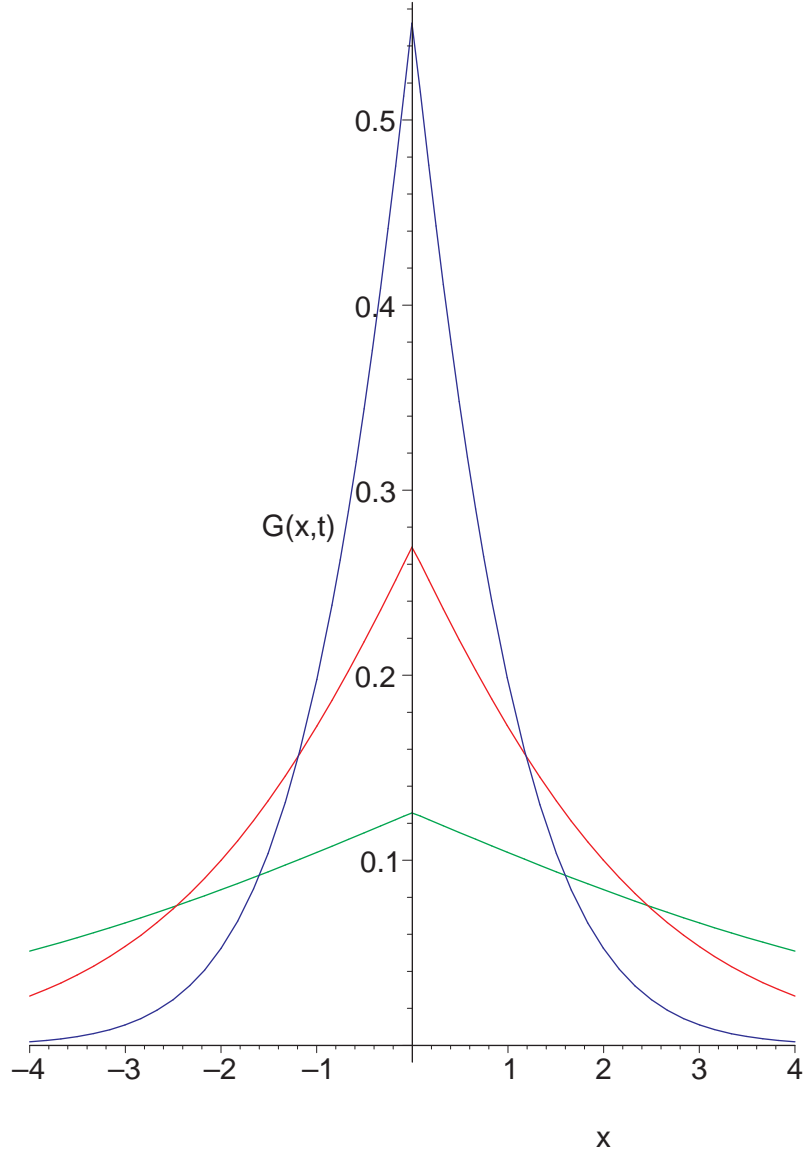


Fig. 1. The solution of the modified fractional diffusion equation in the case  $\alpha = 0.5$  and  $\beta = 0.75$  at the dimensionless times  $t = 0.1, 1, \text{ and } 10$ . The peak height decreases with increasing time.

In summary, in this letter we have found the solution of Sokolov and Klafter's modified fractional diffusion equation [1]. In contrast to the solution of the



