A Sampling Theorem for the Fractional Fourier Transform Without Band-limiting Constraints

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Abstract

The fractional Fourier transform (FRFT), a generalization of the Fourier transform, has proven to be a powerful tool in optics and signal processing. Most existing sampling theories of the FRFT consider the class of band-limited signals. However, in the real world, many analog signals encountered in practical engineering applications are non-bandlimited. The purpose of this paper is to propose a sampling theorem for the FRFT, which can provide a suitable and realistic model of sampling and reconstruction for real applications. First, we construct a class of function spaces and derive basic properties of their basis functions. Then, we establish a sampling theorem without band-limiting constraints for the FRFT in the function spaces. The truncation error of sampling is also analyzed. The validity of the theoretical derivations is demonstrated via simulations.

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1. Introduction

The fractional Fourier transform (FRFT), which generalizes the Fourier transform (FT), has received much attention in recent years due to its numerous applications [1–5], including in the areas of optics, signal and image processing, communications, etc. The FRFT of a continuous signal or function $f(t)$ is defined as [2]

$$F_\alpha(u) = \mathcal{F}^\alpha \{f(t)\}(u) = \int_{\mathbb{R}} f(t)\,\mathcal{K}_\alpha(u,t)\,dt \tag{1}$$

where $\mathcal{F}^\alpha$ denotes the FRFT operator, and kernel $\mathcal{K}_\alpha(u,t)$ is given by

$$\mathcal{K}_\alpha(u,t) = \begin{cases} A_\alpha e^{\left(j\frac{u^2+t^2}{2}\cot\alpha - jut\csc\alpha\right)}, & \alpha \neq k\pi \\ \delta(t-u), & \alpha = 2k\pi \\ \delta(t+u), & \alpha = (2k-1)\pi \end{cases} \tag{2}$$

where $A_\alpha = \sqrt{\frac{1-j\cot\alpha}{2\pi}}$, $k \in \mathbb{Z}$, $\cot\alpha = \cos\alpha/\sin\alpha$, and $\csc\alpha = 1/\sin\alpha$. For $\alpha \in [-\pi,\pi]$, the square root factor A_α can be rewritten without ambiguity as [1]

$$A_\alpha = \frac{e^{-j\left[\frac{\alpha}{2} - \frac{\pi}{4}\operatorname{sgn}(\alpha)\right]}}{\sqrt{2\pi|\sin\alpha|}} \tag{3}$$

where $\mathrm{sgn}(\cdot)$ denotes the sign function. When α is outside the interval $[-\pi, \pi]$, we simply need to replace α by its modulo 2π equivalent lying in this interval and use this value in (3). The u axis is regarded as the fractional Fourier domain. The inverse FRFT with respect to angle α is the FRFT with angle $-\alpha$, i.e., $f(t) = \mathcal{F}^{-\alpha}\left\{F_\alpha(u)\right\}(t) = \int_{\mathbb{R}} F_\alpha(u)\,\mathcal{K}_\alpha^*(u,t)\,du$, where $*$ in the superscript denotes the complex conjugate. In general, we only consider the case of $0 < \alpha < \pi$, since the definition can easily be extended outside the interval $[0, \pi]$ by noting that $\mathcal{F}^{2\pi n}$ is the identity operator for any integer n and that the FRFT operator is additive in angle, i.e., $\mathcal{F}^{\alpha_1+\alpha_2} = \mathcal{F}^{\alpha_1}\mathcal{F}^{\alpha_2}$. Whenever $\alpha = \pi/2$, (1) reduces to the FT given by

$$F(\omega) = \mathfrak{F}\{f(t)\}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-j\omega t} dt \tag{4}$$

with $f(t) \in L^1(\mathbb{R}) \bigcap L^2(\mathbb{R})$, where $\mathfrak{F}$ indicates the FT operator. Conversely, the inverse FT is written as $f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} F(\omega) e^{j\omega t} d\omega$. It follows that the FRFT exists, for α not multiple of π, whenever the FT of $f(t)e^{(j/2)t2\cot\alpha}$ exists. Since the complex exponent in (2) has a constant magnitude, the FRFT can also be defined in most domains in which the FT can be defined.

In digital signal and image processing, digital communications, etc., a continuous signal is usually represented by its discrete samples. Then, a fundamental problem of FRFT theory is how to represent a continuous signal in terms of a discrete sequence. For a fractional band-limited signal $f(t)$, Xia [6] found a Shannon-type sampling theorem for the FRFT. The sampling process of this theorem can be viewed as an approximation procedure in the space of fractional band-limited functions

$$\mathcal{B}_\alpha = \left\{ \sum_{n\in\mathbb{Z}} f[n]\,\mathrm{sinc}(t-n) e^{-j\frac{t^2-n^2}{2}\cot\alpha} \,\middle|\, f[n] \in \ell^2(\mathbb{Z}) \right\} \tag{5}$$

where $\mathrm{sinc}(\cdot) \triangleq \frac{\sin\pi(\cdot)}{\pi(\cdot)}$. Xia's sampling theorem provides an exact representation by the signal's uniform samples $\{f[n]\}_{n\in\mathbb{Z}}$ and has been currently generalized to many other forms. Zayed and García derived a new sampling expansion using the Hilbert transform in [7]. In [8], Stern extended Xia's result to the generalized form of the FRFT, which is called the linear canonical transform [1]. Tao *et al.* discussed sampling and sampling rate conversion of band-limited signals in the fractional Fourier domain in [9]. Bhandari and Marziliano [10] proposed a uniform sampling and reconstruction algorithm for sparse signals in the fractional Fourier domain. Furthermore, authors in [11, 12] studied multi-channel sampling for the FRFT. However, these extensions and modifications of Xia's sampling theorem [6] were derived from the band-limited signal viewpoint. In the real world, many analog signals encountered in practical engineering applications are non-bandlimited. Recently, Liu *et al.* [13] introduced new sampling formulae of the generalized FRFT for non-bandlimited signals by constructing a class of function spaces $B^{h,m}_{M,\Omega_M}(m=1,2,3)$. Unfortunately, as the authors of [13] pointed out, there are no normative rules at present for determining the parameters M, h, Ω_M in practical implementations.

The purpose of this paper is to propose a sampling theorem associated with the FRFT, which can provide a suitable and realistic model of sampling and reconstruction for real applications. First, we introduce a class of function spaces with a single generator and derive basic properties of their basis functions. Then, we derive a sampling theorem for the FRFT in the function spaces. Moreover, the truncation error of sampling and some potential applications of the derived results are presented. The validity of the theoretical derivations is demonstrated via simulations.

The outline of this paper is organized as follows. In Section 2, notations and some facts for the FRFT are first introduced, and then the concept of fractional convolution is given. In Section 3, a sampling theorem for the FRFT without band-limiting constraints is established, and the truncation error of sampling and some potential applications are also discussed. Finally, concluding remarks are given in Section 4.

Nomenclature

FT Fourier transform

DTFT discrete-time Fourier transform

FRFT fractional Fourier transform

DTFRFT discrete-time fractional Fourier transform

$\mathfrak{F}$ FT operator

$\mathcal{F}^{\alpha}$ FRFT operator

$\widetilde{\mathcal{F}}^{\alpha}$ DTFRFT operator

$F(u\csc\alpha)$ FT (with its argument scaled by $\csc\alpha$) of $f(t)$

$\widetilde{F}(u\csc\alpha)$ DTFT (with its argument scaled by $\csc\alpha$) of $f[n]$

$F_{\alpha}(u)$ FRFT of $f(t)$

$\widetilde{F}_{\alpha}(u)$ DTFRFT of $f[n]$

Θ_{α} continuous fractional convolution operator

$\overset{s}{\Theta}_{\alpha}$ semi-discrete fractional convolution operator

2. Preliminaries

2.1. Notation

Throughout this paper, we consider real-valued signals. Continuous signals are denoted with parentheses, e.g., $f(t)$, $t \in \mathbb{R}$, and discrete signals with brackets, e.g., $c[n]$, $n \in \mathbb{Z}$. We denote the inner L^2-product between $f(t)$ and $g(t)$ by

$$\langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(t) g^*(t) dt, \tag{6}$$

and the ℓ^2-inner product between two sequences $c[n]$ and $d[n]$ by

$$\langle c, d \rangle_{\ell^2} = \sum_{n \in \mathbb{Z}} c[n] d^*[n]. \tag{7}$$

Correspondingly, we denote the L^2-norm by $\|f\|^2_{L^2} = \langle f, f \rangle_{L^2}$, and the ℓ^2-norm by $\|c\|^2_{\ell^2} = \langle c, c \rangle_{\ell^2}$.

Let $\mathcal{H}$ be a Hilbert space and $\{\varphi_n(t)\}_{n \in \mathbb{Z}}$ be a complete set of functions in $\mathcal{H}$. The set is a Riesz basis for $\mathcal{H}$ if and only if there exist constants $0 < A \leq B < +\infty$ such that [14]

$$A \, \|c[n]\|^2_{\ell^2} \leq \left\| \sum_{n \in \mathbb{Z}} c[n] \varphi_n(t) \right\|^2_{L^2} \leq B \, \|c[n]\|^2_{\ell^2}, \; \forall \; c[n] \in \ell^2(\mathbb{Z}) \tag{8}$$

with equality if and only if the basis is orthonormal, i.e., when $A = B = 1$.

For a measurable function $f(t)$ on $\mathbb{R}$, let $\|f(t)\|_\infty = \operatorname{ess\,sup} |f(t)|$ and $\|f(t)\|_0 = \operatorname{ess\,inf} |f(t)|$ be the essential supremum and infimum of $|f(t)|$, respectively.

2.2. Some facts for the FRFT

The Parseval formula of the FRFT can be expressed as [1]

$$\|f(t)\|_{L^2}^2 = \|F_\alpha(u)\|_{L^2}^2 . \tag{9}$$

The relationship between the FRFT and FT is given by [15]

$$\mathcal{F}^\alpha\{f(t)\}(u) = \sqrt{2\pi}A_\alpha e^{j\frac{u^2}{2}\cot\alpha}\mathfrak{F}\left\{f(t)e^{j\frac{t^2}{2}\cot\alpha}\right\}(u\csc\alpha) . \tag{10}$$

The discrete-time FRFT (DTFRFT) of $f[n] \in \ell^2(\mathbb{Z})$ is defined as [16]

$$\widetilde{F}_\alpha(u) = \widetilde{\mathcal{F}}^\alpha\{f[n]\}(u) = \sum_{n\in\mathbb{Z}} f[n]\mathcal{K}_\alpha(u,n) \tag{11}$$

where $\mathcal{K}_\alpha(\cdot,\cdot)$ is defined in (2). When $\alpha = \pi/2$, (11) reduces to the discrete time Fourier transform (DTFT) [1]. Correspondingly, the inverse DTFRFT is given by

$$f[n] = \int_I \widetilde{F}_\alpha(u)\mathcal{K}_\alpha^*(u,n)du,\ I \triangleq [0, 2\pi\sin\alpha]. \tag{12}$$

The Parseval formula of the DTFRFT can be expressed as

$$\|f[n]\|_{\ell^2}^2 = \int_I \left|\widetilde{F}_\alpha(u)\right|^2 du \tag{13}$$

which implies that for any $f[n] \in \ell^2(\mathbb{Z})$, $\widetilde{F}_\alpha(u) \in L^2(I)$. Moreover, the

DTFRFT has the following chirp-periodicity [16]:

$$\widetilde{F}_\alpha(u+2k\pi\sin\alpha)e^{-j\frac{(u+2k\pi\sin\alpha)^2}{2}\cot\alpha}=\widetilde{F}_\alpha(u)e^{-j\frac{u^2}{2}\cot\alpha},\ k\in\mathbb{Z}. \tag{14}$$

Due to (11) and (2), we have

$$\widetilde{F}_\alpha(u)e^{-j\frac{u^2}{2}\cot\alpha}=\sum_{n\in\mathbb{Z}}c[n]e^{j\frac{n^2}{2}\cot\alpha-jun\csc\alpha} \tag{15}$$

where $c[n]=A_\alpha f[n]$. Meanwhile, inserting (2) into (12) gives the rise to

$$c[n]=\frac{1}{2\pi\sin\alpha}\int_I\left(\widetilde{F}_\alpha(u)e^{-j\frac{u^2}{2}\cot\alpha}\right)e^{-j\frac{n^2}{2}\cot\alpha+jun\csc\alpha}du. \tag{16}$$

Note that if $\widetilde{F}_\alpha(u)$ is in $L^2(I)$, its product by chirp $e^{-(j/2)u^2\cot\alpha}$ is also in $L^2(I)$. It follows from (15) and (16) that functions belonging to $L^2(I)$ in the fractional Fourier domain can be expanded into a series defined in (15).

2.3. Fractional convolution

It was shown in [5] that there are several definitions of fractional convolution in the literature. The relationships among them were investigated in [5] in detail. We use the one introduced in [12], which has a simple structure.

For two continuous functions $f(t)$ and $\phi(t)$, continuous fractional convolution is denoted by $f(t)\Theta_\alpha\phi(t)$ using the Θ_α symbol [12], i.e.,

$$\begin{aligned} f(t)\Theta_\alpha\phi(t) &= e^{-j\frac{t^2}{2}\cot\alpha}\cdot\left[\left(f(t)e^{j\frac{t^2}{2}\cot\alpha}\right)*\phi(t)\right] \\ &= \int_{\mathbb{R}} f(\tau)\phi(t-\tau)e^{-j\frac{t^2-\tau^2}{2}\cot\alpha}d\tau \end{aligned} \tag{17}$$

where $*$ denotes the ordinary convolution operator, and the continuous fractional convolution theorem is expressed as [12]

$$f(t)\Theta_\alpha\phi(t) \overset{\mathcal{F}^\alpha}{\longleftrightarrow} \sqrt{2\pi}F_\alpha(u)\Phi\left(u\csc\alpha\right). \tag{18}$$

Since $e^{j\frac{t^2}{2}\cot\alpha}$ has a constant magnitude, the continuous fractional convolution can also be defined in most domains, in which the ordinary convolution can be defined. For instance, using Young's inequality [17], for any $f(t) \in L^1(\mathbb{R})$ and $\phi(t) \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, we have $(f(t)\Theta_\alpha\phi(t)) \in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$.