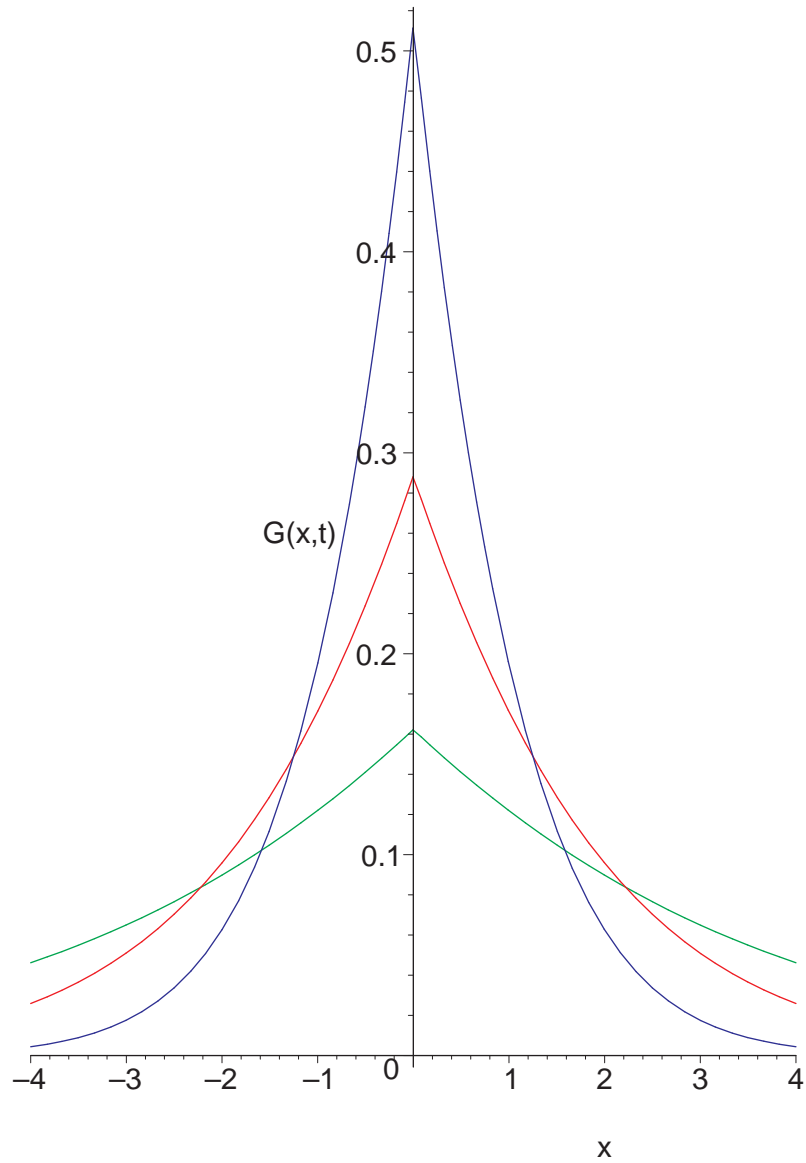


Fig. 2. The solution of the modified fractional diffusion equation in the case  $\alpha = \beta = 0.5$  at the dimensionless times  $t = 0.1, 1$ , and  $10$ . The peak height decreases with increasing time.

traditional fractional diffusion equation, the solution of the modified equation requires a summation of Fox functions instead of a single function. From the representative results we see that there is a crossover between more and less anomalous behaviour.

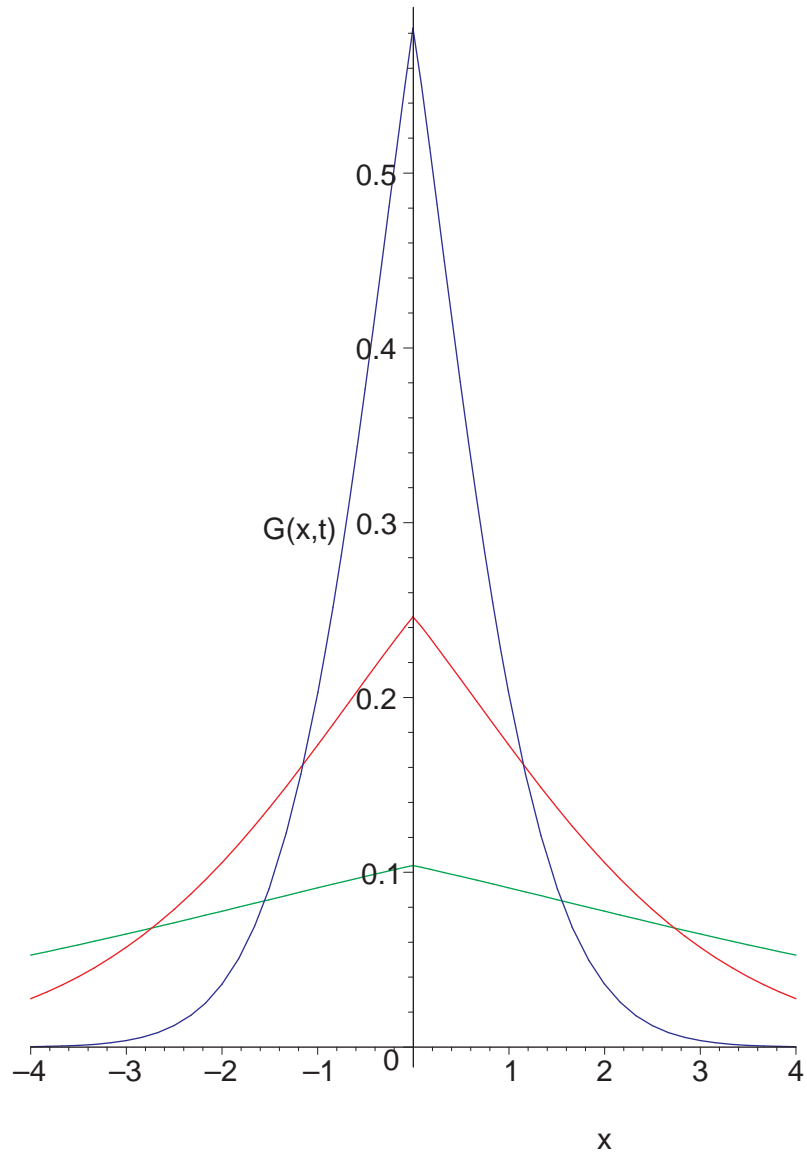


Fig. 3. The solution of the modified fractional diffusion equation in the case  $\alpha = \beta = 0.75$  at the dimensionless times  $t = 0.1, 1$ , and  $10$ . The peak height decreases with increasing time.

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