Let $\phi(t)$ be a continuous function in $L^2(\mathbb{R})$. We assume that the sampling period is equal to one, and we define the semi-discrete fractional convolution operator $\overset{\mathrm{s}}{\Theta}_\alpha$ as a linear map from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{R})$ such that

$$\overset{\mathrm{s}}{\Theta}_\alpha : f[n] \mapsto f[n]\overset{\mathrm{s}}{\Theta}_\alpha\phi(t) = e^{-j\frac{t^2}{2}\cot\alpha}\cdot\left[\left(f[n]e^{j\frac{n^2}{2}\cot\alpha}\right)\overset{\mathrm{s}}{*}\phi(t)\right] \tag{19}$$

where $\overset{\mathrm{s}}{*}$ denotes the semi-discrete ordinary convolution operator. The semi-discrete fractional convolution can be rewritten as

$$f[n]\overset{\mathrm{s}}{\Theta}_{\alpha}\phi(t)=\sum_{n\in\mathbb{Z}}f[n]\phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{20}$$

which satisfies

$$f[n]\overset{\mathrm{s}}{\Theta}_{\alpha}\phi(t)\overset{\mathcal{F}^{\alpha}}{\longleftrightarrow}\sqrt{2\pi}\widetilde{F}_{\alpha}(u)\Phi\left(u\csc\alpha\right). \tag{21}$$

3. Sampling and reconstruction of signals in the fractional Fourier domain without band-limiting constrains

3.1. A sampling theorem for the FRFT without band-limiting constraints

Most known sampling theories of the FRFT consider the class of band-limited functions, which can be expanded in terms of translation and chirp-modulation of the sinc function [6–9, 11, 12, 16], see (5). To derive a sampling theorem for the FRFT without band-limiting constraints, we start with an appropriate function $\phi(t)\in L^2(\mathbb{R})$, and define the function space as

$$\mathcal{V}_{\alpha}(\phi)=\left\{\sum_{n\in\mathbb{Z}}c[n]\phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}\,\middle|\,c[n]\in\ell^2(\mathbb{Z})\right\} \tag{22}$$

where $\phi(t)\in L^2(\mathbb{R})$ is called the generator of $\mathcal{V}_{\alpha}(\phi)$. Intrinsically, the present formulation has the same conceptual simplicity as the band-limited model ($\phi=\mathrm{sinc}$) defined in (5). For simplicity, we let

$$\phi_{n,\alpha}(t)\triangleq\phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}. \tag{23}$$

The definition of the space $\mathcal{V}_\alpha(\phi)$ in (22) makes sense if there exists a positive number $0 < B < +\infty$ such that

$$\left\| \sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t) \right\|_{L^2}^2 \leq B \left\| c[n] \right\|_{\ell^2}^2 \tag{24}$$

for all $c[n] \in \ell^2(\mathbb{Z})$. If, in addition, we have the lower bound

$$A \left\| c[n] \right\|_{\ell^2}^2 \leq \left\| \sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t) \right\|_{L^2}^2 \tag{25}$$

where $A > 0$, then it follows from [14] that $\mathcal{V}_\alpha(\phi)$ is a closed subspace of $L^2(\mathbb{R})$, and the set of functions $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{V}_\alpha(\phi) \subset L^2(\mathbb{R})$. Note that if the sequence $\sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t)$ of elements in $\mathcal{V}_\alpha(\phi)$ is convergent in $L^2(\mathbb{R})$, then the inequality in (25) implies that $\{c[n]\}_{n\in\mathbb{Z}}$ is a Cauchy sequence in $\ell^2(\mathbb{Z})$. Thus, it converges to an element $c'[n] \in \ell^2(\mathbb{Z})$. The inequality in (24) then implies that $\sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t)$ converges to $\sum_{n\in\mathbb{Z}} c'[n]\phi_{n,\alpha}(t) \in \mathcal{V}_\alpha(\phi)$ as n tends to infinity. Finally, if $\sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t) = 0$, then (25) implies that $c[n] = 0$ for all $n \in \mathbb{Z}$. Hence, $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a basis of $\mathcal{V}_\alpha(\phi)$. Our goal is then to find conditions on $\phi(t)$ for $\mathcal{V}_\alpha(\phi)$ to be a well-defined subspace of $L^2(\mathbb{R})$, and for $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ to be its Riesz basis. Towards this end, when taking the FRFT of an element $\sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t) \in \mathcal{V}_\alpha(\phi)$, and using (9), (21), and (14), we arrive at

$$
\begin{aligned}
&\left\|\sum_{n\in\mathbb{Z}} c[n]\phi_{n,\alpha}(t)\right\|_{L^2}^2 \\
&\quad = \int_{\mathbb{R}} \left|\sqrt{2\pi}\widetilde{C}_\alpha(u)\Phi\left(u\csc\alpha\right)\right|^2 du \\
&\quad = 2\pi\sum_{k\in\mathbb{Z}}\int_{I}\left|\widetilde{C}_\alpha(u+2k\pi\sin\alpha)\right|^2\left|\Phi\left(u\csc\alpha+2k\pi\right)\right|^2 du \\
&\quad = 2\pi\int_{I}\left|\widetilde{C}_\alpha(u)\right|^2 G_{\phi,\alpha}^2(u)du
\end{aligned}
\tag{26}
$$

where $G_{\phi,\alpha}(u)$ is defined as

$$
G_{\phi,\alpha}(u) \triangleq \left(\sum_{k\in\mathbb{Z}}\left|\Phi\left(u\csc\alpha+2k\pi\right)\right|^2\right)^{\frac{1}{2}}. \tag{27}
$$

Clearly, if $A \leq 2\pi G_{\phi,\alpha}^2(u) \leq B$, then it follows from (26) and (13) that $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ suffices (8), and (24) and (25) can be established. Overall, we have proved the following theorem.

Theorem 1. *Let $\phi(t)$ be a continuous function in $L^2(\mathbb{R})$. The space $\mathcal{V}_\alpha(\phi)$ is a well-defined, closed subspace of $L^2(\mathbb{R})$ with Riesz basis $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ if and only if there exist two positive constants $0 < A \leq B < +\infty$ such that*

$$
A \leq 2\pi G_{\phi,\alpha}^2(u) \leq B,\ \ a.e.\ u\in\mathbb{R} \tag{28}
$$

Based on the above facts, we have the following sampling theorem for the

FRFT without band-limiting constraints.

Theorem 2. *Let $\phi(t)$ be a continuous function in $L^2(\mathbb{R})$. Suppose that $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis for the subspace $\mathcal{V}_\alpha(\phi)$ of $L^2(\mathbb{R})$ such that the sampling sequence at integers $\{\phi[n]\}_{n\in\mathbb{Z}}$ belongs to $\ell^2(\mathbb{Z})$. Then, there exists a function $s(t) \in L^2(\mathbb{R})$ with $s(t)e^{-j\frac{t^2}{2}\cot\alpha} \in \mathcal{V}_\alpha(\phi)$ such that*

$$f(t) = \sum_{n\in\mathbb{Z}} f[n]s(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{29}$$

holds for all $f(t) \in \mathcal{V}_\alpha(\phi)$ in $L^2(\mathbb{R})$ sense, if and only if

$$\frac{1}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)} \in L^2(I) \tag{30}$$

holds. In this case, $S(u\csc\alpha) = \frac{\Phi(u\csc\alpha)}{\sqrt{2\pi}\widetilde{\Phi}(u\csc\alpha)}$ holds for a.e. $u \in \mathbb{R}$.

Proof. *Sufficiency:* First assume that (30) holds. Thus, $\widetilde{\Phi}\left(u\csc\alpha\right) \neq 0$ holds for a.e. $u \in \mathbb{R}$. By (15), there exists a sequence $\{c[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$ such that

$$\frac{1}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)} = \sum_{n\in\mathbb{Z}} c[n]e^{j\frac{n^2}{2}\cot\alpha - jun\csc\alpha} \tag{31}$$

holds in the $L^2(I)$ sense. Next, since $\widetilde{\Phi}\left(u\csc\alpha\right)$ is $2\pi\sin\alpha$ periodic, we derive

$$\begin{aligned}\int_{\mathbb{R}}\left|\frac{\Phi\left(u\csc\alpha\right)}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)}\right|^2 du &= \sum_{k\in\mathbb{Z}}\int_{I}\left|\frac{\Phi\left(u\csc\alpha+2k\pi\right)}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)}\right|^2 du\\ &= \int_{I}\frac{G^2_{\phi,\alpha}(u)}{\left|\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)\right|^2}du\end{aligned}\tag{32}$$

from which along with (28), it follows that

$$\int_{\mathbb{R}}\left|\frac{\Phi\left(u\csc\alpha\right)}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)}\right|^2 du \leq \|G_{\phi,\alpha}(u)\|^2_\infty\int_{I}\frac{1}{\left|\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)\right|^2}du \tag{33}$$

which implies that $\frac{\Phi(u\csc\alpha)}{\sqrt{2\pi}\widetilde{\Phi}(u\csc\alpha)}\in L^2(\mathbb{R})$. Thus, we can derive

$$S\left(u\csc\alpha\right)=\mathfrak{F}\left\{s(t)\right\}\left(u\csc\alpha\right)\triangleq\frac{\Phi\left(u\csc\alpha\right)}{\sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)}. \tag{34}$$

Then, substituting (31) into (34) gives the rise to

$$S\left(u\csc\alpha\right)=\Phi\left(u\csc\alpha\right)\sum_{n\in\mathbb{Z}}c[n]e^{j\frac{n^2}{2}\cot\alpha-jun\csc\alpha}. \tag{35}$$

Using (10) and (34) yields

$$\begin{aligned}\mathcal{F}^\alpha\left\{s(t)e^{-j\frac{t^2}{2}\cot\alpha}\right\}(u) &= \sqrt{2\pi}A_\alpha e^{j\frac{u^2}{2}\cot\alpha}\mathfrak{F}\left\{s(t)\right\}\left(u\csc\alpha\right)\\ &= \sqrt{2\pi}\Phi\left(u\csc\alpha\right)\sum_{n\in\mathbb{Z}}c[n]\mathcal{K}_\alpha(u,n)\\ &= \sqrt{2\pi}\widetilde{C}_\alpha(u)\Phi\left(u\csc\alpha\right).\end{aligned}\tag{36}$$

Now combining (36), (21), and (20), we derive

$$s(t)e^{-j\frac{t^2}{2}\cot\alpha} = \sum_{n\in\mathbb{Z}} c[n]\phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{37}$$

which implies that $\left(s(t)e^{-j\frac{t^2}{2}\cot\alpha}\right) \in \mathcal{V}_\alpha(\phi)$ since $\left\{\phi(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}\right\}_{n\in\mathbb{Z}}$ forms a Riesz basis for $\mathcal{V}_\alpha(\phi)$. Moreover, by assumption, for any continuous function $f(t) \in \mathcal{V}_\alpha(\phi)$, it follows that

$$f(t) = \sum_{m\in\mathbb{Z}} p[m]\phi(t-m)e^{-j\frac{t^2-m^2}{2}\cot\alpha},\ t \in \mathbb{R} \tag{38}$$

where $p[m] \in \ell^2(\mathbb{Z})$. Then, combining (38), (20), and (21) shows

$$F_\alpha(u) = \sqrt{2\pi}\widetilde{P}_\alpha(u)\Phi\left(u\csc\alpha\right) \tag{39}$$

from which along with (34), it follows that

$$F_\alpha(u) = 2\pi\widetilde{P}_\alpha(u)\widetilde{\Phi}\left(u\csc\alpha\right)S\left(u\csc\alpha\right). \tag{40}$$

Next, by (38), we let

$$f[n] = \sum_{m\in\mathbb{Z}} p[m]\phi[n-m]e^{-j\frac{n^2-m^2}{2}\cot\alpha},\ n \in \mathbb{Z}. \tag{41}$$

Then, $\{f[n]\}_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ is well defined since $\{p[n]\}_{n\in\mathbb{Z}}$ and $\{\phi[n]\}_{n\in\mathbb{Z}}$ are both $\ell^2(\mathbb{Z})$ sequences. In fact,

$$f[n] \to 0 \text{ as } |n| \to \infty. \tag{42}$$

The proof of (42) is as follows. Since $\widetilde{P}_\alpha(u)$ and $\widetilde{\Phi}\,(u\csc\alpha)$ are both in $L^2(I)$, clearly $\widetilde{P}_\alpha(u)\widetilde{\Phi}\,(u\csc\alpha)\,\frac{2\pi e^{-j\frac{u^2}{2}\cot\alpha}}{\sqrt{1-j\cot\alpha}}$ belongs to $L^1(I)$. Then, the Fourier coefficients of this $L^1(I)$ function are derived as

$$\begin{aligned}
&\frac{1}{2\pi\sin\alpha}\int\limits_I \left(\widetilde{P}_\alpha(u)\widetilde{\Phi}\,(u\csc\alpha)\,\frac{2\pi e^{-j\frac{u^2}{2}\cot\alpha}}{\sqrt{1-j\cot\alpha}}\right) e^{jnu\csc\alpha}du \\
&= \frac{\csc\alpha}{\sqrt{1-j\cot\alpha}}\int\limits_I \sum_{m\in\mathbb{Z}} p[m]\mathcal{K}_\alpha(u,m)\widetilde{\Phi}(u\csc\alpha)e^{-j\frac{u^2}{2}\cot\alpha}e^{jun\csc\alpha}du \\
&= \frac{1}{\sqrt{2\pi}}\int\limits_I \left(\sum_{m\in\mathbb{Z}} p[m]e^{j\frac{m^2}{2}\cot\alpha}\right)\widetilde{\Phi}(u\csc\alpha)e^{-j(n-m)u\csc\alpha}d(u\csc\alpha) \\
&= \sum_{m\in\mathbb{Z}} p[m]e^{j\frac{m^2}{2}\cot\alpha}\left(\frac{1}{\sqrt{2\pi}}\int\limits_I \widetilde{\Phi}(u\csc\alpha)e^{-j(n-m)u\csc\alpha}d(u\csc\alpha)\right) \\
&= \sum_{m\in\mathbb{Z}} p[m]\phi[n-m]e^{j\frac{m^2}{2}\cot\alpha} \\
&= f[n]e^{j\frac{n^2}{2}\cot\alpha}.
\end{aligned} \tag{43}$$

Due to the Riemann–Lebesgue Lemma [18], $f[n]e^{j\frac{n^2}{2}\cot\alpha}$ as the Fourier coefficients of the $L^1(I)$ function tends to 0 as n tends to infinity. This implies that (42) holds. Next, taking the inverse FRFT of both sides of (40) yields

$$\begin{aligned}
f(t) &= \int_{\mathbb{R}} 2\pi \widetilde{P}_\alpha(u) \widetilde{\Phi}\left(u \csc\alpha\right) S\left(u \csc\alpha\right) \mathcal{K}_{-\alpha}(u,t) du \\
&= \int_{\mathbb{R}} 2\pi \sum_{m\in\mathbb{Z}} p[m] \mathcal{K}_\alpha(u,m) \frac{1}{\sqrt{2\pi}} \sum_{n'\in\mathbb{Z}} \phi[n'] e^{-jn'u\csc\alpha} \\
&\quad \times S\left(u \csc\alpha\right) \mathcal{K}_{-\alpha}(u,t) du \\
&= \sum_{n'\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} p[m]\phi[n'] e^{-j\frac{t^2-m^2}{2}\cot\alpha} \\
&\quad \times \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} S\left(u \csc\alpha\right) e^{j(t-n'-m)u\csc\alpha} du \csc\alpha \\
&= \sum_{n'\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} p[m]\phi[n'] e^{-j\frac{t^2-m^2}{2}\cot\alpha} s(t-n'-m) \\
&= \sum_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} p[m]\phi[n-m] e^{-j\frac{t^2-m^2}{2}\cot\alpha} s(t-n)
\end{aligned} \tag{44}$$

Then, substituting (41) into (44) gives the rise to (29).

Necessity: On the contrary, suppose that there exists a function $s(t) \in L^2(\mathbb{R})$ with $s(t)e^{-j\frac{t^2}{2}\cot\alpha} \in \mathcal{V}_\alpha(\phi)$ such that (29) holds in the $L^2(\mathbb{R})$ sense. Since $\phi_{n,\alpha}(t) \in \mathcal{V}_\alpha(\phi)$ holds for any $n \in \mathbb{Z}$, it follows that $\phi_{0,\alpha}(t) \in \mathcal{V}_\alpha(\phi)$, i.e., $\phi(t)e^{-j\frac{t^2}{2}\cot\alpha} \in \mathcal{V}_\alpha(\phi)$. Then, replacing $f(t)$ with $\phi(t)e^{-j\frac{t^2}{2}\cot\alpha}$ in (29) yields

$$\phi(t)e^{-j\frac{t^2}{2}\cot\alpha} = \sum_{n\in\mathbb{Z}} \left(\phi[n]e^{-j\frac{n^2}{2}\cot\alpha}\right) s(t-n) e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{45}$$

which reduces to

$$\phi(t)e^{-j\frac{t^2}{2}\cot\alpha} = \sum_{n\in\mathbb{Z}} \phi[n]s(t-n)e^{-j\frac{t^2}{2}\cot\alpha}. \tag{46}$$

Now, taking the FRFT of both sides of (46) yields

$$\begin{aligned}\sqrt{2\pi}A_\alpha e^{j\frac{u^2}{2}\cot\alpha}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \phi(t)e^{-jtu\csc\alpha}dt& \\ = 2\pi A_\alpha e^{j\frac{u^2}{2}\cot\alpha}\frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\phi[n]e^{-jnu\csc\alpha}& \\ \times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} s(t-n)e^{-j(t-n)u\csc\alpha}dt&\end{aligned} \tag{47}$$

Then, we derive

$$\Phi\left(u\csc\alpha\right) = \sqrt{2\pi}\widetilde{\Phi}\left(u\csc\alpha\right)S\left(u\csc\alpha\right) \tag{48}$$

which implies that $\operatorname{supp}\Phi\left(u\csc\alpha\right) \subset \operatorname{supp}\widetilde{\Phi}\left(u\csc\alpha\right)$ holds for a.e. $u \in \mathbb{R}$, i.e., $\operatorname{supp}\Phi\left(u\csc\alpha + 2k\pi\right) \subset \operatorname{supp}\widetilde{\Phi}\left(u\csc\alpha\right)$ holds for all $k \in \mathbb{Z}$ and for a.e. $u \in \mathbb{R}$ due to the fact that $\widetilde{\Phi}\left(u\csc\alpha\right)$ is $2\pi\sin\alpha$ periodic. We show

$$\bigcup_{k\in\mathbb{Z}} \operatorname{supp}\Phi\left(u\csc\alpha + 2k\pi\right) = \mathbb{R} \tag{49}$$

holds except for a zero measure subset of $\mathbb{R}$. Otherwise, there is a measurable subset δ with measure $|\delta| \neq 0$ such that $\delta = \mathbb{R} - \bigcup_{k\in\mathbb{Z}} \operatorname{supp}\Phi\left(u\csc\alpha + 2k\pi\right)$. Then, $\Phi\left(u\csc\alpha + 2k\pi\right) = 0$ holds for any $u \in \delta$ and for all $k \in \mathbb{Z}$. Therefore,

$G_{\phi,\alpha}(u) = \left(\sum_{k\in\mathbb{Z}} |\Phi\,(u\csc\alpha + 2k\pi)|^2\right)^{\frac{1}{2}} = 0$ holds for any $u \in \delta$. However, by (28), $G_{\phi,\alpha}(u) \neq 0$ holds for a.e. $u \in \mathbb{R}$. It forces (49) to hold for a.e. $u \in \mathbb{R}$. Hence, $\operatorname{supp}\widetilde{\Phi}\,(u\csc\alpha) \supset \bigcup_{k\in\mathbb{Z}} \operatorname{supp}\Phi\,(u\csc\alpha + 2k\pi)$ holds for a.e. $u \in \mathbb{R}$, i.e., $\widetilde{\Phi}\,(u\csc\alpha) \neq 0$ for a.e. $u \in \mathbb{R}$. Then, (48) can be rewritten as

$$\frac{\Phi\,(u\csc\alpha)}{\sqrt{2\pi}\widetilde{\Phi}\,(u\csc\alpha)} = S\,(u\csc\alpha)\,. \tag{50}$$

Since $S\,(u\csc\alpha) \in L^2(\mathbb{R})$, we have from (50) and (28)

$$\begin{aligned}
\infty > \int_{\mathbb{R}} |S\,(u\csc\alpha)|^2\,du &= \sum_{k\in\mathbb{Z}} \int_I \left|\frac{\Phi\,(u\csc\alpha + 2k\pi)}{\sqrt{2\pi}\widetilde{\Phi}\,(u\csc\alpha)}\right|^2 du \\
&= \int_I \frac{G^2_{\phi,\alpha}(u)}{\left|\sqrt{2\pi}\widetilde{\Phi}\,(u\csc\alpha)\right|^2} du \\
&\geq \|G_{\phi,\alpha}(u)\|_0^2 \int_I \frac{1}{\left|\sqrt{2\pi}\widetilde{\Phi}\,(u\csc\alpha)\right|^2} du
\end{aligned} \tag{51}$$

which implies that (30) holds. This completes the proof of Theorem 2. ☐

Corollary 1. *If the generator $\phi(t)$ of the subspace $\mathcal{V}_\alpha(\phi)$ of $L^2(\mathbb{R})$ is chosen as the sinc function* $\operatorname{sinc}(t)$, *then Theorem 2 reduces to the FRFT sampling theorem for band-limited signals [6].*

Proof. The continuity of $\phi(t)$ implies that the samples $\{\phi[n]\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$. By applying Poisson's summation formula of the FT [1], it follows that $\widetilde{\Phi}\,(u\csc\alpha) = \sum_{k\in\mathbb{Z}} \Phi(u\csc\alpha + 2k\pi)$. Then, we derive $\frac{1}{\sqrt{2\pi}\widetilde{\Phi}(u\csc\alpha)} = 1 \in L^2(I)$. This result implies that Theorem 2 can be applied. Note that

$S(u\csc\alpha)=\frac{\Phi(u\csc\alpha)}{\sqrt{2\pi}\widetilde{\Phi}(u\csc\alpha)}=\Phi(u\csc\alpha)$. Hence, we have $s(t)=\mathrm{sinc}(t)$. Then, the FRFT sampling theorem for band-limited signals [6] is established. $\square$

Sampling in space $\mathcal{V}_\alpha(\phi)$ that is not band-limited is a suitable and realistic model for a variety of real applications, e.g., for real acquisition and reconstruction devices, for modeling signals with smoother spectrum than is the case with band-limited functions, or for numerical implementation. These requirements can often be met by choosing an appropriate generator $\phi(t)$ of $\mathcal{V}_\alpha(\phi)$. This may mean that $\phi(t)$ has a shape corresponding to a particular "impulse response" of a device, or that it is compactly supported, or that it has a spectrum $|\Phi(u\csc\alpha)|$ that decays smoothly to zero as $|u|\to\infty$.

3.2. Truncation error

When the sampling theorem is applied to recover signals, we should know how many items we need to calculate so that the recovered signal is close to the original one as excepted. Then, the truncation error defined as

$$e(t)=\sum_{|n|\geq N} f[n]s(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha} \tag{52}$$

for $f(t)\in\mathcal{V}_\alpha(\phi)$ should be estimated. However, we need a slightly stronger constraint to be imposed on the generator $\phi(t)$ of $\mathcal{V}_\alpha(\phi)$ than in Theorem 2.

Theorem 3. *Let $\phi(t)$ be a continuous function in $L^2(\mathbb{R})$. Assume that $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis for the subspace $\mathcal{V}_\alpha(\phi)$ of $L^2(\mathbb{R})$ such that $\{\phi[n]\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ and $\frac{1}{\widetilde{\Phi}(u\csc\alpha)}\in L^\infty(I)$. Then, the truncation error is* *bounded by*

$$\|e(t)\|_{L^2} \leq \left(\sum_{|n|\geq N} |f[n]|^2 \right)^{\frac{1}{2}} \left\| \frac{G_{\phi,\alpha}(u)}{\widetilde{\Phi}(u \csc \alpha)} \right\|_\infty . \tag{53}$$

Proof. Taking the FRFT of both sides of (52) gives the rise to

$$E_\alpha(u) = \mathcal{F}^\alpha \left\{ e\left(t\right) \right\}(u) = \sqrt{2\pi} \sum_{|n|\geq N} f[n]\mathcal{K}_\alpha\left(u, n\right) S(u \csc \alpha) \tag{54}$$

from which along with (9), it follows that

$$\begin{aligned} \|e(t)\|_{L^2}^2 &= \left\| \sqrt{2\pi} \sum_{|n|\geq N} f[n]\mathcal{K}_\alpha(u, n) S(u \csc \alpha) \right\|_{L^2}^2 \\ &= \int_{\mathbb{R}} \left| \sqrt{2\pi} \sum_{|n|\geq N} f[n] \sqrt{\frac{1 - j \cot \alpha}{2\pi}} e^{j\frac{n^2}{2}\cot \alpha} e^{-jnu \csc \alpha} e^{j\frac{u^2}{2}\cot \alpha} S(u \csc \alpha) \right|^2 du \\ &= \csc \alpha \int_{\mathbb{R}} \left| \sum_{|n|\geq N} f[n] e^{j\frac{n^2}{2}\cot \alpha} e^{-jnu \csc \alpha} \right|^2 \left|S(u \csc \alpha)\right|^2 du. \end{aligned} \tag{55}$$

For simplicity, we let $\tilde{f}[n] = f[n]e^{j\frac{n^2}{2}\cot \alpha}$. Since $e^{-jnu \csc \alpha}$ is $2\pi \sin \alpha$ periodic,

(55) can be rewritten as

$$
\begin{aligned}
\|e(t)\|_{L^2}^2 &= \csc\alpha \sum_{k\in\mathbb{Z}} \int_I \left| \sum_{|n|\geq N} \tilde{f}[n] e^{-jn(u+2k\pi\sin\alpha)\csc\alpha} \right|^2 |S(u\csc\alpha + 2k\pi)|^2\, du \\
&= \csc\alpha \sum_{k\in\mathbb{Z}} \int_I \left| \sum_{|n|\geq N} \tilde{f}[n] e^{-jnu\csc\alpha} \right|^2 |S(u\csc\alpha + 2k\pi)|^2\, du \\
&= \csc\alpha \int_I \left| \sum_{|n|\geq N} \tilde{f}[n] e^{-jnu\csc\alpha} \right|^2 \sum_{k\in\mathbb{Z}} |S(u\csc\alpha + 2k\pi)|^2\, du.
\end{aligned}
\tag{56}
$$

Then, applying (50), (28), and Parseval's formula of the DTFT [1] yields

$$
\begin{aligned}
\|e(t)\|_{L^2}^2 &= \csc\alpha \int_I \left| \frac{1}{\sqrt{2\pi}} \sum_{|n|\geq N} \tilde{f}[n] e^{-jnu\csc\alpha} \right|^2 \frac{G^2_{\phi,\alpha}(u)}{\left|\widetilde{\Phi}(u\csc\alpha)\right|^2} du \\
&\leq \left\| \frac{G_{\phi,\alpha}(u)}{\widetilde{\Phi}(u\csc\alpha)} \right\|_\infty^2 \int_I \left| \frac{1}{\sqrt{2\pi}} \sum_{|n|\geq N} \tilde{f}[n] e^{-jnu\csc\alpha} \right|^2 d\,(u\csc\alpha) \\
&= \left\| \frac{G_{\phi,\alpha}(u)}{\widetilde{\Phi}(u\csc\alpha)} \right\|_\infty^2 \sum_{|n|\geq N} |\tilde{f}[n]|^2 = \left\| \frac{G_{\phi,\alpha}(u)}{\widetilde{\Phi}(u\csc\alpha)} \right\|_\infty^2 \sum_{|n|\geq N} |f[n]|^2
\end{aligned}
\tag{57}
$$

so that (53) can be established. □

3.3. Applications

In many problems of practical interest one is interested in the samples of a chirp signal $f(t)$ which is ubiquitous in radar, sonar, and communications systems [2, 19]. The FRFT sampling theory states [9] that the Nyquist rate for sampling a chirp signal is lower than the one used in conventional Fourier theory. The signal $f(t)$ can be derived as follows [6]:

$$f(t) = \sum_{n\in\mathbb{Z}} f[n]\,\mathrm{sinc}(t-n)e^{-j\frac{t^2-n^2}{2}\cot\alpha}. \tag{58}$$

In practice one only has a finite number of samples of the function of interest, and therefore, this methodology is rarely used in real applications because of the slow decay of the sinc function. Now, we focus on the same problem based on the established FRFT sampling theory.

Based on the derived results, we can rewrite $f(t)$ as an expansion in terms of a general interpolation function $s(t)$ in (29). By Corollary 1, (58) can be viewed as a special case of (29) with $\phi(t)$ chosen as $\mathrm{sinc}(t)$ in Theorem 2. Our objective is then to choose a generator $\phi(t)$ of $\mathcal{V}_\alpha(\phi)$, which has faster time decay than the sinc function. For instance, we choose $\phi(t) = \beta^3(t)$, which is the cubic spline [20]. By applying the results of [20], we derive $\Phi(u\csc\alpha) = \mathrm{sinc}^4\left(\frac{u\csc\alpha}{2\pi}\right)$. By Theorem 4.5 of [22], it follows that $A_3 \leq G_{\phi,\alpha}(u) \leq 1$, where A_3 is the positive number defined in (4.2.21) with $m = 3$ in [22]. Hence, $\beta^3(t)$ satisfies the conditions for a Riesz basis in $\mathcal{V}_\alpha(\phi)$. Additionally, $\beta^3(t)$ has support on $[0, 4)$, and its values on the integers are $\beta^3(1) = \beta^3(3) = \frac{1}{6}$, $\beta^3(2) = \frac{2}{3}$ [20]. Then, it follows that $\widetilde{\Phi}(u\csc\alpha) = \frac{1}{6}e^{-ju\csc\alpha}(1 + 4e^{-ju\csc\alpha} + e^{-j2u\csc\alpha})$ which has no zeros for real u. Indeed, the polynomial $1 + 4z + z^2$ has zeros at $z_1 = -2-\sqrt{3}$, $z_2 = -2+\sqrt{3}$. Hence, $1/\widetilde{\Phi}(u\csc\alpha)$ may be found by using its Laurent series [21], and the interpolation function $s(t)$ can be derived as

$$s(t) = \sqrt{3}\sum_{n=0}^{\infty}(\sqrt{3}-2)^{n+1}\beta^3(t-n+1)+\sqrt{3}\sum_{n=1}^{\infty}(\sqrt{3}-2)^{n-1}\beta^3(t+n+1). \quad (59)$$

For the purpose of illustration, we observe a signal given by

$$f(t) = [2\sin(0.4\pi t) + 5\sin(0.5\pi t) + 7\sin(0.6\pi t)]\, e^{-j0.5kt^2} \quad (60)$$

where $k = 2$. It is band-limited in the fractional Fourier domain with $\alpha = \operatorname{arccot}(k)$. The maximum FRFT-frequency value of the signal is $0.6\pi \sin\alpha$. Following Xia's result in [6], the sampling rate Δ_α should satisfy $\Delta_\alpha \leq \frac{\pi \sin\alpha}{2\times 0.6\pi \sin\alpha}$. In our example, we choose $\Delta_\alpha = \frac{1}{2}$. The original signal $f(t)$ and its corresponding samples are shown in Fig. 1.

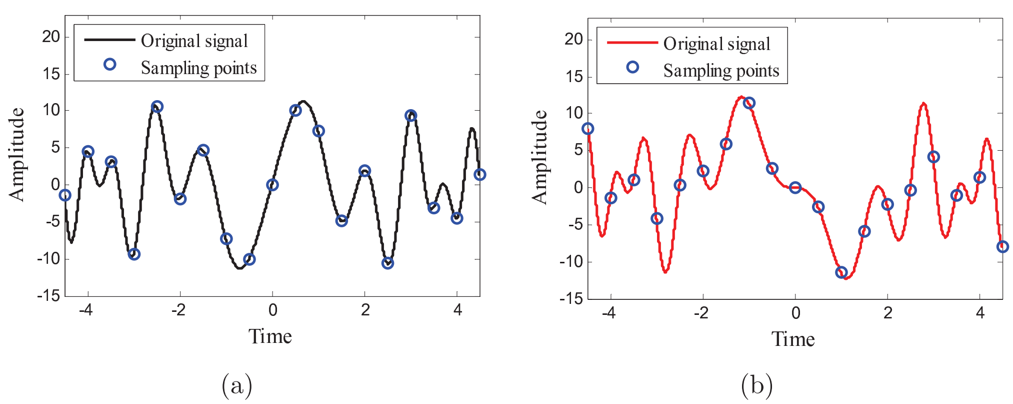
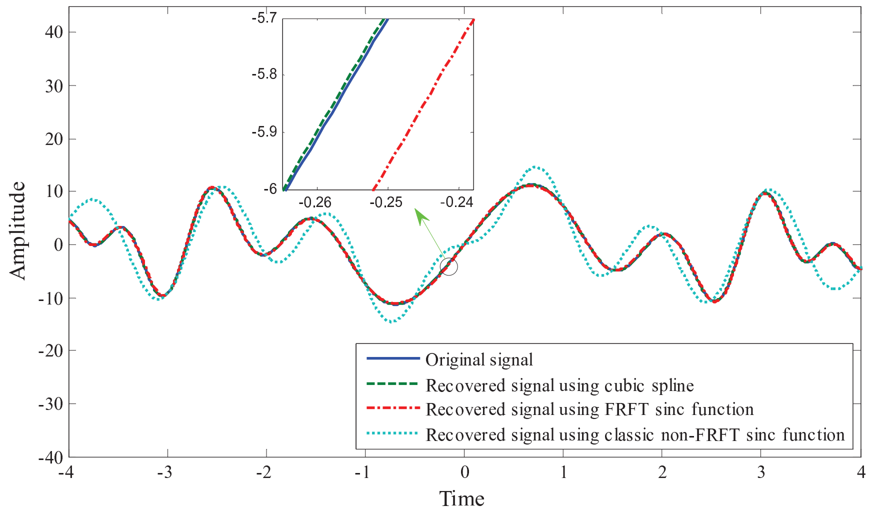
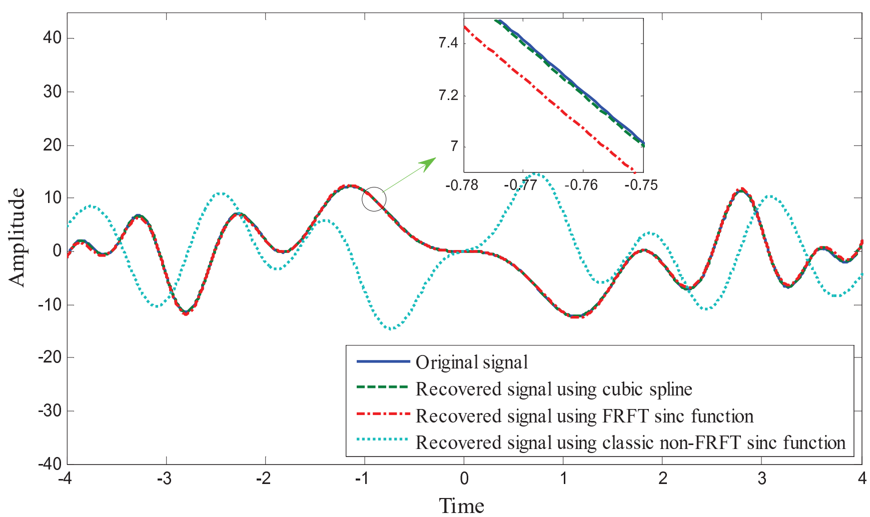


Fig. 1: The original signal and sampling points: (a) Real parts and (b) Imaginary parts.

Now, we try to recover $f(t)$, $t \in [-4, 4]$ using (29) (with ϕ chosen as a cubic spline) against the case of sinc interpolation (or $\phi = \text{sinc}$) under the condition that the number of sampling points is constrained to 19. The original and recovered signals are plotted in Fig. 2. Note that in Fig. 2,



(a)



(b)

Fig. 2: The original and recovered signals: (a) Real parts and (b) Imaginary parts.

the experimental results clearly show that for a finite number of samples and truncated sinc function, the proposed sampling and reconstruction method clearly outperforms the conventional sampling series expression given in (58).

The normalized mean-square error (NMSE) of the proposed method is 6.725×10^{-5}, where the NMSE is defined as NMSE $= \frac{\|\hat{f}(t)-f(t)\|^2_{L^2}}{\|f(t)\|^2_{L^2}}$, and $\hat{f}(t)$ denotes the recovered signal. By comparison, when using (58) and the classic non-FRFT sinc reconstruction, the NMSE are 2.0×10^{-3} and 0.4814, respectively.

Moreover, some of the ideas presented in this paper may be extended to study problems associated with shift-invariant spaces (SISs) [14]. Let $\mathcal{V}(\phi)$ be the SIS generated by the L^2-closure of the linear combination of $\{\phi(t-n)\}_{n\in\mathbb{Z}}$. The relationship between $\mathcal{V}_\alpha(\phi)$ and $\mathcal{V}(\phi)$ is given by

$$f(t) \in \mathcal{V}_\alpha(\phi) \Leftrightarrow f(t)e^{j\frac{t^2}{2}\cot\alpha} \in \mathcal{V}(\phi). \tag{61}$$

Consequently, $\{\phi_{n,\alpha}(t)\}_{n\in\mathbb{Z}}$ is a Riesz basis (or a frame) for $\mathcal{V}_\alpha(\phi)$ if and only if $\{\phi(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis (or a frame) for $\mathcal{V}(\phi)$, for which there are many known results, e.g., nonuniform sampling and reconstruction, wavelets and multiresolution analysis, oversampling, compressed sensing, and frames of translates [14, 24–29].

4. Conclusion

In this paper, we first introduced a class of function spaces and derived basic properties of their basis functions. Then, we established a sampling theorem for the FRFT in the function spaces without band-limiting constraints. The truncation error of sampling and some potential applications of the derived results were also presented.

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References

[1] H.M. Ozaktas, Z. Zalevsky, M.A. Kutay, The Fractional Fourier Transform with Applications in Optics and Signal Processing, Wiley, New York, 2001.

[2] E. Sejdić, I. Djurović, L. Stanković, Fractional Fourier transform as a signal processing tool: an overview of recent developments, Signal Processing 91 (2011) 1351–1369.

[3] S.-C. Pei, J.-J. Ding, Relations between fractional operations and time-frequency distributions, and their applications, IEEE Transactions on Signal Processing 49 (8) (2001) 1638–1655.

[4] J. Shi, X. Liu, N. Zhang, On uncertainty principle for signal concentrations with fractional Fourier transform, Signal Processing 92 (2012) 2830–2836

[5] J. Shi, X. Sha, X. Song, N. Zhang, Generalized convolution theorem associated with fractional Fourier transform, Wireless Communications and Mobile Computing (2012), doi: 10.1002/wcm.2254.

[6] X.-G. Xia, On bandlimited signals with fractional Fourier transform, IEEE Signal Processing Letters 3 (3) (1996) 72–74.

[7] A.I. Zayed, A.G. García, New sampling formulae for the fractional Fourier transform, Signal Processing 77 (1999) 111–114.

[8] A. Stern, Sampling of linear canonical transformed signals, Signal Processing 86 (2006) 1421–1425.

[9] R. Tao, B. Deng, W.-Q. Zhang, Y. Wang, Sampling and sampling rate conversion of bandlimited signals in the fractional Fourier transform domain, IEEE Transactions on Signal Processing 56 (1) (2008) 158–171.

[10] A. Bhandari, P. Marziliano, Sampling and reconstruction of sparse signals in fractional Fourier domain, IEEE Signal Processing Letters 17 (3) (2010) 221–224.

[11] K.K. Sharma, S.D. Joshi, Papoulis-like generalized sampling expansions in fractional Fourier domains and their application to superresolution, Optics Communications 278 (2007) 52–59.

[12] J. Shi, Y. Chi, N. Zhang, Multichannel sampling and reconstruction of bandlimited signals in fractional Fourier domain, IEEE Signal Processing Letters 17 (11) (2010) 909–912.

[13] Y.-L. Liu, K.-I. Kou, I.-T. Ho, New sampling formulae for non-bandlimited signals associated with linear canonical transform and nonlinear Fourier atoms, Signal Processing 90 (2010) 933–945.

[14] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, MA, 2003.

[15] A.I. Zayed, On the relationship between the Fourier and the fractional Fourier transforms, IEEE Signal Processing Letters 3 (12) (1996) 310–311.

[16] T. Erseghe, P. Kraniauskas, G. Cariolaro, Unified fractional Fourier transform and sampling theorem, IEEE Transactions on Signal Processing 47 (12) (1999) 3419–3423.

[17] R.M. Young, An Introduction to Nonharmonic Fourier Series, Academic Press, New York, 1980.

[18] M.J. Lighthill, An Introduction to Fourier Analysis and Generalised Functions, Cambridge University Press, London, 1959.

[19] P. Flandrin, Time-frequency and chirps, Proceeding of SPIE 4391 (2001) 161–175.

[20] M. Unser, A. Aldroubi, M. Eden, B-spline signal processing-Part I: Theory, IEEE Transactions on Signal Processing 41 (2) (1993) 821–833.

[21] J.H. Mathews, R.W. Howell, Complex Analysis for Mathematics and Engineering, Jones and Bartlett, Boston, 1997.

[22] C.K. Chui, An Introduction to Wavelets, Academic Press, Boston, 1992.

[23] W. Chen, S. Itoh, A sampling theorem for shift-invariant subspace, IEEE Transactions on Signal Processing 46 (10) (1998) 2822–2824.

[24] A. Aldroubi, K. Gröhenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Review 43 (4) (2001) 585–620.

[25] M. Bownik, G. Garrigós, Biorthogonal wavelets, MRA's and shift-invariant spaces, Studia Mathematica 160 (2004) 231–248.

[26] W. Chen, B. Han, R.-Q. Jia, On simple oversampled A/D conversion in shift-invariant spaces, IEEE Transactions on Information Theory 51 (2) (2005) 648–657.

[27] Y.C. Eldar, Compressed sensing of analog signals in shift-invariant spaces, IEEE Transactions on Signal Processing 57 (8) (2009) 2986–2997.

[28] J. Shi, N. Zhang, X. Liu, A novel fractional wavelet transform and its applications, SCIENCE CHINA Information Sciences 55 (2012) 1270–1279.

[29] P.G. Casazza, O. Christensen, N.J. Kalton, Frames of translates, Collectanea Mathematica 52 (1) (2001) 35–54